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# From Topological Spaces to Smooth Manifolds

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**Abstract** A **topological space** is one with enough structure for defining continuity. A **topological manifold** is intermediate between a topological space and a smooth manifold. A **smooth manifold** is one with enough structure for defining derivatives. The smooth manifold $\mathbb{R}^N$, the set of $N$-tuples of real numbers equipped with the standard smooth structure, is a familiar example from which all others can be constructed patchwise. This article is a brief reminder of the basic ideas. A list of relationships between various types of topological spaces is given at the end, summarized graphically by a Venn diagram.

## Contents

# 1 Topological spaces and continuity

A topological space is a space on which some notion of continuity is defined. More precisely: a set $X$ is called a **topological space** if it comes with a topology, and a **topology** is a list of distinguished subsets called **open sets** with these properties:

- Any union of open sets is also an open set.
- The intersection of any finite number of open sets is also an open set.
- The empty set and $X$ itself are both open sets.

For constructing examples of topological spaces, the concept of a basis is helpful. A **basis** for the topology of $X$ is a collection of open sets $B_k \subset X$ such that every other open subset of $X$ can be expressed as a union of $B_k$s. The topology is said to be **generated by** the $B_k$s. More vocabulary:

- An element of $X$ is called a **point**.
- For any point $p$, an open set containing $p$ is called a **neighborhood** of $p$.
- A subset of $X$ is called **closed** if its complement is open.

Again, a topological space provides enough structure for defining continuity. Here's the definition: If $X$ and $Y$ are topological spaces, then a map

$$X \xrightarrow{f} Y$$

is called **continuous** if, for each open set $O \subset Y$, the pre-image $f^{-1}(O) \subset X$ is also open. Section 2 shows an example.

## 2 Example: the topological space $\mathbb{R}$

Let $I(a,b)$ denote the set of all real numbers $x$ satisfying $a < x < b$. Let $\mathbb{R}$ denote the set of real numbers with the topology **generated** by the sets $I(a,b)$ – that is, declare each interval $I(a,b)$ to be open, along with all other sets that can be obtained from these by unions and finite intersections. This is the standard topology of $\mathbb{R}$.

These examples illusrate the definition of continuity:

- The function $f(x) = x^2 + 1$ is continuous. In particular,
  - The pre-image of the open set $I(5,10)$ is the union of $I(2,3)$ and $I(-3,-2)$, which is open.
  - The pre-image of the open set $\{x > 0\}$ is $\mathbb{R}$, which is open.
- In contrast, the function

$$f(x) = \begin{cases} x+1 & \text{if } \; x \geq 0 \\ x-1 & \text{if } \; x < 0 \end{cases}$$

  is *not* continuous. In particular, the pre-image of the open set $\{x > 0\}$ is the set $\{x \geq 1\}$, which is *not* open: it includes 1 but doesn't include anything less than 1, and a union or finite intersection of open intervals $I(a,b)$ cannot have that property.

# 3 Generalization: the topological space $\mathbb{R}^N$

Let $x$ denote an $N$-tuple of real numbers:

$$x = (x_1, x_2, ..., x_N).$$

Given an $N$-tuple $y \in \mathbb{R}^N$ and a positive number $R$, define the **(open) ball** $B(y, R)$ to be the set of all $N$-tuples $x \in \mathbb{R}^N$ for which

$$\sum_{n=1}^{N} (x_n - y_n)^2 < R^2.$$

Intuitively, this is the interior of a ball of radius $R$ centered on the point $y$.[1] The topological space $\mathbb{R}^N$ is the set of all $N$-tuples $x$, with the topology generated by the balls, which are themselves taken to be open sets. The example in the previous section is the special case $N = 1$.

[1] Be careful with this intuition, though, because geometric notions like distance are *not defined* by topology alone.

# 4 Homeomorphisms

A homeomorphism[2] is a map[3] between two topological spaces that preserves all of the properties that make them topological spaces. More precisely, if $X$ and $Y$ are topological spaces, then a **homeomorphism** is a continuous map $f : X \to Y$ with a continuous inverse $f^{-1} : Y \to X$. (Calling $f^{-1}$ the **inverse** of $f$ means that the composition of $f$ with $f^{-1}$, in either order, is the identity map.) If such a map exists, then $X$ and $Y$ are called **homeomorphic** or **topologically equivalent** to each other.

If two topological spaces are homeomorphic to each other, then they are *the same* as far as topology is concerned. They may differ from each other in other ways, but not in ways that affect the concept of continuity. Here's an example from the previous section:[4] all of the balls in $\mathbb{R}^N$ are homeomorphic to each other and also to $\mathbb{R}^N$. They are all equivalent as far as topology is concerned, even though they are different subsets of $\mathbb{R}^N$.

[2]Notice the **e** in hom**e**omorphism. Don't confuse this with homomorphism (no **e**), which is a map between algebraic objects (like groups or rings) that preserves the relevant algebraic structure.

[3]The words *map* and *function* are often used interchangeably, but sometimes the word *function* is reserved for a map into a set of numbers.

[4]Lee (2011), page 29, example 2.25

# 5 Topological manifolds

A topological manifold is a special kind of topological space, one with especially nice properties. A topological space $X$ is called an **$N$-dimensional topological manifold** if it satisfies all of these conditions:[5]

- Every point of $X$ has a neighborhood homeomorphic to $\mathbb{R}^N$. (In other words, $X$ is **locally euclidean**.)
- Every two distinct points of $X$ have neighborhoods that don't intersect each other. (In other words, $X$ is **Hausdorff**.) Intuitively, this means that there are "enough" open sets.
- A countable basis for $X$ exists.[6] (In other words, $X$ is **second-countable**.) Intuitively, this means that there are not "too many" open sets.

Basic properties of any $N$-dimensional topological manifold include:

- Every point is closed.[7]
- Any open subset is an $N$-dimensional topological manifold by itself.[8]
- If a nonempty topological manifold $X$ is homeomorphic to an $N$-dimensional topological manifold $Y$, then $X$ is also $N$-dimensional.[9]
- Every topological manifold is homeomorphic to a subset of $\mathbb{R}^N$ for some sufficiently large $N$.[10]

Topological manifolds with *boundaries* are defined in article 44113.

[5] Lee (2000), chapter 2, page 33; Adachi (1993), definition 1.1; Badzioch (2018), definition 13.1; and Davis and Petrosyan (2012), page 2

[6] Some authors omit this condition (Cohen (2023), definition 1.1; Hatcher (2001), section 3.3, page 231).

[7] Lee (2011), page 32, proposition 2.37

[8] Lee (2011), page 39, proposition 2.53

[9] Lee (2011), page 40, theorem 2.55

[10] Lee (2011), page 41

# 6 Examples of topological manifolds

The topological space $\mathbb{R}^N$ described in section 3 is an example of $N$-dimensional topological manifold.

Another important example is the $(N-1)$-dimensional **sphere**, denoted $S^{N-1}$. Roughly, this is the surface of an $N$-dimensional ball – just the surface, excluding the interior.[11] More precisely, we can define $S^{N-1}$ to be the set of points $x \in \mathbb{R}^N$ satisfying

$$\sum_{n=1}^{N} x_n^2 = 1,$$

with this topology: every intersection of an open ball $B \in \mathbb{R}^N$ with $S^{N-1}$ is open, and the topology of $S^{N-1}$ is generated by these open sets.[12]

Another important example is the $N$-dimensional **torus**, denoted $T^N$. It can be defined as the set of $N$-tuples of real numbers, each of which is in the interval $0 \leq x_n < 1$, with a topology that treats 0 and 1 as equivalent (so that the interval "wraps" back into itself instead of having endpoints). To define the topology precisely, start with the map $f : \mathbb{R}^N \to T^N$ defined by

$$f(x_1, ..., x_N) = (x_1 \bmod 1,\ ..., x_N \bmod 1),$$

and take a subset $O \subset T^N$ to be open if and only if $f^{-1}(O) \subset \mathbb{R}^N$ is open.[13] The simplest example is the **circle** $T^1$, which is homeomorphic to $S^1$. The next simplest example is the torus $T^2$, which can be visualized as the surface of a donut/bagel, excluding the interior. This is *not* homeomorphic to $S^2$, which can be visualized as the surface of a 3d ball, excluding the interior.

[11] The open ball defined in section 3 is just the interior, excluding the surface.

[12] This is a special case of a construct called the **subspace topology**.

[13] This is a special case of a construct called the **quotient topology**.

# 7 Topological manifolds and metric spaces

This section introduces the concept of a *metric space* and uses it to give an equivalent definition of *topological manifold*.

A **metric space** is a set $X$ equipped with a **metric**,[14] which is a function that assigns a real number $m(x, y)$ to every pair of elements $x, y \in X$ and that satisfies these conditions:[15]

- $m(x, y) = m(y, x)$,
- $m(x, y) \geq 0$,
- $m(x, y) = 0$ if and only if $x = y$,
- $m(x, z) \leq m(x, y) + m(y, z)$ for all $x, y, z \in X$.

Given a point $b \in X$ and a positive real number $r > 0$, a subset $B \subset X$ consisting of all points $x$ with $m(b, x) < r$ is called an **open ball**. For any metric space with metric $m$, the topology **induced** by $m$ is defined by declaring every union of any number of open balls to be an open set.[16] A topological space is called **metrizable** if it admits a metric $m$ for which the topology induced by $m$ is the same as the original topology.[17] Metrizability is invariant under homeomorphisms.[18]

Every topological manifold is metrizable.[19,20] In fact, a *topological manifold* may be defined as a locally euclidean metric space,[21] and this definition turns out to be equivalent to the one that was given in section 5.[22]

[14] This should not be confused with the concept of a (pseudo)riemannian metric (articles 21808 and 48968).

[15] Badzioch (2018), definition 2.3

[16] Badzioch (2018), example 3.14

[17] Badzioch (2018), definition 3.19

[18] Badzioch (2018), exercise E7.18

[19] This is theorem 13.20 in Badzioch (2018), which allows the manifold to have a boundary (article 44113).

[20] Corollary 13.30 in Lee (2013) confirms that every smooth manifold (with boundary) is metrizable.

[21] Rolfsen (1976) uses this definition (page 33).

[22] This follows from the fact that every metric space is Hausdorff (Badzioch (2018), note 9.5) combined with the fact that if a connected topological space is Hausdorff and locally euclidean, then it is second countable if and only if it is metrizable (Gauld (2009), theorem 2, items 1 and 26).

# 8 Smooth manifolds

Topology provides enough structure for defining continuity, but it doesn't provide enough structure for defining derivatives. An **$N$-dimensional smooth manifold** is an $N$-dimensional topological manifold equipped with a **smooth structure**, which is enough extra structure for defining derivatives. The smooth structure is defined like this:[23]

- A **chart** $(U, \sigma)$ is an open subset $U \subset M$ together with a homemorphism $\sigma$ from $U$ to an open subset of $\mathbb{R}^N$. The subset $U$ is called the **domain** of the chart. A chart is often called a **coordinate chart**, because it labels the points of its domain with $N$-tuples of real numbers, which we can use as **coordinates** for points within $U$.

- Two charts $(U, \sigma)$ and $(U', \sigma')$ are called **smoothly compatible** with each other if the homeomorphism from $\sigma(U \cap U')$ to $\sigma'(U \cap U')$ that we get by composing $\sigma'$ with $\sigma^{-1}$ has well-defined partial derivatives (of all orders) with respect to each coordinate $x_n$.

- A **smooth atlas** is a collection of smoothly compatible charts whose domains cover $M$. A **smooth structure** is maximal smooth atlas, one that already contains every chart that is smoothly compatible with all of its charts.[24] Every smooth atlas is contained in a unique maximal smooth atlas.[25]

Smooth manifolds with boundaries are defined in article 44113.

The topological manifold $\mathbb{R}^N$ can be covered by a single chart, so it's clearly a smooth manifold. The sphere $S^{N-1}$ and the torus $T^{N-1}$ (section 6) are also smooth manifolds, with smooth structures inherited in a natural way from $\mathbb{R}^N$.

[23] Lee (2013), pages 11-13, after exploiting proposition 1.17

[24] We could use any smooth atlas to define a smooth structure, but using a *maximal* smooth atlas allows replacing phrases like "...for some chart that is smoothly compatible with those in $M$'s smooth structure" with simpler phrases like "...for some chart in $M$'s smooth structure."

[25] Lee (2013), proposition 1.17

# 9 Smooth maps and diffeomorphisms

The appropriate notion of equivalence between topological spaces is *homeomorphism* (section 4). Similarly, the appropriate notion of equivalence between smooth manifolds is called *diffeomorphism*.

A **smooth map** $f : X \to Y$ is defined by the condition that every point $p \in X$ is contained in a smooth chart $(U, \sigma)$, and its image $f(p) \in Y$ is contained in a smooth chart $(U', \sigma')$ with $f(U) \subseteq U'$, such that the composite map

$$\sigma(U) \xrightarrow{\sigma^{-1}} U \xrightarrow{f} U' \xrightarrow{\sigma'} \sigma'(U') \tag{1}$$

is smooth. If $X$ and $Y$ are smooth manifolds, a **diffeomorphism** is a smooth map $f : X \to Y$ with a smooth inverse $f^{-1} : Y \to X$. If a diffeomorphism $X \to Y$ exists, then $X$ and $Y$ are called **diffeomorphic** to each other.[26]

Beware of this subtlety in the definitions: two smooth structures may be distinct from each other even if they are diffeomorphic to each other. In more detail: if $X$ is a topological manifold and $\alpha$ and $\alpha'$ are two different smooth structures for $X$, then the smooth manifolds $(X, \alpha)$ and $(X, \alpha')$ may be diffeomorphic to each other even if $\alpha$ and $\alpha'$ are not smoothly compatible with each other.[27]

A given topological manifold may also admit different smooth structures that are not diffeomorphic to each other. In other words, two smooth manifolds may be homeomorphic to each other even if they're not diffeomorphic to each other.[28,29]

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[26]Lee (2013), pages 34 and 38, and previewed on page 11

[27]Lee (2013) demonstrates this in example 1.23 on page 17, which is continued on pages 39-40. Exercise 3.8 in Crainic (2023) describes a family of examples.

[28]Examples exist in each dimension $\geq 4$ (Lee (2013), page 40; Scorpan (2000); Scorpan (2004)). Theorem 1.1.8 in Gompf and Stipsicz (1999) addresses the dimension-dependence of this phenomenon for compact manifolds.

[29]Here's an excerpt from pages 39-40 in Lee (2013): "... as long as $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure (up to diffeomorphism); but $\mathbb{R}^4$ has uncountably many distinct smooth structures, no two of which are diffeomorphic to each other!" This special feature of $\mathbb{R}^4$ is also mentioned in Manolescu (2020), section 1, and in Crainic (2023), chapter 3, section 5. Brans (1994) highlights some implications for spacetime geometries.

# 10 CW complexes

A **CW complex** is another special kind of topological space. Roughly, a CW complex can be constructed by gluing $n$-dimensional cells together, for various $n$.

For each $n \geq 1$, let $B^n$ denote an $n$-dimensional ball. Its boundary $\partial B^n$ is an $n$-sphere, and $\text{int}(B^n)$ will denote its interior. Examples: $B^1$ is a line segment and $\partial B^1$ is a pair of points, $B^2$ is a disc and $\partial B^2$ is a circle, and so on. A topological space $X$ is called a **CW complex** if it is Hausdorff[30] and can be decomposed into subsets called **cells** satisfying these conditions:[31]

- The cells do not intersect each other.
- Each cell is either a point (zero-dimensional cell) or is homeomorphic to the interior of an $n$-dimensional ball, for some $n \geq 1$.
- For each $n$-dimensional cell with $n \geq 1$, a continuous map $c : B^n \to X$ exists with these properties:
  1. $c : \text{int}(B^n) \to X$ is a topological embedding onto the given cell.[32]
  2. $c(\partial B^n)$ is contained in a finite union of lower-dimensional cells.
- A subset $Y \subset X$ is closed in $X$'s topology if and only if its preimage $c^{-1}(Y)$ is closed in the topology of $B^n$ for every map $c : B^n \to X$ that was used above, for every $n$.

If a CW complex has at least one $n$-dimensional cell but no cells of higher dimension, then it's called ***n*-dimensional**. If it has $n$-dimensional cells with arbitrarily large $n$, then it's called **infinite-dimensional**.[33]

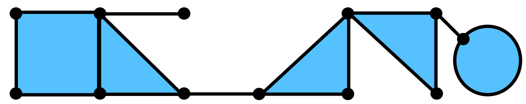
[30] A topological space $X$ is called **Hausdorff** if every two distinct points of $X$ are contained in two open neighborhoods that don't intersect each other.

[31] Mitchell (1997); Hatcher (2001), proposition A.2

[32] Article 44113 defines *topological embedding*.

[33] Mitchell (1997), page 1

# 11 Examples



This picture above shows an example of a connected 2-dimensional CW complex with five 2-cells, sixteen 1-cells, and twelve 0-cells. It is not a manifold.

Here's a more natural example of a CW complex that is not a manifold. Choose some large value of $n$ and consider the points in $\mathbb{R}^n$ whose coordinates are all integers. Take this set of points to be the set of 0-cells. Then declare the interior of each line segment that connects two neighboring points (so that the line segment has length 1) to be a 1-cell. So far, this is a 1-dimensional CW complex, also called a **graph**. If we declare the interior of each square whose edges are 1-cells to be a 2-cell, then we get a 2-dimensional CW complex. Continuing up to $n$-cells would give all of $\mathbb{R}^n$, which is a manifold, but stopping earlier gives a lower-dimensional CW complex that is not a manifold.

For a fun example of a CW complex that is also a manifold, consider the 3d real projective space $\mathbb{RP}^3$. We can think of $\mathbb{RP}^3$ as a CW complex with one 3-cell, one 2-cell, one 1-cell, and one 0-cell. To arrive at this description, start by thinking of $\mathbb{RP}^3$ as $S^3$ with antipodal points identified. Thanks to these identifications, we only need one hemisphere of $S^3$, which is a 3-dimensional ball $B^3$ except that antipodal points on its boundary $\partial B^3 = S^2$ are identified with each other.[34] The interior of this ball is a 3-cell. Its boundary is $S^2$ with antipodal points identified, so we only need one hemisphere of $S^2$, which is a 2-dimensional disk $B^2$ except that antipodal points on its boundary $\partial B^2 = S^1$ are identified with each other. The interior of this disk is a 2-cell. Its boundary is $S^1$ with antipodal points identified, so we only need half of $S^1$, which is a line segment except that its endpoints are identified with each other. The interior of this line segment is a 1-cell, and its endpoints – after identifying them with each other – give a single 0-cell. The union of these cells, one of each dimension from 0 to 3, is the manifold $\mathbb{RP}^3$ that we started with.

[34]The manifold $\mathbb{RP}^3$ does not have a boundary, but we can construct it by starting with a manifold that does have a boundary (namely $B^3$) and then gluing antipodal points on the boundary to each other.

# 12 Manifolds and complexes

Many published results that may be applied to manifolds are stated for other spaces, like CW complexes (which includes all smooth manifolds, in the sense that any smooth manifold may be given the structure of a CW complex). To help determine which published results hold for which types of manifold, section 13 summarizes some relationships between some of the most commonly studied spaces.

The preceding sections defined *topological space*, *topological manifold*,[35] *smooth manifold*, and *CW complex*. Other commonly studied spaces include:

- **simplicial complexes** and their **geometric realizations**,[36,37]
- **Piecewise linear (PL) manifolds**.[38]

A manifold of any type (topological, PL, or smooth) is called **triangulable** if it is homeomorphic to the geometric realization of a simplicial complex.[39]

Section 13 lists some relationships between these spaces. Some relationships are manifest in the definitions, and some of them are difficult theorems. The relationships are not all independent of each other: some of them may be inferred from others. All of them are summarized in a Venn diagram in section 14.

[35] Some authors use a more general definition of *topological manifold* (footnote 6 in section 5).

[36] These are defined in Hatcher (2001), section 2.1, page 107; Manolescu (2016), section 2.1; and Hocking and Young (1961), sections 5.4 and 5.7.

[37] A geometric realization of a simplicial complex is called a **polyhedron** (Lee (2011), text above proposition 5.33; https://ncatlab.org/nlab/show/polyhedron).

[38] These are defined in Davis and Petrosyan (2012), page 2.

[39] Lee (2000), chapter 5, page 100; and Manolescu (2016), section 2.1

# 13 Manifolds and complexes: some relationships

The Venn diagram in section 14 summarizes these relationships:

- Every smooth manifold is a topological manifold.[40]
- Every topological manifold is a topological space.[41]
- Every smooth manifold is homeomorphic to a PL manifold.[42]
- Every PL manifold is a topological manifold.[42]
- Every simplicial complex is a CW complex.[43,44]
- Every CW complex is a topological space.[45]
- Every PL manifold is triangulable.[46]
- Every smooth manifold is triangulable.[47]
- Some topological spaces are not topological manifolds.[48]
- Some topological manifolds don't admit a PL structure.[49]

[40] Section 8

[41] Section 5

[42] Davis and Petrosyan (2012), page 2

[43] This combines two facts from Hatcher (2001):, every simplicial complex is a $\Delta$-complex (page 107), and every $\Delta$-complex is a CW complex (page 534).

[44] https://math.stackexchange.com/questions/1528005/simplicial-complex-vs-delta-complex-vs-cw-complex compares different types of complexes.

[45] Hatcher (2001), appendix, page 519

[46] Manolescu (2016), section 2.2; https://ncatlab.org/nlab/show/triangulation+theorem

[47] Davis and Petrosyan (2012), page 2; and Manolescu (2016), section 2.2

[48] Example: if $M$ is the union of the x-axis and the y-axis in $\mathbb{R}^2$, then $M \subset \mathbb{R}^2$ with the subspace topology (which is defined in article 44113) is a topological space space but not a topological manifold (Lee (2000), problem 4-2).

[49] Davis and Petrosyan (2012), examples 2.1 and 3.9; Manolescu (2016), section 2.2; Rudyak (2001), example 21.3

- Some topological manifolds are not triangulable.[50,51]
- Some triangulable topological manifolds don't admit a PL structure.[52]
- Every $n$-dimensional topological manifold with $n \neq 4$ is homeomorphic to a CW complex, but the situation for $n = 4$ is unknown.[53,54]
- Some topological spaces are not homeomorphic to any CW complex.[55]
- Some CW complexes are not homeomorphic to any simplicial complex.[56,57]
- Some simplicial complexes are not homeomorphic to a topological manifold.[58]
- Some topological manifolds that are homeomorphic to a CW complex are not triangulable.[59]
- Some topological manifolds are not smoothable.[60]
- Some PL manifolds are not smoothable.[61,62]

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[50]Manolescu (2016), theorem 1.1, with more detail in the answer to question 2 in section 2.2; Rudyak (2001), example 21.6; Hatcher (2001), text below corollary A.12

[51]*Not triangulable* means not homeomorphic to any simplicial complex, but every $n$-dimensional topological manifold is homotopy equivalent to an $n$-dimensional simplicial complex (Manolescu (2016), section 2.2). Homotopy equivalence is a more relaxed equivalence relation than homeomorphism (article 61813).

[52]Rudyak (2001), example 21.4

[53]Manolescu (2016), section 2.2; Hatcher (2001), text below corollary A.12

[54]Every compact manifold with boundary is homotopy equivalent to a finite CW complex (Lurie (2014), claim 2).

[55]Example: `https://ncatlab.org/nlab/show/Hawaiian+earring+space`

[56]Manolescu (2016), section 2.1

[57]Every CW complex is homotopy equivalent to a simplicial complex (`https://ncatlab.org/nlab/show/CW+complex`, thoerem 3.6), and every compact manifold and every finite CW complex is a retract of a simplicial complex (Hatcher (2001), text below theorem 2C.3).

[58]Example of a geometric simplicial complex that isn't a topological manifold: three distinct points each connected to a fourth point by line segments (1-simplexes).

[59]This combines two facts from Manolescu (2016), section 2.2: some $n$-dimensional manifolds with $n \geq 5$ are not triangulable (page 3), and every $n$-dimensional manifold with $n \geq 5$ is homeomorphic to a CW complex (page 4).

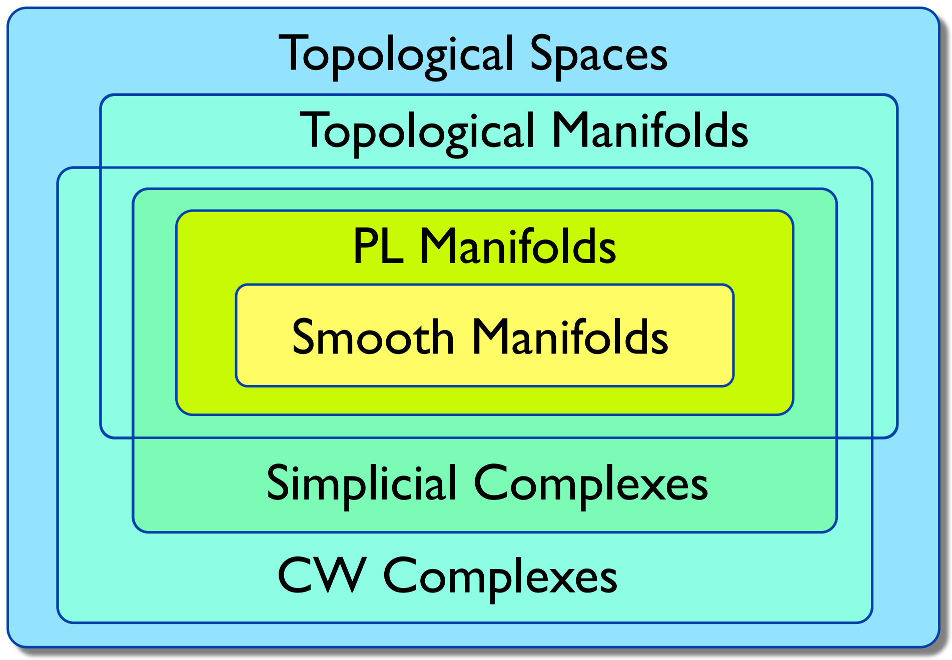
[60]Lee (2013), text above proposition 1.17 and page 40 in chapter 2

[61]Crowley and Hambleton (2013)

[62]Every $n$-dimensional PL manifold with $n \leq 7$ is smoothable (Milnor (2011), theorem 2).

## 14 Manifolds and complexes: Venn diagram

This Venn diagram depicts the relationships that were listed in section 13.[63]




[63]This diagram allows for the possible existence of (4-dimensional) topological manifolds that aren't homeomorphic to a CW complex.

# 15 Other generalizations

As defined in this article, manifolds (topological or smooth) don't have boundaries. Article 44113 introduces manifolds (topological or smooth) with boundaries.

As defined in this article, manifolds are finite-dimensional. The definition can be generalized to allow infinite-dimensional manifolds.[64] One such generalization is the concept of a **Hilbert manifold**,[65] which uses a Hilbert space as a local model instead of using $\mathbb{R}^n$.

**Category theory** is one of the most important unifying themes in math. Roughly, a category is the collection of all mathematical objects of a given type, together with all type-preserving **morphisms** (maps) between them. The category of topological manifolds and continuous maps does not have all of the convenient properties that many other natural categories have, nor does the larger category of all topological spaces and continuous maps. A nicer category of topological spaces – using **compactly generated spaces** and continuous maps – is commonly used in algebraic topology.[66] Every CW complex is compactly generated,[67] and every topological manifold is compactly generated.[68]

Similarly, the category of smooth manifolds with smooth maps as morphisms does not have all of the nice properties that many other natural categories have, but the definitions can be generalized to give a more convenient category. Two such generalizations, **Chen spaces** and **diffeological spaces**, are reviewed in Baez and Hoffnung (2011).[69]

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[64] Schmeding (2022); Michor (1991); https://ncatlab.org/nlab/show/infinite-dimensional+manifold

[65] Meier (2014) (html version: http://www.map.mpim-bonn.mpg.de/Hilbert_manifold)

[66] The beginning of chapter 5 in May (2007) says that this is "the category of spaces in which algebraic topologists customarily work." The end of the same chapter says, "From here on, we agree that all given [topological] spaces are to be compactly generated..." Every compactly generated space $X$ is weakly equivalent to a CW complex $Y$ (May (2007), chapter 10, section 5), which means that a map $X \to Y$ exists that induces isomorphisms of all of their homotopy groups (May (2007), chapter 9, section 6).

[67] Mitchell (1997), section 1

[68] https://ncatlab.org/nlab/show/compactly+generated+topological+space

[69] Baez and Hoffnung (2011) uses the name *smooth space* for both. The introduction says, "Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense. ... We can use the big category for abstract constructions, and the small one for theorems that rely

## 16 References

**Adachi, 1993.** *Embeddings and immersions.* American Mathematical Society

**Badzioch, 2018.** "MTH 427/527 Introduction to Topology I" https://www.nsm.buffalo.edu/~badzioch/MTH427/lecture_notes.html

**Baez and Hoffnung, 2011.** "Convenient Categories of Smooth Spaces" *Trans. Amer. Math. Soc.* **363**: 5789-5825, https://arxiv.org/abs/0807.1704

**Brans, 1994.** "Exotic smoothness and physics" https://arxiv.org/abs/gr-qc/9405010

**Cohen, 2023.** "Bundles, Homotopy, and Manifolds" http://virtualmath1.stanford.edu/~ralph/book.pdf

**Crainic, 2023.** "Manifolds 2017" https://webspace.science.uu.nl/~crain101/manifolds-2017/lecture-notes-23-Oct.pdf

**Crowley and Hambleton, 2013.** "Finite group actions on Kervaire manifolds" *Advances in Math.* **283**: 88-129, https://arxiv.org/abs/1305.6546

**Davis and Petrosyan, 2012.** "Manifolds and Poincaré complexes" https://jfdmath.sitehost.iu.edu/teaching/m623/dp.pdf

**Gauld, 2009.** "Metrisability of manifolds" https://arxiv.org/abs/0910.0885v1

**Gompf and Stipsicz, 1999.** *4-Manifolds and Kirby Calculus.* American Mathematical Society

**Hatcher, 2001.** "Algebraic Topology" https://pi.math.cornell.edu/~hatcher/AT/AT.pdf

**Hocking and Young, 1961.** *Topology.* Dover

on good control over local structure."

**Lee, 2000.** *Introduction to Topological Manifolds.* Springer, https://archive.org/details/springer_10.1007-978-0-387-22727-6/

**Lee, 2011.** *Introduction to Topological Manifolds (Second Edition).* Springer

**Lee, 2013.** *Introduction to Smooth Manifolds (Second Edition).* Springer

**Lurie, 2014.** "Algebraic K-Theory and Manifold Topology (Math 281), lecture 34" https://www.math.ias.edu/~lurie/281.html

**Manolescu, 2016.** "Lectures on the triangulation conjecture" https://arxiv.org/abs/1607.08163

**Manolescu, 2020.** "Four-dimensional topology" https://web.stanford.edu/~cm5/4D.pdf

**May, 2007.** "A Concise Course in Algebraic Topology" http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf

**Meier, 2014.** "Hilbert manifold – definition" http://www.boma.mpim-bonn.mpg.de/data/51print.pdf

**Michor, 1991.** *Gauge Theory for Fiber Bundles (Monographs and Textbooks in Physical Sciences, Lecture Notes 19).* Bibliopolis, https://www.mat.univie.ac.at/~michor/gaubook.pdf

**Milnor, 2011.** "Differential Topology Forty-six Years Later" *Notices A.M.S.* **58**: 804-809, https://www.ams.org/notices/201106/rtx110600804p.pdf

**Mitchell, 1997.** "CW-complexes" https://sites.math.washington.edu//~mitchell/Morse/cw.pdf

**Rolfsen, 1976.** *Knots and Links.* AMS Chelsea Publishing

**Rudyak, 2001.** "Piecewise linear structures on topological manifolds" https://arxiv.org/abs/math/0105047

**Scorpan, 2000.** *The Wild World of 4-Manifolds.* Springer

**Scorpan, 2004.** "The wild world of 4-manifolds" *University of Florida Mathematics Newsletter* **18**: 12-13

**Schmeding, 2022.** *An introduction to infinite-dimensional differential geometry.* Cambridge University Press,
https://www.cambridge.org/core/books/an-introduction-to-infinitedimensi
6483795C98EE417C0F3654F6C192C3BC

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