The Action Principle in Classical Electrodynamics

Randy S

Abstract The action principle for fields was introduced in article 11475. This article introduces the action principle for electrodynamics in flat spacetime.

Contents

1 Review of the equations of motion

Maxwell's equations and the Lorentz force equation are introduced in articles 31738 and 54711. This article uses the same notation and conventions, but using uppercase for the functions that describe the particle's worldline.

The electromagnetic (EM) field is represented by the **Faraday tensor**, whose components are denoted $F_{ab}(x)$. The Faraday tensor is **antisymmetric**, which means

$$F_{ab}(x) = -F_{ba}(x). \tag{1}$$

The relationship between F_{ab} and $\mathbf{E}, \mathbf{B}$ is described in article 31738.

Maxwell's equations can be written as two equations. The first equation[1]

$$\partial_{[a} F_{bc]} = 0 \tag{2}$$

does not depend on a metric field. The second equation

$$\partial_a F^{ab} = -J^b \tag{3}$$

does depend on a metric field, which is assumed here to be the Minkowski metric. The J on the right-hand side accounts for charges and currents.

Now consider a pointlike spin-0 particle of mass m and charge q. The particle's worldline can be described by specifying its coordinates as functions of its proper time: $x^a = X^a(\tau)$. The particle's behavior is governed by the **Lorentz force equation**

$$\frac{dp^c}{d\tau} = \frac{q}{m}\, p^a\, F_{ab}\big(X(\tau)\big)\eta^{bc} \tag{4}$$

where η_{ab} are the components of the Minkowski metric, and

$$p^a \equiv m\frac{dX^a}{d\tau}. \tag{5}$$

The relationship between the current density J and the particle's coordinates X will be addressed later in this article.

[1] Square brackets around indices indicate complete antisymmetrization.

2 The gauge field

Spacetime as we know it is topologically trivial: it can be completely covered by a single coordinate system.[2] In this case, the pair of equations (1) and (2) is equivalent to the single equation

$$F_{ab}(x) = \partial_a A_b(x) - \partial_b A_a(x), \tag{6}$$

where A is called the **gauge field**.[3] Using the gauge field, electrodynamics can be formulated in terms of an action principle with a local lagrangian. The gauge field is not uniquely determined by F_{ab}, because the combination (6) is invariant under the **gauge transformation**

$$A_a(x) \to A_a(x) + \partial_a \theta(x) \tag{7}$$

for any function $\theta(x)$. This shows that the electromagnetic field F_{ab} can represented by a gauge field in many different ways, called different **gauges**.

Physical predictions do not depend on which gauge we use. In physics, part of the task of specifying any model is to specify how the formalism relates to the real world – that is, which things represent *observables*. Observables for the EM field are invariant under gauge transformations (7).

One mathematical consequence of gauge invariance is that Maxwell's equations cannot completely determine the future of the gauge field. To see this, consider any configuration A of the gauge field. By choosing $\theta(x)$ to be zero everywhere except within a bounded region R of spacetime, we obtain another configuration A' that is equal to A everywhere except within R. Even if we specify A everywhere in the past of R, Maxwell's equations cannot determine the behavior of the gauge field in R. They do determine the behavior of F, but A' and A both give the same F, so Maxwell's equations cannot completely determine the behavior of the gauge field.

[2] General relativity encourages us to consider other possibilities, but we'll ignore those possibilities here.

[3] In a model where spacetime has nontrivial topology, we would need to cover the spacetime manifold with topologically trivial patches, using a representation of the form (6) within each patch and using **transition functions** to describe how the patches are related to each other.

3 The action

In classical electrodynamics, the dynamic variables are:

- The electromagnetic field, represented in terms of a gauge field A_a.
- One or more charged particles, each represented in terms of a worldline $X^a(\lambda)$ parameterized by λ. The parameterization is arbitrary: λ is not necessarily the particle's proper time.

With a single charged particle, the action is

$$S = S_A + S_X + S_{XA} \tag{8}$$

with this notation:

- S_A includes all terms that depend on A but not on X,
- S_X includes all terms that depend on X but not on A,
- S_{XA} includes all terms that depend on both X and A.

Explicitly, the field-only part of the action is

$$S_A = -\frac{1}{4}\int d^{D+1}x\ F_{ab}F^{ab} \tag{9}$$

with F_{ab} given by (6). The gauge field is a function of the $D+1$ spacetime coordinates, collectively denoted by a lowercase x. (Uppercase X denotes the particle's worldline.) The particle-only part of the action is

$$S_X = -m\int d\lambda\ \sqrt{\dot{X}^a\dot{X}_a}. \tag{10}$$

An overhead dot denotes a derivative with respect to the parameter λ. The interaction part of the action is

$$\boxed{S_{XA} = q\int d\lambda\; \dot{X}^a A_a\Big(X(\lambda)\Big).} \tag{11}$$

In terms of the action, the equation of motion for the gauge field is

$$\frac{\delta S}{\delta A_b(x)} = 0, \tag{12}$$

and the equation of motion for the particle is

$$\frac{\delta S}{\delta X^b(\lambda)} = 0. \tag{13}$$

Equations (12) and (13) are studied in sections 5 and 6, respectively.

The generalization to multiple charged particles is straightforward: for each particle, the action has one term of the form S_X and one term of the form S_{XA}. Each particle may have a different mass m and charge q. The equation of motion for the nth particle is

$$\frac{\delta S}{\delta X_n^b(\lambda)} = 0,$$

where $X_n^a(\lambda)$ are the coordinates along the nth particle's worldline. Instead of carrying the extra index n to distinguish between different particles, this article considers only the single-particle case.

4 Invariance properties of the action

The action shown above has three invariance properties:

- Lorentz invariance,
- reparameterization invariance,
- gauge invariance.

Article 00418 addresses Lorentz invariance. **Reparameterization invariance** means that the action does not depend on how the particle's worldline is parameterized. If the parameterization is changed by regarding the old parameter λ as a function of a new parameter λ', then S_A is obviously unaffected (because it doesn't involve the particle's worldline at all), and the other two terms also retain the same form because

$$d\lambda = d\lambda' \frac{d\lambda}{d\lambda'} \qquad\qquad \frac{dX^a}{d\lambda} = \frac{d\lambda'}{d\lambda}\frac{dX^a}{d\lambda'},$$

so all of the factors of $d\lambda/d\lambda'$ and $d\lambda'/d\lambda$ cancel each other. (The square-root in S_X is essential for this.) **Gauge invariance** means that the action is invariant under the transformation

$$A_a \to A_a + \partial_a\theta \tag{14}$$

for any function $\theta(x)$ that approaches a constant at infinity. The term S_A is gauge-invariant because F_{ab} is, and the term S_{XA} is gauge-invariant because the effect of the gauge transformation (14) on S_{XA} is

$$\delta S_{XA} = \int d\lambda\; \dot{X}^a\partial_a\theta(X) = \int d\lambda\; \dot{\theta}\Big(X(\lambda)\Big).$$

If $\theta(x)$ approaches the same constant at infinity in all directions in spacetime, then the integral of the derivative is zero. The action is invariant under these gauge transformations.

5 Equations of motion from the action, part 1

This section focuses on the equation of motion for the gauge field – equation (12). The term S_X does not depend on the gauge field, so it does not contribute to that equation. The remaining terms have different structures: S_A is an integral over all of the spacetime coordinates, and S_{XA} is an integral over the worldline's parameter λ. To make the action principle easier to handle, we can start by writing both terms the same way, as integrals over all of the spacetime coordinates, like this:

$$S_A = \int d^{D+1}x\; L_A(x) \qquad\qquad S_{XA} = \int d^{D+1}x\; L_{XA}(x)$$

with[4]

$$L_A(x) = -\frac{1}{4}F_{ab}(x)F^{ab}(x) \tag{15}$$

$$L_{XA}(x) = J^a(x)\,A_a(x) \tag{16}$$

$$J^a(x) \equiv q\int d\lambda\; \dot{X}^a\,\delta^{D+1}\big(x - X(\lambda)\big). \tag{17}$$

Now, just like in article 49705, the action principle leads to the Euler-Lagrange equation

$$\partial_a \frac{\delta L}{\delta\partial_a A_b(x)} = \frac{\delta L}{\delta A_a(x)} \tag{18}$$

with $L = L_A + L_{XA}$. Use equations (15) and (16) in (18) to recover equation (3), where now the current density J in that equation is given by (17).

The metric-independent equation of motion (2) is already implied by equation (6): instead of coming from the action principle, it comes from the relationship of F_{ab} to the gauge field, which is the independent entity in the action principle.

[4] In the multi-particle version of the model, equation (17) is $J^a(x) = \sum_n q_n \int d\lambda\; \dot{X}_n^a\,\delta^{D+1}\big(x - X_n(\lambda)\big)$.

6 Equations of motion from the action, part 2

This section focuses on the equation of motion for the particle – equation (13). For this, we can leave the other terms S_X and S_{XA} in their original form, as integrals over the paramter λ along the particle's worldline. The term S_A does not depend on the particle's worldline $X(\lambda)$, so it doesn't contribute to the particle's equation of motion (13). Equation (13) leads to the Euler-Lagrange equation

$$\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{X}^b(\lambda)} = \frac{\delta L}{\delta X^b(\lambda)} \tag{19}$$

where now $L = L_X(\lambda) + L_{XA}(\lambda)$, with $L_X(\lambda)$ and $L_{XA}(\lambda)$ being the integrands of equations (10) and (11), respectively. The variational derivatives are

$$\begin{aligned} \frac{\delta L}{\delta \dot{X}^b(\lambda)} &= -m\frac{\dot{X}_b}{(\dot{X}^c\dot{X}_c)^{1/2}} + qA_b\big(X(\lambda)\big) \\ \frac{\delta L}{\delta X^b(\lambda)} &= q\dot{X}^a\partial_b A_a\big(X(\lambda)\big). \end{aligned}$$

Use these together with the identity

$$\dot{A}_b\Big(X(\lambda)\Big) = \dot{X}^a\partial_a A_b\Big(X(\lambda)\Big)$$

in the Euler-Lagrange equation to get the equations of motion for the particle:

$$m\frac{d}{d\lambda}\frac{\dot{X}_b}{(\dot{X}^c\dot{X}_c)^{1/2}} = q\dot{X}^a(\lambda)F_{ab}\Big(X(\lambda)\Big). \tag{20}$$

After choosing the worldline parameter λ to be the particle's proper time τ so that $(\dot{X}^c\dot{X}_c)^{1/2} = 1$, this becomes the Lorentz force equation (4).

7 The stress-energy tensor

The (Hilbert) stress-energy tensor is defined by[5]

$$T^{ab}(x) \equiv \frac{-2}{\sqrt{|\det g|}}\,\frac{\delta S}{\delta g_{ab}(x)}. \tag{21}$$

To use the definition (21), the action introduced in section 3 needs to be generalized to an arbitrary metric g_{ab}. The generalization is

$$S = S_A + S_X + S_{AX} \tag{22}$$

with

$$\begin{aligned} S_A &= -\frac{1}{4}\int d^N x\ \sqrt{|\det g|}g^{ab}g^{cd}F_{ac}F_{bd} \\ S_X &= -m\int d\lambda\ \sqrt{g_{ab}\dot{X}^a\dot{X}^b} \\ S_{XA} &= q\int d\lambda\ \dot{X}^a A_a(X(\lambda)). \end{aligned}$$

The term S_{XA} does not involve the metric tensor, so only S_A and S_X contribute to the stress-energy tensor (21). The contribution from S_A can be worked out using the identities shown in article 11475, and the result is shown in article 32191. The contribution from S_X is calculated in article 41182. Altogether, the result is

$$T^{ab} = \frac{1}{4}\eta^{ab}F^{cd}F_{cd} - F^{ac}F^{b}{}_{c} + m\int d\tau\ \dot{X}^a\dot{X}^b\,\delta^4\big(x - X(\tau)\big)$$

after specializing to the Minkowski metric.[6]

[5] Article 11475 introduces the **Hilbert** stress-energy tensor, and article 32191 explains its relationship to the **canonical** stress-energy tensor – the one associated with translation symmetry via Noether's theorem (in flat spacetime).

[6] We can choose a specific metric *after* calculating the variational derivative in the definition (21).

8 Non-relativistic approximation

The non-relativistic Lorentz force equation can be derived in either of two ways: by applying the non-relativistic approximation directly to the Lorentz force equation, as in article 54711, or by applying the non-relativistic approximation to the *action* and then using the action principle. This section uses the second approach.

Use the notation

$$\mathbf{X} \equiv (X^1,\, X^2,\, ...,\, X^D)$$
$$\mathbf{A} \equiv (A_1,\, A_2,\, ...,\, A_D).$$

In the action shown in section 3, the particle's worldline is parameterized by an arbitrary parameter λ. As explained in section 4, the action is invariant under reparameterizations. To facilitate the nonrelativistic approximation, we can take the parameter λ to be the "time" coordinate $t \equiv x^0$. Then S_X and S_{XA} (equations (10) and (11)) become

$$S_X = -m \int dt\ \sqrt{1-\dot{\mathbf{X}}^2} \tag{23}$$

$$S_{XA} = q \int dt\ (A_0 + \dot{\mathbf{X}}\cdot\mathbf{A}) \tag{24}$$

because $\dot{X}^0 = 1$ in this parameterization. (An overhead dot denotes a derivative with respect to the time coordinate t.) The quantities A_0 and $\mathbf{A}$ in the integrand are evaluated at the point on the particle's worldline specified by the parameter t.

The non-relativistic approximation applies when all of the particles' velocities are much less than the speed of light (which is 1 in the units assumed here), so

$$\dot{\mathbf{X}}^2 \ll 1. \tag{25}$$

In this approximation, we can use

$$\sqrt{1-\dot{\mathbf{X}}^2} \approx 1 - \frac{1}{2}\dot{\mathbf{X}}^2$$

to get the approximate action[7]

$$S_{\text{nonrel}} = \int dt\; L_{\text{nonrel}} \qquad\qquad L_{\text{nonrel}} = \frac{\dot{\mathbf{X}}^2}{2} + q\dot{\mathbf{X}}\cdot\mathbf{A} + qA_0 + \text{const.} \tag{26}$$

Requiring that $\mathbf{x}(t)$ satisfy the condition of stationary action $\delta S_{\text{nonrel}}/\delta\mathbf{X} = 0$ gives the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\delta L_{\text{nonrel}}}{\delta\dot{\mathbf{X}}} = \frac{\delta L_{\text{nonrel}}}{\delta\mathbf{X}}.$$

Use

$$\frac{\delta L_{\text{nonrel}}}{\delta\dot{\mathbf{X}}} = m\dot{\mathbf{X}} + q\mathbf{A} \qquad\qquad \frac{\delta L_{\text{nonrel}}}{\delta X^j} = q\dot{\mathbf{X}}\cdot(\nabla_j\mathbf{A}) + q\nabla_j A_0$$

and

$$\frac{d}{dt}\mathbf{A} = \frac{\partial}{\partial t}\mathbf{A} + \dot{\mathbf{X}}\cdot\nabla\mathbf{A}$$

to reduce the Euler-Lagrange equation to

$$m\ddot{X}^j = q(E_j + B_{jk}\dot{X}^k)$$

with[8]

$$\mathbf{E} \equiv \nabla A_0 - \frac{\partial}{\partial t}\mathbf{A} \qquad\qquad B_{jk} \equiv F_{jk}.$$

For $D = 3$, this reduces to

$$\frac{d}{dt}\mathbf{v} = \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{27}$$

$$\mathbf{v} \equiv \dot{\mathbf{X}} \qquad\qquad \mathbf{B} \equiv (B_{23},\, B_{31},\, B_{12}) = \nabla\times\mathbf{A}.$$

This agrees with the result derived in article 54711 for the nonrelativistic approximation to the Lorentz force equation. Equation (27) is exact in the context of the model defined by the action (26), because in this approach the nonrelativistic approximation is already built into the action (26).

[7] Unlike the original action (10)-(11), the action (26) is not reparameterization-invariant. That's okay, because we have already chosen the worldline parameter to be the "time" coordinate, and we won't need to change this.

[8] With this sign convention, A_0 is the negative of the traditional **electric potential** V.

9 References in this series

Article **00418** (https://cphysics.org/article/00418):
"Diffeomorphisms, Tensor Fields, and General Covariance" (version 2022-02-20)

Article **11475** (https://cphysics.org/article/11475):
"Classical Scalar Fields in Curved Spacetime" (version 2022-02-05)

Article **31738** (https://cphysics.org/article/31738):
"The Electromagnetic Field and Maxwell's Equations" (version 2022-02-18)

Article **32191** (https://cphysics.org/article/32191):
"Relationship Between the Stress-Energy Tensors" (version 2022-02-18)

Article **41182** (https://cphysics.org/article/41182):
"Energy and Momentum at All Speeds: Derivation" (version 2022-02-18)

Article **49705** (https://cphysics.org/article/49705):
"Classical Scalar Fields and Local Conservation Laws" (version 2022-02-05)

Article **54711** (https://cphysics.org/article/54711):
"Charged Particles in an Electromagnetic Field" (version 2022-02-18)