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The Curvature of a Conformally Flat Manifold

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Abstract The standard N-dimensional sphere is an example of a smooth manifold that is curved but conformally flat. This article calculates the Ricci tensor for an arbitrary N-dimensional conformally flat manifold with arbitrary signature and then shows that the result simplifies as expected when the space is maximally symmetric. The result is used to show that the scalar curvature for a standard sphere of radius σ is $N(N-1)/\sigma^2$.

Contents

1 Introduction, outline, and notation

A metric is called **conformally flat** if each point has a neighborhood in which the components g_{ab} of the metric can be written

$$g_{ab}(x) = e^{2\phi(x)}\eta_{ab} \tag{1}$$

in some coordinate system x, where η_{ab} is a flat metric. Without loss of generality, we can use a coordinate system in which η_{ab} is independent of the coordinates x. This article calculates the scalar curvature of an arbitrary conformally flat metric, with arbitrary signature and an arbitrary number N of dimensions.

Section 8 will show that the standard metric of the N-dimensional sphere is conformally flat. Section 2 will derive the connection coefficients for an arbitrary conformally flat manifold, and section 4 will use those results to derive the Ricci tensor and scalar curvature. Section 8 will specialize the result to the N-dimensional sphere.

Sections 2-4, in which the signature is arbitrary, will use the standard index notation. Each index takes values in $\{1, 2, ..., N\}$. The ath coordinate will be denoted x^a, using the superscript as an index (not as an exponent). The partial derivative with respect to the ath coordinate will be denoted ∂_a.

The components of the inverse of the metric tensor will be denoted g^{ab}, and the components of the inverse of the flat metric tensor η_{ab} in equation (1) will be denoted η^{ab}. The components of the identity matrix will be denoted δ_a^b. A sum over an index is implied whenever that index appears both as a subscript and as a superscript in the same term, as in the identities $g^{ab}g_{bc} = \eta^{ab}\eta_{bc} = \delta_c^a$. Sometimes a symbol like $\bullet$ or $\times$ will be used as an index, because this can help make equations with several indices easier to parse more quickly.

An index-free notation will be introduced starting in section 5 to streamline the analysis of maximally symmetric spaces.

2 The connection coefficients

Start with the metric (1) of an arbitrary conformally flat manifold, using a coordinate system in which the flat factor η_{ab} is independent of the coordinates x. The general expression for the connection coefficients is[1]

$$\Gamma^c_{ab} = \frac{1}{2} g^{c\bullet}(\partial_a g_{b\bullet} + \partial_b g_{a\bullet} - \partial_\bullet g_{ab}). \tag{2}$$

Use equation (1) to get

$$\Gamma^c_{ab} = \eta^{c\bullet}(\phi_a \eta_{b\bullet} + \phi_b \eta_{a\bullet} - \phi_\bullet \eta_{ab}) = \phi_a \delta^c_b + \phi_b \delta^c_a - \phi^c \eta_{ab} \tag{3}$$

with

$$\phi_a \equiv \partial_a \phi \qquad\qquad \phi^a \equiv \eta^{a\bullet}\partial_\bullet \phi. \tag{4}$$

The abbreviation

$$\phi_{ab} \equiv \partial_a \partial_b \phi \tag{5}$$

will also be used. Beware that ϕ^a is defined from ϕ_a using the flat metric η^{ab} instead of the full metric g^{ab}. The curvature may be calculated efficiently by packaging the connection coefficients like this:

$$\Gamma^c_a \equiv \Gamma^c_{a\bullet}\, dx^\bullet, \tag{6}$$

where $dx^\bullet$ is the differential of the coordinate $x^\bullet$. Use (3) in (6) to get

$$\Gamma^c_a = \phi_a\, dx^c + \delta^c_a\, d\phi - \phi^c\, dx_a \tag{7}$$

with $dx_a \equiv \eta_{a\bullet} dx^\bullet$.

[1] Article 03519

3 Approach to calculating the curvature

The curvature will be calculating using an approach inspired by **Cartan's method**. The relationship between the components of the curvature tensor $R_{abc}{}^d$ and the connection coefficients may be written

$$d\Gamma^c_b + \Gamma^c_\bullet \wedge \Gamma^\bullet_b = \frac{1}{2} R_{\times\bullet b}{}^c dx^\times \wedge dx^\bullet, \tag{8}$$

where d is the **exterior derivative**[2] and $\wedge$ is the **exterior product** or **wedge product**.[3] The exterior product is antisymmetric, so

$$f_{\times\bullet} dx^\times \wedge dx^\bullet = \frac{1}{2}(f_{\times\bullet} - f_{\bullet\times}) dx^\times \wedge dx^\bullet \tag{9}$$

for any f_{ab}. We can use (8) as the definition of the curvature tensor, with the understanding that its first pair of subscripts is antisymmetric. The Ricci tensor is defined by[4]

$$R_{ab} \equiv R_{\bullet ab}{}^\bullet. \tag{10}$$

The goal in this article is to calculate the scalar curvature

$$R \equiv g^{ab} R_{ab}. \tag{11}$$

For this purpose, we only need the diagonal components of the Ricci tensor, because we're using a diagonal metric.[5] The strategy will be to calculate $d\Gamma^c_b$ and $\Gamma^c_\bullet \wedge \Gamma^\bullet_b$ separately and then to extract each of their contributions to the diagonal components of the Ricci tensor, without bothering to extract the full curvature tensor $R_{abc}{}^d$.

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[2] If $\omega = \omega_\bullet dx^\bullet$, then $d\omega = (\partial_\times \omega_\bullet) dx^\times \wedge dx^\bullet$.

[3] Article 81674

[4] Using the relationship (8) in (10) gives $R_{ab} = \partial_\bullet \Gamma^\bullet_{ab} - \partial_a \Gamma^\bullet_{\bullet b} + \Gamma^\times_{\times\bullet}\Gamma^\bullet_{ab} - \Gamma^\times_{a\bullet}\Gamma^\bullet_{\times b}$, which agrees with the sign convention described in articles 03519 and 80838.

[5] Section 1

4 Calculation of the scalar curvature

This section calculates the scalar curvature of an arbitrary conformally flat metric, using the coordinate system described in section 1, the results for the connection coefficients shown in equation (7), and the strategy described in section 3.

Use (7) to get

$$d\Gamma^c_{\ a} = \phi_{a\bullet} dx^\bullet \wedge dx^c - \phi^c_\bullet\, dx^\bullet \wedge dx_a \tag{12}$$

$$\Gamma^c_{\ \bullet} \wedge \Gamma^\bullet_{\ a} = \phi_a\phi_\bullet\, dx^c \wedge dx^\bullet - \phi_\bullet\phi^\bullet\, dx^c \wedge dx_a + \phi^c\phi^\bullet\, dx_\bullet \wedge dx_a, \tag{13}$$

where each index on ϕ denotes a partial derivative, and the indices on ϕ and dx are raised/lowered using the flat metric η.[6] Now we can use equations (8)-(10) and (12)-(13) to extract the diagonal components of the Ricci tensor. To make this easier, re-index equations (12)-(13) like this:

$$d\Gamma^c_{\ b} = (\phi_{b\times}\delta^c_\bullet - \phi^c_\times \eta_{b\bullet}) dx^\times \wedge dx^\bullet$$
$$\Gamma^c_{\ \bullet} \wedge \Gamma^\bullet_{\ b} = (\phi_b\phi_\bullet\delta^c_\times - \phi_\circ\phi^\circ\delta^c_\times\eta_{b\bullet} + \phi^c\phi_\times\eta_{b\bullet}) dx^\times \wedge dx^\bullet.$$

Compare these[7] to equations (8)-(10) to get these results for the contributions of $d\Gamma$ and $\Gamma\wedge\Gamma$ to the components of the Ricci tensor:

$$\boxed{\begin{gathered} R_{ab}(d\Gamma) = (2-N)\phi_{ab} - \eta^{\circ\bullet}\phi_{\circ\bullet}\eta_{ab} \\ R_{ab}(\Gamma\wedge\Gamma) = (N-2)(\phi_a\phi_b - \eta^{\circ\bullet}\phi_\circ\phi_\bullet\eta_{ab}). \end{gathered}} \tag{14}$$

Use these in (11) to get this result for the scalar curvature of an arbitrary conformally flat metric (1):

$$\boxed{R = -e^{-2\phi}\eta^{\circ\bullet}\big(2(N-1)\phi_{\circ\bullet} + (N-1)(N-2)\phi_\circ\phi_\bullet\big).} \tag{15}$$

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[6] Section 2

[7] Remember to antisymmetrize them first, using the identity (9), because the curvature tensor is antisymmetric in its first pair of subscripts, by definition (article 03519).

5 Maximally symmetric spaces

To be concise, this section uses the word **space** for any smooth manifold equipped with a metric tensor. The signature of the metric tensor is arbitrary.

This section constructs examples of *maximally symmetric* spaces. For such a space, symmetry implies $R_{ab} \propto g_{ab}$. Section 6 will show that these examples are conformally flat, and section 7 will check equations (14) by showing that they are consistent with $R_{ab} \propto g_{ab}$ in these examples.

A space is called **maximally symmetric** if it is homogeneous and isotropic,[8] which means[9] that if x and $\tilde{x}$ are any two points and v and $\tilde{v}$ are any two tangent vectors at those points with $g_{ab}v^a v^b = g_{ab}\tilde{v}^a\tilde{v}^b$, then an isometry exists that maps x to $\tilde{x}$ and v to $\tilde{v}$. This section constructs examples of N-dimensional maximally symmetric spaces with the help of a flat ambient space with $N+1$ dimensions. The fact that the resulting spaces are maximally symmetric is clear from this construction.

Let $(w_0, \mathbf{w})$ be coordinates for the ambient space, with $\mathbf{w} = (w^1, w^2, ..., w^N)$. The superscript here is an index, not an exponent. Use the abbreviations

$$\mathbf{w}\cdot\tilde{\mathbf{w}} \equiv \eta_{jk} w^j \tilde{w}^k \qquad\qquad \mathbf{w}^2 \equiv \mathbf{w}\cdot\mathbf{w} \qquad\qquad d\mathbf{w}^2 \equiv d\mathbf{w}\cdot d\mathbf{w},$$

where the sums are over index-values in $\{1, 2, ..., N\}$, and

$$\eta_{jk} = \begin{cases} \pm 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The quantities $\mathbf{w}^2$ and $d\mathbf{w}^2$ are not necessarily positive. Take the line element of the ambient space to be

$$ds^2 \equiv \epsilon\,(dw_0)^2 + d\mathbf{w}^2 \qquad\qquad \epsilon = \pm 1. \tag{17}$$

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[8] Section 14.1 in Blau (2022) uses a different definition but then shows that it's equivalent to this one.

[9] Blau (2022), section 14.1

The quantity ds^2 is not necessarily positive.[10] For any positive real number $\sigma > 0$, the set of points whose coordinates satisfy

$$w_0^2 + \epsilon\,\mathbf{w}^2 = \sigma^2 \tag{18}$$

is an N-dimensional manifold M embedded in the $N+1$-dimensional ambient space. The induced metric on M is given by solving equation (18) for w_0 and substituting the result into (17).

If $N \geq 2$, then the manifold M with this induced metric is an example of a maximally symmetric space.[11,12] To deduce this, use the fact that the ambient flat space is manifestly maximally symmetric, together with the fact that the condition (18) is invariant under isometries of the ambient space that preserve the origin $(w_0, \mathbf{w}) = 0$. Here's a familiar example: if $\epsilon = 1$ and all of the signs in (16) are positive, then M is an N-sphere of radius σ. Other examples, like de Sitter and anti de Sitter spacetime, correspond to other choices of the signs in (16) and (17).

To get an explicit expression for the metric on M, take the differential of (18) to get

$$w_0\,dw_0 + \epsilon\,\mathbf{w}\cdot d\mathbf{w} = 0. \tag{19}$$

Solve this for dw_0 and substitute into (17).[13] After some rearranging, the result may be written

$$ds_M^2 = d\mathbf{w}^2 - \frac{(\mathbf{w}\cdot d\mathbf{w})^2}{\mathbf{w}^2} + \left(\frac{\sigma^2}{\sigma^2 - \epsilon\,\mathbf{w}^2}\right)\frac{(\mathbf{w}\cdot d\mathbf{w})^2}{\mathbf{w}^2}. \tag{20}$$

Section 6 will show that this is conformally flat.

[10] This abbreviation is common. Example: section 14.4 in Blau (2022), which describes the same construction that we're using here.

[11] Blau (2022), section 14.4

[12] If $N = 1$, then the space has two disconnected parts if the ambient space has lorentzian signature. The case $N = 1$ is boring anyway, though, so we don't lose anything interesting by excluding it.

[13] The resulting metric is valid only where $w_0 \neq 0$, but thanks to the space's symmetry, we can use a metric like this in a neighborhood of any point.

6 Proof that (20) is conformally flat

This section shows that the line element (20) may be written in the manifestly conformally flat form

$$ds_M^2 = b^2 d\mathbf{x}^2 \tag{21}$$

with

$$b \equiv \frac{2\sigma}{1+\epsilon\mathbf{x}^2}. \tag{22}$$

The coordinates $\mathbf{x}$ are related to $\mathbf{w}$ by

$$\mathbf{w} = b\mathbf{x}. \tag{23}$$

Start with the line element (21) and define new coordinates $\mathbf{w}$ by (23). Think of b as a function of $\mathbf{x}^2$, so

$$b(\chi) = \frac{2\sigma}{1+\epsilon\chi} \qquad\qquad \chi \equiv \mathbf{x}^2, \tag{24}$$

and use the abbreviation

$$b' \equiv \frac{d}{d\chi}b.$$

The definition (23) of $\mathbf{w}$ implies

$$d\mathbf{w} = b\,d\mathbf{x} + (2\mathbf{x}\cdot d\mathbf{x})b'\mathbf{x}. \tag{25}$$

Use this to get

$$\mathbf{w}\cdot d\mathbf{w} = (b^2 + 2b'b\mathbf{x}^2)\mathbf{x}\cdot d\mathbf{x},$$

which gives

$$\frac{(\mathbf{w}\cdot d\mathbf{w})^2}{\mathbf{w}^2} = (b+2b'\mathbf{x}^2)^2\frac{(\mathbf{x}\cdot d\mathbf{x})^2}{\mathbf{x}^2}. \tag{26}$$

Equation (25) also gives

$$d\mathbf{w}^2 = b^2\,d\mathbf{x}^2 + \big(4\mathbf{x}^2b'b + 4(\mathbf{x}^2b')^2\big)\frac{(\mathbf{x}\cdot d\mathbf{x})^2}{\mathbf{x}^2}. \tag{27}$$

Use (26) and (27) to get the satisfying relationship

$$d\mathbf{w}^2 - \frac{(\mathbf{w}\cdot d\mathbf{w})^2}{\mathbf{w}^2} = b^2\left(d\mathbf{x}^2 - \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{\mathbf{x}^2}\right). \tag{28}$$

Now consider the obvious identity

$$b^2\, d\mathbf{x}^2 = b^2\left(d\mathbf{x}^2 - \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{\mathbf{x}^2}\right) + b^2\frac{(\mathbf{x}\cdot d\mathbf{x})^2}{\mathbf{x}^2}.$$

Use (28) to rewrite the first term, and use (26) to rewrite the second term. This gives

$$b^2\, d\mathbf{x}^2 = d\mathbf{w}^2 - \frac{(\mathbf{w}\cdot d\mathbf{w})^2}{\mathbf{w}^2} + \frac{b^2}{(b+2b'\mathbf{x}^2)^2}\frac{(\mathbf{w}\cdot d\mathbf{w})^2}{\mathbf{w}^2}. \tag{29}$$

To finish, use (24) to get

$$2b' = \frac{-\epsilon}{\sigma}b^2 \tag{30}$$

which gives

$$\begin{aligned}\frac{b^2}{(b+2b'\mathbf{x}^2)^2} &= \frac{b^2}{(b-\epsilon b^2\mathbf{x}^2/\sigma)^2} = \frac{1}{(1-\epsilon b\mathbf{x}^2/\sigma)^2} = \left(\frac{1+\epsilon\mathbf{x}^2}{1-\epsilon\mathbf{x}^2}\right)^2 \\ &= \frac{(1+\epsilon\mathbf{x}^2)^2}{(1+\epsilon\mathbf{x}^2)^2 - 2\epsilon\mathbf{x}^2} = \frac{b^2(1+\epsilon\mathbf{x}^2)^2}{b^2(1+\epsilon\mathbf{x}^2)^2 - 2\epsilon b^2\mathbf{x}^2} = \frac{\sigma^2}{\sigma^2 - \epsilon\mathbf{w}^2}.\end{aligned}$$

Using this in (29) shows that the line element (20) is the same as (21), as claimed.

7 Ricci tensor of a maximally symmetric space

If g_{ab} are the components of the metric tensor of a maximally symmetric space, then the componentes of the Ricci tensor must be[14]

$$R_{ab} \propto g_{ab}.$$

Section 6 showed that the maximally symmetric spaces that were constructed in section 5 are conformally flat. Compare equations (1) and (21) to get

$$\phi = \log \frac{2\sigma}{1 + \epsilon \mathbf{x}^2}.$$

This implies

$$\partial_a \phi = \frac{-2\epsilon \eta_{a\bullet} x^\bullet}{1 + \epsilon \mathbf{x}^2}$$

and

$$\partial_a \partial_b \phi = \frac{-2\epsilon \eta_{ab}}{1 + \epsilon \mathbf{x}^2} + \frac{4\eta_{a\times}\eta_{b\bullet} x^\times x^\bullet}{(1 + \epsilon \mathbf{x}^2)^2}$$

Use these in equations (14) to get this result for the components of the Ricci tensor:[15]

$$R_{ab} = \frac{4(N-1)\epsilon}{(1 + \epsilon \mathbf{x}^2)^2}\, \eta_{ab} = \frac{(N-1)\epsilon}{\sigma^2}\, g_{ab}.$$

This is proportional to the metric tensor, as it should be for a maximally symmetric space. The scalar curvature is

$$R \equiv g^{ab} R_{ab} = \frac{N(N-1)\epsilon}{\sigma^2}. \tag{31}$$

[14] Blau (2022), section 14.2

[15] Blau (2022), equation (14.19)

8 Application: the standard sphere

The standard metric on an N-dimensional sphere is defined to be the metric that it inherits from its embedding as a sphere of radius σ in $N+1$-dimensional flat euclidean space. This is a special case of the construction in section 5, namely the case with all $+1$s in equation (16) and with $\epsilon = 1$ in equations (17)-(18), so equation (31) shows that the scalar curvature of an N-sphere of radius σ is

$$\boxed{R = \frac{N(N-1)}{\sigma^2}.}$$

A more familiar form of the line element of a two-dimensional unit sphere is

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \tag{32}$$

using spherical coordinates θ, ϕ. This agrees with (21)-(22) when $N = 2$, $\sigma = 1$, and $\epsilon = 1$. To show this, write $\mathbf{x}$ as

$$\mathbf{x} = \frac{\sin\theta}{1+\cos\theta}(\cos\phi, \sin\phi).$$

This implies

$$1 + \mathbf{x}^2 = \frac{2}{1+\cos\theta} \tag{33}$$

and

$$d\mathbf{x} = \frac{1}{1+\cos\theta}(\cos\phi, \sin\phi)d\theta + \frac{\sin\theta}{1+\cos\theta}(-\sin\phi, \cos\phi)d\phi. \tag{34}$$

Equation (34) implies

$$d\mathbf{x}^2 = \frac{d\theta^2 + \sin^2\theta\, d\phi^2}{(1+\cos\theta)^2},$$

and then equation (33) implies that the $N = 2$, $\sigma = 1$, $\epsilon = 1$ version of (21)-(22) is equal to (32).

9 References

Blau, 2022. "Lecture Notes on General Relativity" (updated September 18, 2022) http://www.blau.itp.unibe.ch/GRLecturenotes.html

10 References in this series

Article **03519** (https://cphysics.org/article/03519):
"Covariant Derivatives and Curvature" (version 2023-12-11)

Article **80838** (https://cphysics.org/article/80838):
"Sign Conventions in General Relativity" (version 2023-12-09)

Article **81674** (https://cphysics.org/article/81674):
"Can the Cross Product be Generalized to Higher-Dimensional Space?" (version 2022-02-06)