Determining the Relative Sign in the Einstein Field Equation

Randy S

Abstract The Einstein field equation describes the dynamics of the spacetime metric field by relating it to the matter tensor (stress-energy tensor). In this relationship, the relative sign of the matter tensor term should be such that test objects are attracted by positive masses. This article uses that criterion to determine the appropriate sign. The approach used here starts with a simple ansatz for the spacetime metric tensor that is static, spherically symmetric, and diagonal, generalized to an arbitrary number N of spacetime dimensions. The components of the Einstein tensor are calculated, and those results are used to do three things: to determine the appropriate sign of the matter tensor term, to derive an N-dimensional generalization of the Schwarzschild solution (article 24902), and to highlight an exceptional feature of the case $N = 3$.

Contents

1 Introduction

Like Maxwell's equations, the gravitational field equation in general relativity is one of the most iconic equations in physics. When the cosmological constant term is omitted, the equation has the form[1]

$$R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}, \tag{1}$$

where κ is a universal constant,[2] R_{ab} are the components of the the Ricci tensor, g_{ab} are the components of the metric tensor, g^{ab} are the components of the inverse metric tensor, $R \equiv g^{ab} R_{ab}$ are the components of the Ricci scalar,[3] and T_{ab} are the components of the **matter tensor**.[4,5,6]

This article determines the sign of κ by requiring that test objects are gravitationally attracted (not repelled) by a static isolated concentration of positive mass. The result is that if $N \geq 4$, then κ should be positive: $\kappa > 0$.

The analysis uses a simple ansatz for the metric tensor. The number N of spacetime dimensions is arbitrary for most of the calculation, but the final steps require $N \geq 4$. As a byproduct, an N-dimensional generalization of the Schwarzschild solution[7] will be obtained.

[1] This article uses the mostly-minus convention for the metric tensor g_{ab} and standard sign conventions for other related tensors as described in article 80838.

[2] When the number N of spacetime dimensions is 4, the constant κ is usually written as 8π times Newton's gravitational constant. The factor of 8π is convenient in the Newton-model approximation when $N = 4$, but it is inconvenient in general (Blau (2022), end of section 19.4; Robinson (2006); and article 00669).

[3] As usual in general relativity, a sum is implied over any index that appears both as a subscript and as a superscript in the same term.

[4] Various names are used, including *matter tensor* (Martin (1988)), *energy-momentum tensor* (Blau (2022)), *stress-energy-momentum tensor* (Wald (1984)), and *stress-energy tensor* (Misner *et al* (2017)).

[5] Articles 11475 and 78463 describe examples of matter tensors.

[6] In general relativity, the word *matter* usually refers to everything other than the metric field, so *matter* includes the electromagnetic field.

[7] Article 24902 reviews some properties of the Schwarzschild metric in the physically relevant case $N = 4$.

2 The ansatz

This article uses an ansatz for the metric tensor that is static, spherically symmetric, and diagonal. The metric tensor may be specified by writing out the equation for the proper time increment $d\tau$ in N-dimensional spacetime. This equation has the form[8,9]

$$d\tau^2 = \begin{cases} g_{ab}(x)\, dx^a\, dx^b & \text{in the mostly-minus convention,} \\ -g_{ab}(x)\, dx^a\, dx^b & \text{in the mostly-plus convention.} \end{cases}$$

This article will use the mostly-minus convention, but the sign of κ in equation (1) is the same for either convention.[10]

For both conventions, the ansatz is

$$d\tau^2 = A(r)\, dt^2 - B(r)\, dr^2 - r^2\, ds^2 \qquad A(r) > 0 \qquad B(r) > 0, \tag{2}$$

with

$$ds^2 \equiv \sum_k S_k(\phi)\, d\phi_k^2. \tag{3}$$

The independent coordinates are t, r, and ϕ_k with $k \in \{1, 2, ..., N-2\}$. The functions $S_k(\phi)$ depend only on the **angular** coordinates $\phi_1, \phi_2, ..., \phi_{N-2}$. The functions $S_k(\phi)$ are defined so that (3) is the proper distance increment ds for the unit sphere in $(N-1)$-dimensional euclidean space. With that definition of the functions $S_k(\phi)$, the metric defined by (2) is flat when $A = B = 1$. This implies that the curvature tensor is zero when $A = B = 1$, and this is the only thing we'll need to know about the functions $S_k(\phi)$.

[8] Article 48968

[9] As usual in general relativity, $d\tau^2$ is an abbreviation for $(d\tau)^2$, not to be confused with $d(\tau^2)$. The superscript on dx^a is an index, not an exponent.

[10] Article 80838

3 The geodesic equations

Section 5 will express the Ricci tensor in terms of the Levi-Civita connection coefficients Γ^c_{ab}, which are also the coefficients in the geodesic equations.[11] This article derives the geodesic equations using a method introduced in article 33547. Section 4 will extract the connection coefficients from these geodesic equations.

The geodesic equations are

$$\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{t}} = \frac{\delta L}{\delta t} \qquad \frac{d}{d\lambda}\frac{\delta L}{\delta \dot{r}} = \frac{\delta L}{\delta r} \qquad \frac{d}{d\lambda}\frac{\delta L}{\delta \dot{\phi}_k} = \frac{\delta L}{\delta \phi_k}, \tag{4}$$

with the **lagrangian**

$$L = A(r)\,\dot{t}^2 - B(r)\,\dot{r}^2 - r^2\,\dot{s}^2 \qquad \dot{s}^2 \equiv \sum_k S_k(\phi)\,\dot{\phi}_k^2. \tag{5}$$

This corresponds to the ansatz (2). Each overhead dot is a derivative with respect to a worldline parameter λ. where the variational derivatives $\delta/\delta t$ and $\delta/\delta\dot{t}$ are defined by temporarily treating t and $\dot{t}$ as independent variables, and likewise for the other coordinates r and ϕ_k.

Using (5) and the abbreviations $A' \equiv dA/dr$ and $S_{kj} \equiv dS_k/d\phi_j$, the variational derivatives are

$$\frac{\delta L}{\delta \dot{t}} = 2A\dot{t} \qquad \frac{\delta L}{\delta t} = 0$$
$$\frac{\delta L}{\delta \dot{r}} = -2B\dot{r} \qquad \frac{\delta L}{\delta r} = A'\dot{t}^2 - B'\dot{r}^2 - 2r\sum_k S_k\,\dot{\phi}_k^2$$
$$\frac{\delta L}{\delta \dot{\phi}_k} = -2r^2 S_k\dot{\phi}_k \qquad \frac{\delta L}{\delta \phi_k} = -r^2\sum_j S_{jk}\dot{\phi}_j^2.$$

[11] Article 03519

Using those results, the left-hand sides of the geodesic equations (4) are

$$
\begin{aligned}
\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{t}} &= 2\big(A\ddot{t} + A'\dot{r}\dot{t}\big) \\
\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{r}} &= -2\big(B\ddot{r} + B'\dot{r}^2\big) \\
\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{\phi}_k} &= -2\left(r^2 S_k\ddot{\phi}_k + 2rS_k\dot{r}\dot{\phi}_k + r^2\sum_j S_{kj}\dot{\phi}_j\dot{\phi}_k\right).
\end{aligned}
$$

After rearranging, the geodesic equations become

$$
\ddot{t} + \frac{A'}{A}\dot{r}\dot{t} = 0 \tag{6}
$$

$$
\ddot{r} + \frac{B'}{2B}\dot{r}^2 + \frac{A'}{2B}\dot{t}^2 - \frac{r}{B}\sum_k S_k\,\dot{\phi}_k^2 = 0 \tag{7}
$$

$$
\ddot{\phi}_k + \frac{2}{r}\dot{r}\dot{\phi}_k + \sum_j \frac{S_{kj}}{S_k}\dot{\phi}_j\dot{\phi}_k - \frac{1}{2}\sum_j \frac{S_{jk}}{S_k}\dot{\phi}_j^2 = 0. \tag{8}
$$

4 The connection coefficients

The connection coefficients Γ^c_{ab} may be extracted from equations (6)-(8) by comparing to the general form[12]

$$\ddot{x}^c = \Gamma^c_{ab}\dot{x}^a\dot{x}^c = 0$$

with the understanding that $\Gamma^c_{ba} = \Gamma^c_{ab}$. Using the symbol 0 for the index corresponding to t, using the symbol r for the index corresponding to r, using the symbol k for the index corresponding to ϕ_k, the nonzero connection coefficients are

$$\begin{aligned}
&\Gamma^0_{r0} = \Gamma^0_{0r} = \frac{A'}{2A} \\
&\Gamma^r_{rr} = \frac{B'}{2B} \\
&\Gamma^r_{00} = \frac{A'}{2B} \\
&\Gamma^r_{kk} = \frac{-rS_k}{B} \\
&\Gamma^k_{rk} = \Gamma^k_{kr} = \frac{1}{r} && \text{(no sum over } k\text{)} \\
&\Gamma^k_{jk} = \Gamma^k_{kj} = \frac{S_{kj}}{2S_k} && \text{(no sum over } k\text{, and } j \neq k\text{)} \\
&\Gamma^k_{jj} = \frac{-S_{jk}}{S_k} && (j \text{ and } k \text{ may be equal).}
\end{aligned}$$

These imply[13]

$$\Gamma^{\bullet}_{\bullet r} = \frac{A'}{2A} + \frac{B'}{2B} + \frac{N-2}{r}. \tag{9}$$

[12] Article 03519

[13] The symbol $\bullet$ is used here as an index, and a sum over $\bullet$ is implied.

5 The Ricci tensor

The Ricci tensor is[14,15,16]

$$R_{ab} = \partial_\bullet \Gamma^\bullet_{ab} - \partial_a \Gamma^\bullet_{\bullet b} + \Gamma^\times_{\times\bullet}\Gamma^\bullet_{ab} - \Gamma^\times_{a\bullet}\Gamma^\bullet_{\times b}. \tag{10}$$

The functions S_k in (3) are defined so that the ansatz (2) is flat when $A = B = 1$, so the Ricci tensor must be zero when $A = B = 1$. This implies

$$R_{ab} = \delta R_{ab} = \partial_\bullet \delta\Gamma^\bullet_{ab} - \partial_a \delta\Gamma^\bullet_{\bullet b} + \delta\left(\Gamma^\times_{\times\bullet}\Gamma^\bullet_{ab}\right) - \delta\left(\Gamma^\times_{a\bullet}\Gamma^\bullet_{\times b}\right) \tag{11}$$

with $\delta f(A,B) \equiv f(A,B) - f(1,1)$. For R_{00}, use the results from section 4 to get

$$\partial_\bullet \delta\Gamma^\bullet_{00} = \partial_r \Gamma^r_{00} \qquad \partial_0 \delta\Gamma^\bullet_{\bullet 0} = 0 \qquad \delta\left(\Gamma^\times_{\times\bullet}\Gamma^\bullet_{00}\right) = \Gamma^\times_{\times r}\Gamma^r_{00} \qquad \delta\left(\Gamma^\times_{0\bullet}\Gamma^\bullet_{\times 0}\right) = 2\Gamma^r_{00}\Gamma^r_{0r},$$

and use these together with (9) to get

$$\begin{aligned} R_{00} &= \partial_r \frac{A'}{2B} + \left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{N-2}{r}\right)\frac{A'}{2B} - \frac{(A')^2}{2AB} \\ &= \frac{A''}{2B} + \left(-\frac{A'}{2A} - \frac{B'}{2B} + \frac{N-2}{r}\right)\frac{A'}{2B}. \end{aligned} \tag{12}$$

For R_{rr}, use

$$\partial_\bullet \delta\Gamma^\bullet_{rr} = \partial_r \Gamma^r_{rr} \qquad \delta\left(\Gamma^\times_{\times\bullet}\Gamma^\bullet_{rr}\right) = \Gamma^\times_{\times r}\Gamma^r_{rr} \qquad \delta\left(\Gamma^\times_{r\bullet}\Gamma^\bullet_{\times r}\right) = \left(\Gamma^0_{0r}\right)^2 + \left(\Gamma^r_{rr}\right)^2$$

to get

$$\begin{aligned} R_{rr} &= \partial_r \frac{B'}{2B} - \partial_r\left(\frac{A'}{2A} + \frac{B'}{2B}\right) + \left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{N-2}{r}\right)\frac{B'}{2B} - \left(\left(\frac{A'}{2A}\right)^2 + \left(\frac{B'}{2B}\right)^2\right) \\ &= -\frac{A''}{2A} + \left(\frac{A'}{2A} + \frac{N-2}{r}\right)\frac{B'}{2B} + \left(\frac{A'}{2A}\right)^2. \end{aligned} \tag{13}$$

[14] The symbols $\bullet$ and $\times$ is used here as indices, and sums over these indices are implied.

[15] ∂_a denotes the partial derivative with respect to the ath coordinate.

[16] This convention for the overall sign of the Ricci tensor seems to be standard (article 80838).

For R_{kk}, use

$$\partial_\bullet \delta\Gamma^\bullet_{kk} = \partial_r \delta\Gamma^r_{kk} \qquad \partial_k \delta\Gamma^\bullet_{\bullet k} = 0$$
$$\delta\left(\Gamma^\times_{\times\bullet}\Gamma^\bullet_{kk}\right) = \Gamma^\times_{\times r}\Gamma^r_{kk} + (N-2)S_k \qquad \delta\left(\Gamma^\times_{k\bullet}\Gamma^\bullet_{\times k}\right) = 2\Gamma^k_{kr}\delta\Gamma^r_{kk}$$

to get

$$\begin{aligned} R_{kk} &= \partial_r\left(\left(1-\frac{1}{B}\right)rS_k\right) - \left(\frac{A'}{2A}+\frac{B'}{2B}+\frac{N-2}{r}\right)\frac{rS_k}{B} \\ &\quad + (N-2)S_k - 2\left(1-\frac{1}{B}\right)S_k \\ &= \left(\frac{B'}{B}-\frac{A'}{A}\right)\frac{rS_k}{2B} + (N-3)\left(1-\frac{1}{B}\right)S_k. \end{aligned} \tag{14}$$

The off-diagonal components of R_{ab} are zero:[17]

$$R_{ab} = 0 \qquad \text{when } a \neq b.$$

[17]The results for the connection coefficients $\Gamma^\cdot_{\cdot\cdot}$ in section 4 may be used to check that each term in equation (11) is zero when $a \neq b$.

6 The Einstein tensor

The Einstein tensor is

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R$$

with $R \equiv g^{ab} R_{ab}$. Equations (2)-(3) for the metric give

$$G_{00} = \frac{1}{2}\left(R_{00} + \frac{A}{B} R_{rr} + \sum_k \frac{A}{r^2 S_k} R_{kk} \right)$$

$$G_{rr} = \frac{1}{2}\left(R_{rr} + \frac{B}{A} R_{00} - \sum_k \frac{B}{r^2 S_k} R_{kk} \right)$$

$$G_{kk} = \frac{1}{2}\left(R_{kk} - \sum_{j\neq k} \frac{S_k}{S_j} R_{jj} + \frac{r^2 S_k}{A} R_{00} - \frac{r^2 S_k}{B} R_{rr} \right)$$

and then equations (12), (13), and (14) for the Ricci tensor give[18,19]

$$G_{00} = \frac{N-2}{2r}\left(\frac{B'}{B} + \frac{N-3}{r}(B-1) \right) \frac{A}{B} \tag{15}$$

$$G_{rr} = \frac{N-2}{2r}\left(\frac{A'}{A} - \frac{N-3}{r}(B-1) \right) \tag{16}$$

$$G_{kk} = \frac{r^2 S_k}{2B}\left(\frac{A''}{A} - \left(\frac{A'}{A} + \frac{B'}{B} \right)\frac{A'}{2A} + \frac{N-3}{r}\left(\frac{A'}{A} - \frac{B'}{B} \right) - \frac{(N-3)(N-4)}{r^2}(B-1) \right). \tag{17}$$

[18]When $N = 4$, these results agree with equations (6.2.3)-(6.2.5) in Wald (1984) except for overall factors of A and $1/B$ in equations (6.2.3) and (6.2.4), respectively. This apparent discrepancy is because Wald (1984) uses a non-coordinate basis (shown in equations (6.1.6)) and this article uses a coordinate basis. Wald (1984) uses the mostly-plus convention for the metric tensor, and this article uses the mostly-minus convention, but changing the overall sign of the metric tensor doesn't affect the Ricci tensor or the Einstein tensor (article 80838).

[19]For any metric, the Einstein tensor is identically zero when $N = 2$ (Martin (1988), chapter 10, problem 2). Equations (15)-(16) are consistent with this. The components G_{kk} are absent when $N = 2$ (section 2).

7 Determining the sign of κ

In terms of the Einstein tensor, the gravitational field equation (1) is

$$G_{ab} = \kappa T_{ab}. \tag{18}$$

This section determines the sign of κ using this criterion: a test object should be gravitationally attracted toward a static and spherically symmetric concentration of positive mass.

The sign will be determined by analyzing just one configuration of matter. That's sufficient because equation (18) – with the same value of κ – governs all configurations. This configuration will be used:

- T_{00} is positive[20] for $r < r_0$ and zero for $r > r_0$, for some $r_0 > 0$.
- For $r < r_0$, the mass density T_{00} has spherical symmetry about $r = 0$.
- The other components of T_{ab} are zero everywhere.

To simplify the analysis, an approximation will be used: terms that are quadratic in the dimensionless quantities $A - 1$ and $B - 1$ (and their derivatives) will be neglected. This is not a good approximation for all configurations, but it is for some configurations,[21] and that's sufficient for determining the sign of κ.

For the configuration described above, equation (18) gives

$$G_{rr} = 0 \qquad\qquad G_{kk} = 0.$$

In the approximation described above, equation (17) implies

$$G_{kk} \approx \frac{r^2 S_k}{2}\left(A'' + \frac{N-3}{r}\,(A' - B') - \frac{(N-3)(N-4)}{r^2}(B-1)\right).$$

[20] The mass density is assumed to be small enough so that a black hole has not formed. Section 9 will show that this is consistent with the conditions $A > 0$ and $B > 0$ in the ansatz (2).

[21] This can be anticipated intuitively from the fact that the ansatz (2) is flat when $A = B = 1$.

After using $G_{rr}=0$ to express $B-1$ in terms of $A'/A \approx A'$, this becomes

$$\begin{aligned}G_{kk} &\approx \frac{r^2 S_k}{2}\left(A'' + \frac{N-3}{r}\,(A'-B') - \frac{(N-4)}{r}\,A'\right)\\ &= \frac{r^2 S_k}{2}\left(A'' + \frac{A'}{r} - \frac{N-3}{r}\,B'\right),\end{aligned}$$

and G_{00} becomes

$$G_{00} \approx \frac{N-2}{2r}\,(B'+A')\,. \tag{19}$$

If $N \neq 3$, then the condition $G_{kk}=0$ implies

$$B' \approx \frac{r}{N-3}\left(A'' + \frac{A'}{r}\right).$$

Use this in equation (19) to get

$$G_{00} \approx \frac{N-2}{2(N-3)}\left(A'' + \frac{N-2}{r}A'\right) = \frac{N-2}{2(N-3)r^{N-2}} \times \frac{d}{dr}\left(r^{N-2}A'\right).$$

Use this in the 00 component of (18) to get

$$\frac{N-2}{2(N-3)} \times \frac{d}{dr}\left(r^{N-2}A'\right) \approx \kappa r^{N-2} T_{00}, \tag{20}$$

and integrate over r from $r=0$ to $r=s$ to get

$$\frac{N-2}{2(N-3)} \times s^{N-2}A'(s) \approx \kappa \int_0^s dr\; r^{N-2}T_{00}(r). \tag{21}$$

We can choose $s > r_0$ so that the left-hand side is evaluated at a point where matter is absent. Now we can apply the criterion from the beginning of this section: a test object should be gravitationally attracted toward a static and spherically symmetric concentration of positive mass. In particular, a test object which is initially at rest in this coordinate system should start to fall toward $r=0$. According to geodesic equation (7), this requires $A' > 0$. The integrand on the right-hand side of (21) is also positive, so if $N \geq 4$, then κ must be positive.

8 The exceptional case $N = 3$

Now suppose $N = 3$. In a region of spacetime where $T_{ab} = 0$, the field equation (18) says $G_{ab} = 0$. The exact results (15)-(17) for the Einstein tensor G_{ab} then give $A' = B' = 0$ when $N = 3$, so the quantities A and B in the ansatz (2) are both constants. This implies that the curvature tensor $R_{abc}{}^d$ is zero, so this spacetime is flat. Using $A' = 0$ in the geodesic equation (7) shows that a test object that is initially at rest in this coordinate system does not fall.

In fact, when $N = 3$, the field equation (18) implies that the metric must be flat wherever $T_{ab} = 0$.[22] This feature of the case $N = 3$ showed up in section 7 as an obstruction to the existence of a nontrivial weak-curvature approximation.

In Newton's model of gravity,[23] the force on a test object is proportional to ∇V, the gradient[24] of a function V that is related to the mass density T_{00} by $\nabla^2 V \propto T_{00}$, where ∇^2 is the laplacian. When $N \geq 4$, general relativity reproduces Newton's model under the appropriate approximations.[25,26] In contrast, when $N = 3$, general relativity does not reproduce the $N = 3$ version of Newton's model.[25] This is another manifestation of the fact that when $N = 3$, the equation (18) requires the metric to be flat wherever $T_{ab} = 0$.

Even though A and B are constants when $N = 3$, they may still be different from 1. For $B > 1$ the metric (2) has a **conical singularity** at $r = 0$, so falling objects can meet twice if they travel past the central mass on opposite sides.[27,28,29] In this sense, the central mass affects the motion of test objects in general relativity even when $N = 3$, even though the spacetime outside the mass is flat.

[22] Martin (1988), section 8.3

[23] Article 50710

[24] Here, ∇_k denotes the partial derivative with respect to the kth space coordinate, not a covariant derivative.

[25] Robinson (2006)

[26] Equation (20) confirms this, thanks to the identity $r^{N-2}\nabla^2 A(r) = \frac{d}{dr}\left(r^{N-2}A'(r)\right)$.

[27] Jackiw (1990)

[28] The name *conical singularity* comes from an analogy with the surface of a cone: geodesics (straight lines) drawn on the surface of a cone can intersect each other twice, even though the surface of the cone is intrinsically flat.

[29] Derivation: write $B\,dr^2 + r^2\,d\phi^2 = d\tilde{r}^2 + \tilde{r}^2\,d\tilde{\phi}^2$ with $\tilde{r} \equiv B^{1/2}r$ and $\tilde{\phi} = \phi/B^{1/2}$. The original coordinates satisfy $r > 0$ and $\phi \sim \phi + 2\pi$, so the new coordinates satisfy $\tilde{r} > 0$ and $\tilde{\phi} \sim \tilde{\phi} + 2\pi/B^{1/2}$, where "$\sim$" means equivalence.

9 The Schwarzschild-Tangherlini solution

This section determines the form the functions $A(r)$ and $B(r)$ in the part of spacetime where $T_{ab} = 0$, without using any approximations. Where $T_{ab} = 0$, equation (18) says $G_{ab} = 0$. Combine this with equations (15)-(16) to get

$$\frac{A'}{A} + \frac{B'}{B} = 0, \tag{22}$$

and use this in $G_{rr} = 0$ to get

$$\frac{B'}{B} + \frac{N-3}{r}(B-1) = 0 \qquad \Rightarrow \qquad \left(1 - \frac{1}{B}\right)' + \frac{N-3}{r}\left(1 - \frac{1}{B}\right) = 0$$

which implies

$$\frac{1}{B} = 1 - \frac{\beta}{r^{N-3}}. \tag{23}$$

for some constant β. Equation (22) implies $A \propto 1/B$. The proportionality factor must be positive so that the ansatz (2) has lorentzian signature, and then the proportionality factor may be absorbed into the definition of the coordinate t, so we might as well set $A = 1/B$.

This holds for all $N \geq 3$. If $N > 3$, then the condition $A' > 0$ implies $\beta < 0$, so

$$\boxed{A = 1 - \left(\frac{r_s}{r}\right)^{N-3} \qquad\qquad B = A^{-1}} \tag{24}$$

for some $r_s > 0$ called the **Schwarzschild radius**. This solution is called the **Schwarzschild-Tangherlini** metric.[30] It generalizes the Schwarzschild metric to an arbitrary number of dimensions. The $N = 4$ version of this metric has real-world applications as a good approximation near (but outside) an isolated non-rotating

[30] Myers (2012), section 5.1

spherical body whose mass density is not too extreme.[31,32] The constant β in (23) is related to the body's total mass.[33]

The result (24) was derived using only two components of the field equation (18), namely $G_{00} = 0$ and $G_{rr} = 0$. The result (24) is also consistent with $G_{kk} = 0$, which we can check using equation (17), and the off-diagonal components of G_{ab} are zero for arbitrary $A(r)$ and $B(r)$. This shows that when $N \geq 4$ and $T_{ab} = 0$, the ansatz (2) satisfies the field equations (18) for all $r > 0$ when A and B are given by (24).

For the purpose of determining the sign of the constant κ in the field equation (18), section (2) asserted that $A > 0$ and $B > 0$ in the ansatz (2). This is consistent with the approximation used in section 7, because the condition $A \approx 1$ implies $r_s/r \ll 1$. However, the solution (24) is also valid for $0 < r < r_s$, where A and B are both negative. In this case, the signature of (2) is still lorentzian, but the ∂_t and ∂_r directions are spacelike and timelike, respectively – the opposite of the case where A and B are both positive. Article 24902 clarifies the relationship between the regions $0 < r < r_s$ and $r > r_s$, using $N = 4$ as an example.

[31] Article 24902

[32] It is also valid for non-rotating spherical bodies with extreme mass densities, but real bodies with extreme mass densities – like real neutron stars – are expected to be rotating rapidly.

[33] Wald (1984), section 6.2

10 References

Blau, 2022. "Lecture Notes on General Relativity" (updated September 18, 2022) http://www.blau.itp.unibe.ch/GRLecturenotes.html

Jackiw, 1990. "Topics in planar physics" *Nuclear Physics B (Proceedings Supplement)* **18A**: 107-170

Martin, 1988. *General Relativity: A guide to its Consequences for Gravity and Cosmology.* John Wiley & Sons

Misner *et al*, 2017. *Gravitation.* Princeton University Press

Myers, 2012. "Chapter 5: Myers-Perry black holes." Pages 101-133 in *Black Holes in Higher Dimensions*, edited by Horowitz (Cambridge University Press) https://arxiv.org/abs/1111.1903

Robinson, 2006. "Normalization conventions for Newtons constant and the Planck scale in arbitrary spacetime dimension" https://arxiv.org/abs/gr-qc/0609060

Wald, 1984. *General Relativity.* University of Chicago Press

11 References in this series

Article **00669** (https://cphysics.org/article/00669):
"Units in Electrodynamics" (version 2022-06-04)

Article **03519** (https://cphysics.org/article/03519):
"Covariant Derivatives and Curvature" (version 2023-12-11)

Article **11475** (https://cphysics.org/article/11475):
"Classical Scalar Fields in Curved Spacetime" (version 2023-12-16)

Article **24902** (https://cphysics.org/article/24902):
"The Ideal Non-Rotating Black Hole" (version 2023-12-09)

Article **33547** (https://cphysics.org/article/33547):
"Free-Fall, Weightlessness, and Geodesics" (version 2022-10-23)

Article **48968** (https://cphysics.org/article/48968):
"The Geometry of Spacetime" (version 2022-10-23)

Article **50710** (https://cphysics.org/article/50710):
"Newton's Model of Gravity" (version 2022-02-05)

Article **78463** (https://cphysics.org/article/78463):
"Energy, Momentum, and Angular Momentum in Classical Electrodynamics" (version 2023-12-15)

Article **80838** (https://cphysics.org/article/80838):
"Sign Conventions in General Relativity" (version 2023-12-09)