’t Hooft Operators from GNO Configurations

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Abstract Quantum models with gauge fields typically include observables called **Wilson operators** and **’t Hooft operators**. These operators are nominally localized on lower-dimensional submanifolds of spacetime, in the same sense that electric and magnetic field operators are nominally localized at individual points in spacetime. This article introduces a type of ’t Hooft operator that is nominally localized on a $(d-3)$-dimensional closed submanifold Γ in d-dimensional spacetime, treating spacetime as discrete to make the math clear. The gauged group G can be any compact connected Lie group. The operators are constructed by modifying the action in a tubular neighborhood of Γ so that the path integral is dominated by configurations of the gauge field the approach a prescribed **GNO configuration** near Γ. The GNO configuration is based on a specially chosen connection for a principal G-bundle. Article 49708 explains how to construct the G-bundle and the connection and explains the reason for the name **GNO**.

Contents

1 Introduction

Article 22721 gives an overview of Wilson and 't Hooft operators, distinguishing two different types based on the nature of the submanifold X of d-dimensional spacetime on which the operator is nominally localized:[1,2]

operator	notation	number of dimensions of X	type of submanifold	this article
Wilson, type 1	W°	1	proper and neat	
Wilson, type 2	$W^\bullet$	2	proper	
't Hooft, type 1	T°	$d-3$	proper and neat	✓
't Hooft, type 2	$T^\bullet$	$d-2$	proper	

Article 22721 reviews the motivation for considering these operators and explains the meaning of the last column.

Other articles in this series describe Wilson operators of type 1 (article 89053) and 't Hooft operators of type 2 (article 82508). Those operators are natural even when spacetime is discretized, in the sense that their constructions don't involve any arbitrary choices beyond those that were already made when discretizing spacetime. This article describes the other type of 't Hooft operator (type 1).[3] In this case, the construction involves additional arbitrariness that goes away only near the smooth-spacetime limit for $d \in \{3, 4\}$.[4]

This article uses the path integral formulation. Spacetime is discretized so the construction is unambiguous.[5] The gauged group G can be any compact connected Lie group.

[1]In this article, "nominally localized on X" means "localized in an arbitrarily small neighborhood of X."

[2]The names **type 1** and **type 2** and the superscripts $\circ$ and $\bullet$ are not standard.

[3]Disclaimer: I have not found any other accounts of these operators in discrete spacetime.

[4]The construction requires choosing a tubular neighborhood τ of the submanifold on which the operator is nominally localized (section 5). In the smooth-spacetime limit, the width of τ goes to zero in physical units.

[5]Currently, the only known well-defined nonperturbative constructions of nonabelian Yang-Mills models in 4-dimensional spacetime involve discretizing space or spacetime.

2 Notation and conventions

This section summarizes some of the notation and conventions that will be used in this article.

- G is a compact connected Lie group.
- The speed of light and $\hbar$ are both equal to 1. Aside from $\hbar = 1$, the units convention is the same as in article 26542.
- d is the number of dimensions of spacetime.

Spacetime is discretized as described in article 46333. The smooth spacetime manifold M is partitioned into d-dimensional polyhedra called d-cells, whose boundary is made of $(d-1)$-cells, and so on. The names **point**, **link**, and **plaquette** are synonyms for 0-cell, 1-cell, and 2-cell, respectively. Each k-cell can be endowed with either of two **orientations**.[6] The model uses only the discrete structure, but some things will be described by referring to the underlying smooth manifold M because this simplifies the descriptions. This article doesn't try to define the **smooth-spacetime limit** precisely, but the idea is to make ϵ approach zero in physical units, where ϵ is a representative distance between neighboring points in discrete spacetime.

The path integral is an integral over G-valued **link variables** $u(\ell)$, one for each link ℓ in discrete spacetime, subject to $u(\ell^{-1}) = (u(\ell))^{-1}$ where ℓ^{-1} is the orientation-reversed version of ℓ. The collection of link variables constitutes the **gauge field**. A **plaquette variable** with basepoint x is the G-valued quantity defined by

$$u(x, \square) \equiv \prod_{\ell \in \square} u(\ell)$$

where the factors in the product are ordered sequentially around the perimeter of the oriented plaquette $\square$, starting with the given basepoint x (which must be one

[6] Article 46333 reviews what this means.

of the plaquette's corners). This is discrete-spacetime version of a holonomy. The normalized trace of a plaquette variable is the complex-valued quantity

$$w(\square) \equiv \frac{\text{trace}\big(u(x,\square)\big)}{\text{trace}(1_G)} \tag{1}$$

where 1_G is the identity element of G and the trace is defined using a faithful representation of G. The trace makes $w(\square)$ independent of the basepoint x.

This article is about the definition and basic properties of a type of 't Hooft operator localized on a closed $(d-3)$-dimensional manifold Γ of spacetime.[7] Such an operator will be denoted $T^\circ(\Gamma)$. Article 22721 explains the superscript $\circ$. The construction involves quantities $\hat{u}(\ell)$ and $\hat{u}(x,\square)$ that will be defined in section 8. They are prescribed G-valued quantities, and the construction is designed so that the path integral near the smooth-spacetime limit is dominated by configurations of the gauge field with $u(x,\square) \approx \hat{u}(x,\square)$.

When an operator (a linear operator on the Hilbert space) is constructed by modifying the integrand of the path integral, the modification can have important properties that would be lost if we thought of it as nothing more than a linear operator on the Hilbert space.[8] In the literature about higher-form symmetries, the word *operator* is often used for the thing with those additional properties, not reduced to a mere linear operator on the Hilbert space.[9] This article uses the word *operator* in that more liberal sense.[10]

[7] If the spacetime manifold M has a boundary ∂M, then Γ may have a boundary $\partial\Gamma$ contained in ∂M, but for simplicity this article assumes Γ is a closed manifold (compact and boundaryless).

[8] Article 02242

[9] Sometimes the word *defect* is used instead.

[10] Article 09181 uses a special notation to distinguish between two different versions of = (equality), one that accounts for the additional properties and one that does not. That distinction is less important in this article because $T^\circ(\Gamma)$ is not a topological operator.

3 The path integral

The path integral has the form[11]

$$\Psi'[u]_{t'} \propto \int_{<t'}[du]\ e^{iS[u]}\Psi[u]_t \tag{2}$$

where

- Ψ and Ψ' are the initial and final states,
- $[u]_t$ denotes the set of link variables whose endpoints are both at time t,
- the integral is over of the link variables that have at least one endpoint in the range $\geq t$ and $< t'$ (with no more than one endpoint at time t'),
- each link variable is integrated over the gauged group G.

Section 4 will list the properties of the action $S[u]$ that are important in this article.

[11] Article 89053

4 Important properties of the action

The properties of the action $S[u]$ that will be important in this article include:[12]

- It depends on link variables $u(\ell)$ only through traced plaquette variables $w(\Box)$.[13]
- It is a sum of terms that each depends on only one traced plaquette variable:

$$S[u] = \sum_{\Box} c(\Box)\big(1 - w(\Box)\big) \tag{3}$$

  where $c(\Box)$ are fixed complex-valued coefficients. This allows for Wick rotation to a **euclidean action**.[14]
- When the overall coefficient of S is large, the path integral is dominated by configurations of the link variables that come close[15] to minimizing the euclidean action.

[12]The Wilson action introduced in article 89053 has these properties.

[13]This implies that $S[u]$ is invariant under gauge transformations.

[14]Article 89053

[15]*How close* is a detail that won't be addressed here. The goals here are to give precise definitions in discrete spacetime and to give intuition about the continuum limit, not to give precise results about the continuum limit.

5 Where the path integral will be modified

The operators introduced in this article are constructed by modifying the action (3) in a neighborhood τ of a closed $(d-3)$-dimensional submanifold Γ of d-dimensional spacetime. This section describes the neighborhood τ. Section 9 will describe how the path integral is modified within τ.

Choose a closed $(d-3)$-dimensional submanifold $\Gamma \subset M$ of the d-dimensional spacetime manifold M, and let τ denote a tubular neighborhood[16] of Γ. The recipe in section 9 for modifying the path integral inside τ assumes that the action S in equation (2) is the only thing in the path integral that depends on link variables in τ, so the neighborhood τ should be treated as a **keep-out zone**:[17] it should not intersect the initial time, the final time, or the localization regions of any other operators in the same path integral. The size of τ in directions size transverse to Γ will be called its **thickness**. Its thickness should be large compared to the discretization scale. In the smooth-spacetime limit, its thickness should diverge compared to the discretization scale but should become infinitesimal in physical units.

[16] Article 53600 defines **tubular neighborhood**.

[17] Article 02242

6 Prescribed configuration in smooth spacetime

The 't Hooft operator $T^\circ(\Gamma)$ will be defined by modifying the action (3). The modification is such that the smooth-spacetime limit of the path integral is dominated by configurations of the gauge field the approach a prescribed configuration near Γ. Article 49708 describes the prescribed configuration in smooth spacetime. This section gives a brief review.

Define τ as in section 5, and consider a principal $U(1)$-bundle over $\tau\setminus\Gamma$ for which the net flux on any 2-sphere linked with Γ (with a given orientation) is 2π.[18,19,20] Arbitrarily close to Γ, this condition on the flux implies that the field strength necessarily becomes arbitrarily large, but we should choose the connection to keep the field strength elsewhere small in units of $1/\epsilon$.[21,22] We can do that by choosing a connection that minimizes the euclidean action in τ, subject to the constraint that the connection is consistent with the given principal G-bundle.[23]

Now let G be any compact connected Lie group. Every such group G has at least one subgroup isomorphic to $U(1)$. Let $\rho : U(1) \to G$ be a homomorphism whose image is such a subgroup.[24] Article 49708 shows that ρ converts a connection on a principal $U(1)$-bundle to a connection on a principal G-bundle. Applying this to the connection described in the previous paragraph gives the G-bundle connection that will be used to construct the 't Hooft operator $T^\circ_\rho(\Gamma)$ in section 9.

[18] The **flux** is the integral of the field strength 2-form over the specified surface.

[19] The existence of such a bundle depends on the topology of $\tau \setminus \Gamma$ (article 36626).

[20] In a principal $U(1)$-bundle, the set of possible values for the net flux on a 2-sphere is discrete (article 36626). With the units adopted in section 2, the minimum nonzero magnitude for the net flux on an oriented 2-sphere is 2π.

[21] Section 2 defines ϵ.

[22] If the field strength were allowed to violate this condition, then every principal $U(1)$-bundle on the underlying smooth spacetime would be consistent with the given configuration of the gauge field in discrete spacetime (article 11617).

[23] Minimizing the euclidean action gives the (euclidean) classical Yang-Mills equation of motion for the gauge field (article 49708). Using the euclidean action (instead of lorentzian) for this seems natural because the path integral near the smooth-spacetime limit is dominated by configurations that minimize the euclidean action. The concept of a connection does not depend on the spacetime metric, so once a connection has been chosen (even if a metric-dependent criterion is used), it may be used with any spacetime metric.

[24] The image of any homomorphism $\rho : U(1) \to G$ is either a subgroup isomorphic to $U(1)$ or the trivial subgroup consisting of only one element (the identity element).

7 Example: $G = U(1)$

This section uses the case $G = U(1)$ to illustrate two of the concepts that were mentioned in section 6.

First, consider the concept of choosing a connection that minimizes the euclidean action in τ. Take spacetime to be 3-dimensional, so Γ is 0-dimensional (a finite set of points). If Γ is a single point, then τ is a ball around that point. The euclidean action in τ has the form $\sim \int_\tau F^2$. The 2-form F is determined by a connection on a principal $U(1)$-bundle, so the action also is a function of that connection, not a function of an arbitrary 2-form. This enforces $\int F = 2\pi k$ with $k \in \mathbb{Z}$. Suppose the euclidean spacetime metric has spherical symmetry. Then, for a given k, minimizing the euclidean action subject to the constraint $\int F = 2\pi k$ would give F^2 =constant on each 2-sphere with constant radius from the point Γ. This illustrates the general idea that configurations which minimize the euclidean action have a field strength that is small in units of $1/\epsilon$ at distances $\gg \epsilon$ from Γ.

Next, consider the concept of a homomorphism $\rho : U(1) \to G$. When $G = U(1)$, any integer n gives a homomorphism $\rho : U(1) \to G$ that maps each element $g \in U(1)$ to $g^n \in U(1)$, and every homomorphism from $U(1)$ to itself has this form. We could denote the 't Hooft operator as $T_n^\circ(\Gamma)$ in this case, but this article uses the notation $T_\rho^\circ(\Gamma)$ because it works for any compact connected group G.[25]

[25] We could use a list of r integers to label the 't Hooft operator, where r is the **rank** of the Lie group G. Article 49708 explains how that works. Article 92035 defines the *rank* of a Lie group. The group $U(1)$ has rank 1, so a single integer is sufficient when $G = U(1)$.

8 Prescribed configuration in discrete spacetime

The 't Hooft operator $T^\circ(\Gamma)$ will be defined by modifying the action (3). The modification is such that the smooth-spacetime limit of the path integral is dominated by configurations of the gauge field the approach a prescribed configuration near Γ. Section 6 described the prescribed configuration in smooth spacetime. This section transfers it to discrete spacetime.

Let G denote the gauged group, which may be any compact connected Lie group. Discretize spacetime as reviewed in section 2. The gauge field consists of G-valued link variables, one for each oriented link in the lattice. Write $\square \in \tau$ to indicate that the corners of the plaquette $\square$ are all inside τ, and write $\square \notin \tau$ to indicate that at least one of $\square$'s corners is not inside τ.

For each oriented link ℓ whose endpoints are both in τ, let $\hat{u}(\ell)$ denote the element of G given by parallel transport along the link ℓ using the connection in section 6. These quantities $\hat{u}(\ell)$ constitute the prescribed configuration of the gauge field inside τ. It depends on a homomorphism $\rho : U(1) \to G$ as explained in section 6, even though the notation does not explicitly indicate this dependence. For all links with one or two endpoints outside τ, define $\hat{u}(\ell) \equiv 1_G$. This will enforce the condition in section 5 about where the path integral is modified.

For each oriented plaquette $\square$ whose corners are all inside τ, define

$$\hat{u}(x, \square) = \hat{u}(\ell_1)\hat{u}(\ell_2)\cdots$$

where $\ell_1, \ell_2, ...$ is the sequence of oriented links around the perimeter of $\square$, starting at the corner x. For plaquettes inside τ, $\hat{u}(x, \square)$ is a G-valued holonomy defined by the prescribed configuration.

Section 9 will use the quantities $\hat{u}(x, \square)$ for all plaquettes, not just those inside τ, to construct the 't Hooft operators $T^\circ(\Gamma)$. Remember that $\hat{u}(x, \square) = 1_G$ whenever the corners of $\square$ are all outside τ.

9 ’t Hooft operators in path integrals

Let $|\square|$ denote the number of points in the plaquette $\square$.[26] Let $\hat{u}(x,\square)$ denote the quantities defined in section 8, and define[27]

$$\hat{w}(x,\square) \equiv \frac{\text{trace}\left(u(x,\square)\big(\hat{u}(x,\square)\big)^{-1}\right)}{\text{trace}(1_G)} \tag{4}$$

and

$$\hat{w}(\square) \equiv \frac{1}{|\square|}\sum_{x\in\square}\hat{w}(x,\square). \tag{5}$$

The **’t Hooft operator** $\boldsymbol{T^\circ_\rho(\Gamma)}$ is defined by replacing the action (3) with

$$\hat{S}[u] = \sum_{\square} c(\square)\big(1-\hat{w}(\square)\big). \tag{6}$$

Sections 10-11 will show that the operator defined by this modification of the path integral is gauge invariant even though the modified action (6) is not.[28]

Section 12 will show that near the smooth-spacetime limit, the path integral is dominated by configurations with $u(x,\square)\approx\hat{u}(x,\square)$.[29] Section 15 will relate this to a common way of describing these operators when the model is imagined to be defined directly in smooth spacetime.[30]

[26] When spacetime is discretized as described in article 46333, each plaquette is a polygon with at least three corners, so $|\square|\geq 3$. In a conventional (hyper)cubic lattice, each plaquette is a square, so $|\square|=4$.

[27] If all corners of the plaquette $\square$ are outside τ, then $\hat{u}(x,\square)\equiv 1_G$, which implies $\hat{w}(x,\square)=\hat{w}(\square)=w(\square)$. This is consistent with section 5.

[28] The gauge invariance properties derived in sections 10-11 would hold even without the sum over basepoints. The sum over basepoints (equation (5)) is included only to make the path integral manifestly basepoint-independent.

[29] We might consider an alternative definition in which the link variables $u(\ell)$ for links ℓ inside τ are simply set equal to $\hat{u}(\ell)$ (which would enforce $u(x,\square)=\hat{u}(x,\square)$ exactly) instead of being treated as integration variables. Section 18 will explain why that alternative definition doesn’t quite work as desired.

[30] The word *imagined* is used here because such a definition is never actually given, at least not when dim $=4$ and G is nonabelian, but that imagined model is believed to emerge in the smooth-spacetime limit of the discrete-spacetime model used here. Article 07611 reviews one of the reasons for this belief.

10 Gauge invariance, part 1

This section shows that, despite appearances, the operator defined in section 9 is invariant under gauge transformations of the prescribed quantities $\hat{u}$, even though the modified action (6) is not. Notation:

- Let t_1 be the initial time and t_2 the final time. The difference $t_2 - t_1$ may be large compared to the lattice time-step.
- Let $\hat{u}$ be the prescribed quantities $\hat{u}(\ell)$ defined in section 8 for links in τ. Thanks to the conditions on τ in section 5, none of these links have any endpoints at times t_1 or t_2.
- Let u_k denote the set of spacelike link variables whose endpoints are both at time t_k.
- Let u denote the set of all link variables over which the path integral integrates. This includes u_1 but not u_2.
- Let Ψ be a gauge invariant initial state.
- Let μ denote the factor $e^{i\hat{S}}$ that replaces the e^{iS} in equation (2).

The path integral produces the final state

$$\Psi'[u_2, \hat{u}] \propto \int [du]\ \mu[u_2, \hat{u}, u]\Psi[u_1]. \tag{7}$$

The notation for the final state allows for the possibility that it depends on the prescribed values $\hat{u}$, but we will show that it is actually independent of those values. The modified action $\hat{S}$ (and therefore the function μ) is invariant if the same gauge transformation is applied to the link variables (u and u_2) and to the prescribed quantities $\hat{u}$:

$$\mu[u_2^{(h)}, \hat{u}^{(h)}, u^{(h)}] = \mu[u_2, \hat{u}, u]. \tag{8}$$

Choose a G-valued gauge transformation function $h(x)$ that is trivial at the initial and final times, so $h(x) = 1$ whenever x is at time t_1 or t_2. This does not restrict the effect of the gauge transformation on $\hat{u}$ because τ does not intersect the initial time.[31] Use a superscript (h) to denote the result of applying a gauge transformation to a link variable $u(\ell)$ or to a prescribed quantity $\hat{u}(\ell)$. With that notation, the calculation is easy:

$$\begin{aligned}
\Psi'[u_2, \hat{u}^{(h)}] &\propto \int [du]\ \mu[u_2, \hat{u}^{(h)}, u]\Psi[u_1] && \text{(equation (7))} \\
&\propto \int [du]\ \mu[u_2^{(h^{-1})}, \hat{u}, u^{(h^{-1})}]\Psi[u_1] && \text{(equation (8))} \\
&\propto \int [du]\ \mu[u_2^{(h^{-1})}, \hat{u}, u]\Psi[u_1^{(h)}] && \text{(shift invariance of Haar measure)} \\
&\propto \int [du]\ \mu[u_2, \hat{u}, u]\Psi[u_1] && (h(x) = 1 \text{ if } x \text{ is at time } t_1 \text{ or } t_2) \\
&= \Psi'[u_2, \hat{u}] && \text{(equation (7))}.
\end{aligned}$$

This shows that the operator defined in section 9 is actually invariant under gauge transformations of the prescribed quantities $\hat{u}$, even though the modified action (6) not.

[31] Section 5

11 Gauge invariance, part 2

This section shows that an operator defined by shifting the values of some plaquette variables (section 9) really is a linear operator on the Hilbert space of gauge invariant states: it maps each gauge invariant initial state to a gauge invariant final state. This is probably not surprising given the result that was derived in section 10, and the steps used in that derivation could be adapted to derive the result asserted here. This section uses a different approach, starting with a general lemma that is not tied to the specific construction in section 9. Notation:

- Let t_1 be the initial time and t_2 the final time, and suppose that the difference $t_2 - t_1$ is a single step in the spacetime lattice.
- Let u_k denote the set of spacelike link variables whose endpoints are both at time t_k.
- Let u_0 denote the set of timelike link variables that each have endpoints at both times t_1 and t_2.
- Let μ be a gauge invariant function of the link variables.
- Let Φ be any function of the link variables u_1. It might not be gauge invariant.

The path integral produces the final state

$$\Psi'[u_2] \propto \int [du_0][du_1]\; \mu[u_0, u_1, u_2]\Phi[u_1]. \tag{9}$$

Let $u^{(h)}$ denote the result of applying a gauge transformation to each link variable: $u(x, y) \to h(x)u(x, y)h^{-1}(y)$. Choose the G-valued gauge transformation function $h(x)$ to have support only at time t_2, so $h(x) = 1$ whenever x is not at time t_2. The fact that μ is gauge invariant implies that the final state is also gauge invariant:

$$\begin{aligned}
\Psi'\left[u_2^{(h)}\right] &\propto \int [du_0][du_1]\;\mu\left[u_0,u_1,u_2^{(h)}\right]\Phi[u_1] && \text{(equation (9))}\\
&= \int [du_0][du_1]\;\mu\left[u_0^{(h^{-1})},u_1^{(h^{-1})},u_2\right]\Phi[u_1] && \text{(μ is gauge invariant)}\\
&= \int [du_0][du_1]\;\mu[u_0,u_1,u_2]\Phi\left[u_1^{(h)}\right] && \text{(shift invariance of Haar measure)}\\
&= \int [du_0][du_1]\;\mu[u_0,u_1,u_2]\Phi[u_1] && \text{($h(x)=1$ if x is at time t_1)}\\
&= \Psi'[u_2] && \text{(equation (9)).}
\end{aligned}$$

This result says that time evolution automatically produces a gauge invariant state after a single time-step, even if the initial state was not gauge invariant.

That lemma clearly implies that the operator defined in section 9 maps gauge invariant initial states to gauge invariant final states, because we can take the function Φ to be the result of starting with a gauge invariant initial state Ψ at some earlier time $t < t_1$ and evolving it forward to time t_1 using a path integral in which some of the plaquette variables at times $t < t_1$ are shifted. The lemma derived above says that the final state Ψ' is gauge invariant, so the operator defined in section 9 really is an operator on the Hilbert space of gauge invariant states even though the modified action is not gauge invariant.[32]

[32] Article 07611 uses essentially the same lemma to show that the expectation value of an untraced product of link variables is proportional to the expectation value of the product's trace.

12 Configurations near the smooth-spacetime limit

The path integral integrates over all values of every link variable. In other words, it integrates over all configurations of the gauge field, where *configuration* means an assignment of specific values to the link variables. This section shows that near the smooth-spacetime limit, the path integral is dominated by configurations that approach the prescribed configuration $\hat{u}$ near Γ.

To take the smooth-spacetime limit of the path integral, some amount of Wick rotation must be used.[33] Then magnitude of the quantity $e^{iS[u]}$ in equation (2) is $e^{-f[u]}$ where $f[u]$ is proportional to the euclidean action.[34] If the integrand of the path integral includes an 't Hooft operator as described in section 9, then[33]

$$f[u] \propto \sum_{\square} \sum_{x \in \square} \left(1 - \frac{\hat{w}(x,\square) + \hat{w}^*(x,\square)}{2}\right) \tag{10}$$

with a positive proportionality factor, where $\hat{w}$ is defined by (5). Near the smooth-spacetime limit, the factor $e^{-f[u]}$ implies that the path integral is dominated by configurations of the gauge field that are close to minimizing (10).[35,36] The maximum possible magnitude of $\hat{w}(x,\square)$ is 1, and this maximum occurs only when $u(x,\square) = \hat{u}(x,\square)$,[37] so the path integral is dominated by configurations with

$$u(x,\square) \approx \hat{u}(x,\square) \qquad \text{for all } x, \square, \tag{11}$$

as long as such configurations are not too suppressed by the initial state, and section 14 will show that they are not.

Recall that $\hat{u}(x,\square) = 1_G$ outside τ. The neighborhood τ should be thick enough to ensure $\hat{u}(x,\square) \approx 1_G$ when approaching $\partial\tau$ from the inside. That way, the condition (11) does not require any sudden jumps in the value of the plaquette variables $u(x,\square)$ when crossing from one side of $\partial\tau$ to the other.

[33] Article 89053

[34] Footnote 14 in section 4

[35] Article 40191 reviews the reason and conditions for this.

[36] This statement also assumes that no operators are inserted in the integrand.

[37] Equation (5) and article 89053

13 A possible ambiguity

The construction of $T^\circ(\Gamma)$ is well-defined (unambiguous), but the intepretation of the resulting operator is potentially ambiguous.[38] This interpretational ambiguity occurs only under special circumstances in discrete spacetime – so special that contriving such a circumstance would take some effort. This section acknowledges the possible (though unlikely) ambiguity and explains how the smooth-spacetime limit eliminates the ambiguity.

If all the quantities $\hat{u}(x, \square)$ happen to be integer powers of some element $g \in G$ and the identity element of G is also an integer power of g, then an integer N exists for which the Nth power of each of those plaquette-variable shifts is the identity element. In that case, replacing ρ with ρ^{N+1} gives the same quantities $\hat{u}(x, \square)$, even though it changes the underlying principal G-bundle in smooth spacetime.

Even though this scenario is possible, a slight perturbation of the connection[39] or of the discrete spacetime points would eliminate the ambiguity. Even without such perturbations, we can eliminate the ambiguity in the smooth-spacetime limit by growing the thickness of the tubular neighborhood τ of Γ without bound compared to the discretization scale while taking the smooth-spacetime limit.[40] Even if the scenario described above happens to be realized at each value of the thickness for some thickness-dependent value of N, this growth causes the value of N to increase without bound,[41] eliminating the ambiguity in the continuum limit. This shows that the smooth-spacetime limit produces the desired operator $T^\circ_\rho(\Gamma)$, not $T^\circ_{\rho^{N+1}}(\Gamma)$ for any $N \neq 1$, despite a possible (though exceedingly unlikely) ambiguity when the thickness of τ is finite compared to the discretization scale.

[38] The potential ambiguity is in the value of it's **GNO charge**, using the vocabulary reviewed in article 49708.

[39] Section 6

[40] Section 5

[41] This growth in N occurs because the construction in section 6 implies that the quantities $\hat{u}(x, \square)$ asymptotically approach 1_G as the transverse distance of $\square$ from Γ increases compared to the discretization scale.

14 Initial states in the smooth-spacetime limit

Section 12 assumed that the initial state does not suppress configurations that satisfy (11). To show that this assumption is correct, recall[42] that τ doesn't intersect the time of the initial state, so equation (4) and the definition of $\hat{u}(x, \square)$ in section 8 say that terms in the action involving link variables at the initial time are not modified. Physically meaningful states have close to the minimum possible energy, where *close* is compared to $1/\epsilon$,[43] so the relationship between the hamiltonian (energy operator) and the euclidean action[44] implies that any physically meaningful initial state is mostly supported on configurations satisfying (11). This shows that physically meaningful initial states do not suppress those configurations.

[42]Section 5

[43]Section 2 defines ϵ.

[44]Article 89053

15 The singularity as a characteristic feature

The smooth-spacetime limit of $T^\circ(\Gamma)$ would not be well-defined (as an ordinary operator on the Hilbert space) without the help of some kind of smearing to alleviate the singular behavior of the field strength near Γ.[45] The region over which the singularity is smeared can be arbitrarily small in physical units, though, so the behavior of the field arbitrarily close to Γ can still be used as a way of characterizing $T^\circ(\Gamma)$. This characterization is commonly used when spacetime is not discretized: it is "defined" by requiring the gauge field variables in the path integral to approach a prescribed **singularity** on Γ.[46]

Operators defined by inserting a function of the field variables into the integrand of the path integral are sometimes called **order operators**, and operators defined by requiring the field variables in the path integral to approach a prescribed singularity are sometimes called **disorder operators**.[47,48] An 't Hooft operator of the type constructed in this article is an example of a disorder operator.

[45]Article 10690 shows that the smooth-spacetime limit of a Wilson operator $W^\circ(C)$ would be ill-defined without smearing.

[46]Kapustin (2006), section 3.2; Kapustin and Witten (2007), section 6.2; Gomis *et al* (2009), section 2

[47]Kapustin and Witten (2007), section 6.2

[48]This distinction between order and disorder operators might be ambiguous, because section 1 in Kapustin (2006) suggests that the idea of using prescribed singularities to characterized operators nominally localized on lower-dimensional submanifolds of spacetime is not limited to operators like T° whose "singularity can be detected from afar for topological reasons." (Article 40191 explains how T° can be detected that way.)

16 Avoiding singularities elsewhere

The construction in section 9 modifies the path integral by prescribing the configuration of the gauge field inside the tubular neighborhood τ of Γ. In the smooth-spacetime limit, the prescribed configuration becomes singular on Γ. Singularities elsewhere can be avoided if the principal $U(1)$-bundle in $\tau \setminus \Gamma$ can be extended to a principal $U(1)$-bundle everywhere in $M \setminus \Gamma$, where M is the spacetime manifold. To demonstrate that this condition can be satisfied, article 36626 constructs examples of nontrivial principal $U(1)$-bundles over $M \setminus \Gamma$ for various pairs (M, Γ).

17 Principal G-bundles and the path integral

Section 15 mentioned one of the ways T° is often described, namely by making the gauge field approach a prescribed singularity on Γ. It is also often described another way, namely excising the interior of the neighborhood τ from spacetime and imposing GNO monopole boundary conditions on its boundary $\partial\tau$.[49,50,51,52] Either way, the smooth-spacetime limit of the path integral notionally includes[53] a sum over connections and bundles compatible with the specified conditions, whether on the boundary of an excised region or asymptotically close to the singularity.[54,55]

[49] Article 02242 describes a way of thinking about quantum field theory that naturally accommodates this way of describing an operator.

[50] Examples: Argyres and Ünsal (2012), text before equation (2.26); Harlow and Ooguri (2021), text around equations (2.89)-(2.91); Witten (1997a), text below equation (2.21)

[51] Figure 6.2 in Atanasov (2018) depicts this (for 3-dimensional spacetime).

[52] Section 3.1 in Kapustin (2006) describes an example of an operator localized on a line in a scalar CFT and explains that it can also be described either as a prescribed singularity in the fields along the line or using prescribed boundary conditions on a neighborhood of the excised line.

[53] Article 11617

[54] The text after equation (2.30) in Argyres and Ünsal (2012) says, "Inserting this magnetic probe operator in the path integral means that we should integrate over all gauge fields with the boundary condition (2.26)..."

[55] Witten (1997a) hints at this in the text below equation (2.21).

18 Why not freeze the link variables in τ?

As an alternative to the construction in section 9, we might consider constructing the 't Hooft operator $T^\circ(\Gamma)$ by setting the link variables $u(\ell)$ equal to $\hat{u}(\ell)$ for links ℓ inside τ instead of integrating over them. For reference in this section, this will be called the **freezing** construction, because it freezes the values of the link variables in τ. The freezing construction clearly isn't equivalent to the original construction in section 9 when spacetime is discrete, but equation (11) suggests that it can achieve a similar effect in the smooth-spacetime limit. The freezing construction also seems consistent with the idea mentioned in section 17 where $T^\circ(\Gamma)$ is defined by excising a neighborhood of Γ from spacetime and constraining the behavior of the field on the boundary of the excised region.

When the homomorphism $\rho : G \to U(1)$ described in section 6 is nontrivial, the freezing construction seems to work as desired. The derivation in section 10 of invariance under gauge transformations of $\hat{u}$ still holds if the quantity denoted $\hat{u}$ in that section is reinterpreted as the fixed values of the link variables in τ. The derivation in section 11 still holds just as it is.

When ρ is the trivial homomorphism $\rho : U(1) \to 1_G$, though, the freezing approach doesn't quite work as desired. The operator $T^\circ_\rho(\Gamma)$ should reduce to the identity operator when ρ is trivial. The construction in section 9 clearly has this property,[56] but the freezing construction does not, not even in the smooth-spacetime limit. To confirm that it doesn't have that property in the smooth-spacetime limit, consider 4-dimensional spacetime, and consider a slightly deformed version Γ' of Γ such that Γ' is still inside τ but doesn't intersect Γ. The identity operator doesn't modify the integrand of the path integral in τ, so it allows the holonomy around Γ' to differ from 1_G. This is true even in the smooth-spacetime limit because small differences between $u(x,\square)$ and $\hat{u}(x,\square) = 1_G$ that are allowed by equation (11) can accumulate around the loop Γ'.[57] In contrast, the freezing construction forces

[56]This is clear because $\hat{u}(x,\square) = 1_G$ when ρ is trivial (section 8).

[57]In the smooth-spacetime limit, the size of the loop Γ (and therefore of Γ') becomes arbitrarily large compared to the discretization scale, so this accumulation can still occur even though the approximation (11) becomes arbitrarily good.

the holonomy around Γ' to be equal to 1_G, and taking the smooth-spacetime limit cannot change this. This shows that if $T^\circ_\rho(\Gamma)$ were constructed using freezing, then it would not reduce to the identity operator when ρ is trivial.

Compared to the freezing construction, the construction in section 9 also has another advantage: it still works as desired when charged matter fields are added to the model. The freezing construction doesn't, at least not without making it more complicated. In the freezing construction, the kinetic terms for the matter fields wouldn't be gauge invariant if the link variables in them were fixed, and if they weren't fixed then the euclidean action would not enforce (11) when approaching the smooth-spacetime limit. We could try to compensate for this by fixing the values of the matter fields inside τ, too, but that would complicate the construction.

19 References

(Open-access items include links.)

Argyres and Ünsal, 2012. "The semi-classical expansion and resurgence in gauge theories: new perturbative, instanton, bion, and renormalon effects" *JHEP* **2012(08)**: 063, https://doi.org/10.1007/JHEP08(2012)063

Atanasov, 2018. "Magnetic Monopoles, 't Hooft Lines, and the Geometric Langlands Correspondence" https://abatanasov.com/Files/Thesis.pdf

Gomis *et al*, 2009. "Quantum 't Hooft operators and S-duality in $\mathcal{N} = 4$ super Yang–Mills" *Adv. Theor. Math. Phys.* **13**: 1941-1981, https://projecteuclid.org/euclid.atmp/1282054381

Harlow and Ooguri, 2021. "Symmetries in Quantum Field Theory and Quantum Gravity" *Communications in Mathematical Physics* **383**: 1669-1804, https://arxiv.org/abs/1810.05338

Kapustin, 2006. "Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality" *Phys.Rev. D* **74**: 025005, https://arxiv.org/abs/hep-th/0501015

Kapustin and Witten, 2007. "Electric-Magnetic Duality And The Geometric Langlands Program" *Communications in Number Theory and Physics* **1**: 1-236, https://link.intlpress.com/JDetail/1805800346326876162

Witten, 1997a. "Lecture II-7: abelian duality" https://www.math.ias.edu/QFT/spring/witten7.ps

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