The Global Structure of Lie Groups

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Abstract This article summarizes some results about the global structure (topology) of Lie groups.

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1 Some prerequisites

This article assumes that these concepts are familiar:[1]

- a **group**,
- a **homomorphism** from one group to another,
- an **isomorphism** of two groups,
- a **subgroup** of a group.

This article also assumes that these concepts are familiar:[2]

- a **topological space**,
- a **continuous map** from one topological space to another,
- a **homeomorphism** of two topological spaces,
- a **topological manifold**,
- a **smooth manifold**,
- a **smooth map** from one smooth manifold to another,
- a **diffeomorphism** of two smooth manifolds.

[1] These are reviewed in article 29682.
[2] These are reviewed in article 93875.

2 Some notation

Some notation:

- $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ are the real numbers, rational numbers, and integers.
- $\mathbb{Z}_n$ is the integers modulo n. (Another common way to write $\mathbb{Z}_n$ is $\mathbb{Z}/n\mathbb{Z}$.)
- $\mathbb{R}$ is the field of real numbers.
- $\mathbb{R}^n$ is n-dimensional euclidean space.
- S^n is the n-dimensional sphere, the boundary of an $(n+1)$-dimensional ball.
- $\mathbb{R}\mathrm{P}^n$ is n-dimensional real projective space.
- If G and H are algebraic structures (like groups), then the notation $G \simeq H$ means that G and H are isomorphic to each other.
- If X and Y are topological spaces, then $X \times Y$ is their cartesian product with the product topology.
- Article 28539 defined $\times$, $\oplus$, and $\otimes$ for abelian groups.
- $\pi_k(X)$ is the kth **homotopy group**[3] of a topological space X.
- $H_k(X)$ is the kth **homology group**[4] of a topological space X.
- $H^k(X)$ is the kth **cohomology group**[4] of a topological space X.
- $T(G)$ is the **torsion**[4] of an abelian group G.

[3] Article 61813

[4] Article 28539

3 Some general definitions

A **topological group** is a group that is also a topological space and whose group operations (multiplication and inverse) are continuous with respect to its given topology.[5] A **Lie group** is a topological group that is also a smooth manifold and whose group operations (multiplication and inverse) are smooth with respect to its given smooth structure.[6]

If G and H are Lie groups, then a **Lie group homomorphism** $G \to H$ is both a homomorphism (when G and H are regarded as groups) and a smooth map (when G and H are regarded as smooth manifolds).[7,8] If $G \to H$ is an ordinary group homomorphism that is also continuous, then it is automatically smooth.[9] Two Lie groups G and H are called **isomorphic** to each other if they are isomorphic to each other as groups and diffeomorphic to each other as smooth manifolds.[10]

A **Lie subgroup** of a Lie group is a subgroup that is also a Lie group and also satisfies another condition that won't be reviewed here.[11] If a subgroup of a Lie group is a closed subset in the topological sense, then it is automatically a Lie subgroup and an embedded submanifold.[12,13] This is called the **closed subgroup theorem**. All of the subgroups used in this article will be closed.

A Lie group is called **connected** if it is connected in the topological sense. A set of totally disconnected points is a boring but legitimate example of a smooth manifold, so a discrete group is an example of a (non-connected) Lie group.[14] This article is mostly about connected Lie groups.

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[5] Lee (2011), chapter 3, page 77; Mimura and Toda (1991), section 1.1, page 5

[6] Lee (2013), chapter 7, page 151; Mimura and Toda (1991), section 1.5, page 39; Fulton and Harris (1991), §7.1

[7] Lee (2013), chapter 7, page 153; Mimura and Toda (1991), section 1.5, page 39; Fulton and Harris (1991), §7.1

[8] If the category (Lie groups in this case) is understood from the context, then it may simply be called a **morphism**.

[9] Mimura and Toda (1991), chapter 1, theorem 5.16

[10] Lee (2013), chapter 7, page 153

[11] Lee (2013), chapter 7, page 156

[12] Lee (2013), theorem 20.12, previewed on page 161

[13] Even better: If G is a Lie group and H is any subgroup, then H is closed in G if and only if H is an embedded Lie subgroup of G (Lee (2013), corollary 20.13).

[14] Harlow and Ooguri (2021), end of section 1.1

4 Some generalities about compact Lie groups

Recall[15] that a topological space X is called **compact** if any collection of open sets that covers X includes a finite number of open sets that already cover X. A Lie group is called **compact** if it is compact in the topological sense.

Every closed subgroup of a compact Lie group is compact,[16] and every compact subgroup of a Lie group is closed.[17] In both cases, the subgroup is a Lie group and an embedded submanifold.[18]

Group theory plays a key role in the study of symmetry, and we could also say that symmetry plays a role in the study of group theory. Examples: every finite group is the automorphism group of some convex polytope,[19] and every compact connected Lie group is the automorphism group (geometry-preserving group) of some compact connected riemannian manifold.[20]

[15] Article 44113

[16] More generally, every closed subset of a compact space is compact (Lee (2011), proposition 4.36).

[17] More generally, every compact subset of a Hausdorff space is closed (Lee (2011), proposition 4.36). Every Lie group is a smooth manifold, so every Lie group is a Hausdorff space (article 93875).

[18] This follows from the closed subgroup theorem (section 3).

[19] Chirvasitu (2008), abstract

[20] Chirvasitu (2008), section 1

5 Some families of compact Lie groups

This section reviews the definitions of three families of compact Lie groups: orthogonal groups, unitary groups, and symplectic groups.

Let $\mathbb{F}$ denote any of these associative division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. If $x \in \mathbb{F}$, then x^* denotes the conjugate obtained by reversing the signs of the non-real parts. The product $xx^* = x^*x$ is always a nonnegative real number.

Let $M(n, \mathbb{F})$ denote the matrix algebra in which each matrix A has components $A_{jk} \in \mathbb{F}$ with $j, k \in \{1, ..., n\}$. Let A^* denote the matrix obtained from A by taking the transpose of the matrix and replacing each component by its conjugate, so the components of A^* are $(A^*)_{jk} \equiv (A_{kj})^*$. Let I_n denote the unit matrix in $M(n, \mathbb{F})$.

The **orthogonal**, **unitary**, and **symplectic** groups are defined by[21,22]

$$\begin{aligned} O(n) &\equiv \big\{A \in M(n,\mathbb{R}) \,\big|\, AA^* = I_n\big\}, \\ U(n) &\equiv \big\{A \in M(n,\mathbb{C}) \,\big|\, AA^* = I_n\big\}, \\ \mathrm{Sp}(n) &\equiv \big\{A \in M(n,\mathbb{H}) \,\big|\, AA^* = I_n\big\}, \end{aligned}$$

respectively. The integer n is called the **order** of the group.[22] The subgroups

$$SO(n) \subset O(n) \qquad\qquad SU(n) \subset U(n)$$

are obtained by keeping only the matrices whose determinant is equal to 1. They're called the **special orthogonal** and **special unitary** groups, respectively. All of the groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, and $\mathrm{Sp}(n)$ are compact Lie groups.[21,23]

The symplectic group $\mathrm{Sp}(n)$ was defined here as a subgroup of $M(n, \mathbb{H})$, but it is also isomorphic (equivalent as an abstract Lie group) to a subgroup of $M(2n, \mathbb{C})$. That subgroup will be described in section 6.

[21] Adams (1996), chapter 1, pages 1-3

[22] Mimura and Toda (1991), section 1.2, pages 19 (for O, U) and 22 (for Sp)

[23] Mimura and Toda (1991), chapter 1, theorems 2.8 (for O, SO, U, SU) and 2.18 (for Sp)

6 Complex Lie groups

A *Lie group* is a smooth manifold whose group operations are smooth. Similarly, *complex Lie group* is a complex manifold whose group operations are holomorphic functions.[24] This section introduces two families of complex Lie groups, namely $O(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{C})$.

Every complex Lie group is also a Lie group, but not conversely. Using complex numbers in a matrix representation of a Lie group doesn't make it a complex Lie group. The groups $U(n)$ that were defined in section 5 are not complex Lie groups: the defining condition for a matrix to belong to $U(n)$ involves complex conjugation, which is not allowed for a complex Lie group.

The **complex orthogonal group** $O(n,\mathbb{C})$ is defined by

$$O(n,\mathbb{C}) \equiv \big\{A \in M(n,\mathbb{C}) \,\big|\, AA^T = I_n\big\},$$

where A^T is the transpose of A. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, define

$$\mathrm{Sp}(n,\mathbb{F}) \equiv \{A \in M(2n,\mathbb{F}) \mid A^T J_n A = J_n\} \qquad\qquad J_n \equiv \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

The group $\mathrm{Sp}(n,\mathbb{R})$ is called the **real symplectic group**, and $\mathrm{Sp}(n,\mathbb{C})$ is called the **complex symplectic group**.[25] The groups $O(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{C})$ are complex Lie groups:[26] the conditions for membership in these groups do not involve complex conjugation.[27]

To distinguish it from the variants defined above, the group $\mathrm{Sp}(n)$ that was defined in section 5 may be called the **compact symplectic group**.[28] That group is isomorphic to a subgroup of $\mathrm{Sp}(n,\mathbb{C})$:[25]

$$\mathrm{Sp}(n) \simeq \mathrm{Sp}(n,\mathbb{C}) \cap U(2n).$$

[24] Fulton and Harris (1991), section 7.1, page 95; https://ncatlab.org/nlab/show/complex+Lie+group

[25] Mimura and Toda (1991), section 1.2, page 23

[26] Fulton and Harris (1991), section 7.2, page 97

[27] A matrix to belongs to $O(n,\mathbb{C})$, if its inverse is equal to its transpose, so $O(n,\mathbb{C})$ is a complex Lie group. A matrix belongs to $U(n)$ if its inverse is the complex conjugate of its transpose, so $U(n)$ is not a complex Lie group.

[28] Fulton and Harris (1991), section 7.2, page 99

7 The general structure of noncompact Lie groups

A topological space that is not compact is called **noncompact**. A Lie group is called **noncompact** if it is noncompact in the topological sense.

Most of this article is about compact Lie groups. That focus is partly motivated by this result: every noncompact connected Lie group G is homeomorphic to $H\times\mathbb{R}^n$ for some n, where H is the maximal compact subgroup of G.[29,30] This implies that G is homotopy equivalent to its maximal compact subgroup H.[31] Intuitively, this means that H holds essentially all of the topologically interesting information about G, even though H has fewer dimensions than G.

A group is called **abelian** if all of its elements commute with each other. When G is abelian, then the topology is especially simple: any connected abelian Lie group is homeomorphic to $H\times\mathbb{R}^n$, where H is a torus (a cartesian product of circles).[32]

Section 8 will list some examples for nonabelian Lie groups.

[29]Mimura and Toda (1991), section 1.4, page 36

[30]This is related to the **Iwasawa decomposition** of G.

[31]Article 61813 reviews the definition of **homotopy equivalence**.

[32]Mimura and Toda (1991), section 5.3, theorem 3.5; Hilgert and Neeb (2012), section 9.5, pages 337-338

8 Examples

The groups that were defined in section 6 are not compact. These relationships illustrate the general result that was highlighted in section 7:[33,34,35]

$$
\begin{aligned}
O(n,\mathbb{C}) &\text{ is homeomorphic to } O(n)\times\mathbb{R}^{n(n-1)/2}\\
\mathrm{Sp}(n,\mathbb{R}) &\text{ is homeomorphic to } U(n)\times\mathbb{R}^{n(n+1)}\\
\mathrm{Sp}(n,\mathbb{C}) &\text{ is homeomorphic to } \mathrm{Sp}(n)\times\mathbb{R}^{n(2n+1)}.
\end{aligned}
$$

When p and q are both ≥ 1, the **indefinite orthogonal groups** $O(p,q)$ are noncompact subgroups of $O(n,\mathbb{C})$.[36] Topologically,[37]

$$
O(p,q) \text{ is homeomorphic to } \big(O(p)\times O(q)\big)\times\mathbb{R}^{pq}
$$

When $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, the **general linear group** and the **special linear group** are

$$
\begin{aligned}
GL(n,\mathbb{F}) &\equiv \big\{A\in M(n,\mathbb{F})\,\big|\,A\text{ is invertible}\big\},\\
SL(n,\mathbb{F}) &\equiv \big\{A\in GL(n,\mathbb{F})\,\big|\,\det A = 1\big\}.
\end{aligned}
$$

These groups are not compact. Topologically,[38]

$$
\begin{aligned}
GL(n,\mathbb{R}) &\text{ is homeomorphic to } O(n)\times\mathbb{R}^{n(n+1)/2}\\
GL(n,\mathbb{C}) &\text{ is homeomorphic to } U(n)\times\mathbb{R}^{(n^2)}\\
SL(n,\mathbb{R}) &\text{ is homeomorphic to } SO(n)\times\mathbb{R}^{n(n+1)/2}\\
SL(n,\mathbb{C}) &\text{ is homeomorphic to } SU(n)\times\mathbb{R}^{(n^2)}.
\end{aligned}
$$

---

[33] Mimura and Toda (1991), chapter 1, corollary 4.12

[34] The groups $O(n)$ and $O(n,\mathbb{C})$ that were defined in sections (5) and 6 are not connected, but the homeomorphism listed here still holds when restricted to their respective connected subgroups. A similar comment applies to other examples in this section that involve the groups $O(\cdot)$.

[35] Each of the groups $SO(n)$, $U(n)$, and $\mathrm{Sp}(n)$ is connected (Mimura and Toda (1991), chapter 1, corollary 3.12).

[36] These groups are defined in article 08264 and in Hilgert and Neeb (2012), section 4.3, page 74.

[37] Hilgert and Neeb (2012), proposition 17.2.5

[38] Mimura and Toda (1991), chapter 1, theorem 4.11 and corollary 4.12

9 Normal subgroups

A **normal subgroup** H of a group G in one for which $gHg^{-1} = H$ for all $g \in G$. Every group G with more than one element has at least two normal subgroups, namely the **trivial** group (the one-element group consisting of only the identity element) and G itself. The concept of a *normal subgroup* is important because of these results:

- H is a normal subgroup of G if and only if H is the kernel of a homomorphism from another group into G.[39]

- If H is a normal subgroup of G, then a group G/H called the **quotient group** may be defined.

The concept of a *quotient group* will be used extensively in the rest of this article. Article 29682 reviews the definition. Intuitively, G/H is G modulo H.

If a topological group is not connected, then the connected component that contains the identity element is a normal subgroup.[40]

[39] Article 29682 states the *if* part explicitly, and the *only if* part is implied by the quotient-group construction.

[40] Mimura and Toda (1991), chapter 1, theorem 1.12

10 Simple Lie groups

When applied to a group G, the word **simple** can mean either of two different things:

1. Usually, it means that G doesn't have any normal subgroups other than the trivial group and G itself.[41]

2. When G is a Lie group, it often means that G doesn't have any *connected* normal subgroups of G other than the trivial group and G itself.[42,43] With that meaning, a simple Lie group G is allowed have other normal subgroups, but they must be discrete.

In this article, when the word **simple** is applied to a Lie group, it will always have the second meaning.[44] Beware that a Lie group may be simple in the second sense even if it's not simple in the first sense.

[41]Scott (1987), section 2.5

[42]Bredon (1972), pages 29-30

[43]This usage of the word *simple* comes from the fact that such a Lie group's Lie algebra is *simple* in the sense that it doesn't have any nontrivial ideals (Salamon (2022), text above theorem 11.1; Hilgert and Neeb (2012), definition 12.1.15).

[44]This is done for the sake of consistency with the sources that will be cited.

11 The center of a Lie group

If G is any group, the **center** of G is the subgroup $Z(G) \subset G$ defined by this property: $h \in Z(G)$ if and only if $hg = gh$ for every $g \in G$. Section 17 will list the centers of some compact simply-connected Lie groups. If G is a Lie group and Γ is any discrete subgroup of the center of G, then G/Γ is another Lie group.[45] Sections 18-19 will show some examples. If G is a connected Lie group, then:

- Every discrete normal subgroup of G is contained in $Z(G)$.[46]
- The center of $G/Z(G)$ is the trivial group.[47] In other words, $G/Z(G)$ is a **centerless** Lie group.[48]
- If $Z(G)$ is discrete, then $G/Z(G)$ is isomorphic to $\tilde{G}/Z(\tilde{G})$, where $\tilde{G}$ is the universal covering group of G.[49,50]

If G is a compact connected Lie group, then:

- $Z(G)$ is the kernel of the **adjoint representation** (not reviewed here).[51]
- $Z(G)$ is finite if and only if G is semisimple.[52,53]

[45] Fulton and Harris (1991), proposition 7.10

[46] Fulton and Harris (1991), exercise 7.11a

[47] Fulton and Harris (1991), exercise 7.11b

[48] Examples of sources that use the name *centerless* include Wolf and Gray (1968a) and Wolf and Gray (1968b).

[49] Fulton and Harris (1991), exercise 7.12

[50] Article 61813 reviews the definition of **universal covering space**. The universal covering space $\tilde{G}$ of a connected Lie group G is has a natural Lie group structure (Hilgert and Neeb (2012), corollary 9.4.7; and Mimura and Toda (1991), chapter 2, lemma 4.5). This Lie group $\tilde{G}$ is called the the **universal covering group** or the **simply-connected covering group**.

[51] Mimura and Toda (1991), chapter 5, theorem 4.18

[52] Mimura and Toda (1991), chapter 5, theorem 5.29; Borel (1955), section 18, page 426

[53] If G is compact and connected, then this may be used as the definition of *semisimple*. Section 12 will mention another (equivalent) definition.

12 The general structure of compact Lie groups

If G is a compact Lie group, then G is isomorphic to a closed subgroup of $O(n)$ if n is sufficiently large.[54,55] That statement remains true if $O(n)$ is replaced by $U(n)$, because $O(n)$ is a subgroup of $U(n)$.[56]

A compact connected abelian Lie group G is always a **torus**,[57,58] which means that it is isomorphic to a cartesian product of copies of $U(1)$:

$$G \simeq U(1) \times U(1) \times \cdots \times U(1).$$

Every compact connected not-necessarily-abelian Lie group G has the form[59]

$$G = \frac{S_1 \times S_2 \times \cdots \times S_m \times T}{Z} \tag{1}$$

where the groups S_k, T, and Z satisfy these conditions:

- Each S_k is a compact, connected, simply-connected, and simple[60] Lie group.
- T is a torus.
- Z is a discrete subgroup of the center of $S_1 \times S_2 \times \cdots \times S_m \times T$.

If the factor T is absent, then G is called **semisimple**.[61,62]

[54] Mimura and Toda (1991), chapter 5, lemma 2.13 and theorem 2.14

[55] Example: article 86175 shows that the group $\text{Spin}(n)$ that will be mentioned in section 15 is a subgroup of $O(2^{(n/2)+1})$ or $O(2^{(n+1)/2})$ if n is even or odd, respectively.

[56] This should be clear from the definitions in section 5.

[57] Mimura and Toda (1991), chapter 5, corollary 3.6; Bröcker and tom Dieck (1985), chapter 2, section 8, page 107

[58] A connected abelian Lie group, whether compact or not, is isomorphic to $T \times \mathbb{R}^n$ for some n, where T is a torus (Mimura and Toda (1991), chapter 5, theorem 3.5).

[59] Mendes (2004), text above theorem B; Bredon (1972), theorems 6.9 and 6.10; Mimura and Toda (1991), chapter 5, corollary 5.31; Fulton and Harris (1991), section 26.1, page 439

[60] Here, *simple* has the second meaning in section 10.

[61] Bredon (1972), text below theorem 6.9

[62] A not-necessarily-compact Lie group is called **semisimple** if its Lie algebra is semisimple (Hilgert and Neeb (2012), definition 12.1.15).

13 Examples

These examples illustrate equation (1):

- For $n \geq 2$, $SU(n)$ is simply-connected and simple: it has the form (1) with $T = 1$ and $Z = 1$ and only one factor S_k, namely $SU(n)$ itself.[63]

- The group $U(n)$ is compact and connected but not semisimple:[64]

  $$U(n) \simeq \frac{SU(n) \times U(1)}{\mathbb{Z}_n}.$$

  The denominator $\mathbb{Z}_n$ is the subgroup of $SU(n) \times U(1)$ consisting of the n elements of the form (zI, z^*), where I is the identity matrix in $SU(n)$ and z is a complex number satisfying $z^n = 1$.

- $SO(n)$ is semisimple for all n, and it's simple for $n \neq 4$ but not for $n = 4$.[63] For $n = 4$, it has the form

  $$SO(4) \simeq \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}. \tag{2}$$

  The denominator $\mathbb{Z}_2$ is the subgroup of $SU(2) \times SU(2)$ consisting of (I, I) and $(-I, -I)$, where I is the identity element of $SU(2)$.[65]

[63] Section 15
[64] Boya (1989), equation II.8
[65] The isomorphism (2) may be deduced using quaternions (Gallier (2023)).

14 The rank of a Lie group

Even if a compact Lie group is semisimple, so that the torus factor in (1) is absent, it still has a subgroup isomorphic to a torus. If G is any connected Lie group, a subgroup that is isomorphic to a torus is called a **maximal torus** if it is not contained in any larger subgroup isomorphic to a torus. The number of dimensions (number of $U(1)$ factors) of any maximal torus in a compact connected Lie group G is called the **rank** of G.[66] Examples:[67]

- $U(n)$ has rank n, and the subgroup consisting of all diagonal matrices in $U(n)$ is a maximal torus.
- $SU(n)$ has rank $n-1$, and the subgroup consisting of all diagonal matrices with unit determinant is a maximal torus.
- $SO(2k)$ and $SO(2k+1)$ both have rank k, and a maximal torus for an $SO(2k)$ subgroup of $SO(2k+1)$ is also a maximal torus for $SO(2k+1)$. An example of a maximal torus for $SO(2k)$ is the subgroup generated by rotations about the origin in a fixed collection of k mutually orthogonal planes.[68]

If G is a compact connected Lie group, then:[69]

- Every element of G is contained in a maximal torus.
- All maximal tori are conjugate to each other. This means that if T and T' are two maximal tori in G, then $T' = g^{-1}Tg$ for some G.

[66] Bröcker and tom Dieck (1985), section 2.1, page 165; Mimura and Toda (1991), section 5.3, page 261 (their assumption that G is compact and connected is established on page 257); https://ncatlab.org/nlab/show/rank+of+a+Lie+group

[67] The maximal tori in these examples are given in Mimura and Toda (1991), chapter 1, equations (3.8)-(3.9).

[68] Bröcker and tom Dieck (1985), chapter 4, theorem 3.4

[69] Bröcker and tom Dieck (1985), chapter 4, theorem 1.6; Mimura and Toda (1991), chapter 5, theorem 3.15 and corollary 3.16

15 Compact simply-connected simple Lie groups

Every compact, connected, simply-connected, simple[70] Lie group is isomorphic to one of these:[71,72,73]

$$
\begin{array}{ll}
A_n \equiv SU(n+1) & n \geq 1 \\
B_n \equiv \text{Spin}(2n+1) & n \geq 2 \\
C_n \equiv \text{Sp}(n) & n \geq 3 \\
D_n \equiv \text{Spin}(2n) & n \geq 4 \\
E_n & n \in \{6,7,8\} \\
F_4 & \\
G_2 &
\end{array}
$$

The cases A_n, B_n, C_n, D_n are called **classical** Lie groups, and the others are called **exceptional** Lie groups.[74] The subscript in each case is the rank of the Lie group.[75] For the classical Lie groups, the values n in the list are restricted because:[76]

- Spin(2) is isomorphic to $U(1)$, so it's not simple.
- Spin(3) and Sp(1) are both isomorphic to $SU(2)$, which is already included.
- Spin(4) is isomorphic to $SU(2) \times SU(2)$, so it's not simple.
- Sp(2) is isomorphic to Spin(5), which is already included.
- Spin(6) is isomorphic to $SU(4)$, which is already included.[77]

[70] Here, *simple* has the second meaning in section 10.

[71] Salamon (2022), theorem 11.1; Adams (1996), chapter 1, page 10

[72] The fact that the groups $SU(\cdot)$ and $\text{Sp}(\cdot)$ that were defined in section 5 are simply-connected is stated in Mimura and Toda (1991), chapter 2, theorem 4.12.

[73] Article 08264 explains how to construct the groups $\text{Spin}(\cdot)$. That construction shows that $\text{Spin}(n)$ has a $\mathbb{Z}_2$ subgroup for which $\text{Spin}(n)/\mathbb{Z}_2$ is isomorphic to $SO(n)$. Topologically, $\text{Spin}(n)$ is a double cover of $SO(n)$.

[74] Mendes (2004), section 1, text above theorem B

[75] Borel (1955), section 5; Koch (2022), near the end of section 9.11

[76] The first four reasons listed here are given in Salamon (2022), in the text below theorem 11.1.

[77] Figueroa-O'Farrill (2017), lemma 8.1; Adams (1996), proposition 5.1

16 Dimensions

A Lie group is, among other things, a smooth manifold. This table lists the number of dimensions of the smooth manifold for each of the Lie groups that was listed in section 15:[78,79]

group	number of dimensions
$SU(n)$	n^2-1
$\mathrm{Spin}(n)$	$n(n-1)/2$
$\mathrm{Sp}(n)$	$(2n+1)n$

group	number of dimensions
G_2	14
F_4	52
E_6	78
E_7	133
E_8	248

The results on the left may be derived from the definitions in section 5 by working in a neighborhood of the identity matrix I. Details:

- For $A = I + B \in O(n)$,[80] to first order in B, the defining condition $AA^* = I$ implies that B is real and antisymmetric, so $\dim O(n) = n(n-1)/2$.

- For $A = I + B \in U(n)$,[81] to first order in B, the defining condition $AA^* = I$ implies $B = B_0 + iB_1$ where B_0 is a real antisymmetric matrix and B_1 is a real symmetric matrix, so $\dim U(n) = n(n-1)/2 + n(n+1)/2 = n^2$.

- For $A = I + B \in \mathrm{Sp}(n)$, to first order in B, the defining condition $AA^* = I$ implies $B = B_0 + iB_1 + jB_2 + kB_3$ where i, j, k are linearly independent square roots of -1, B_0 is a real antisymmetric matrix, and each of B_1, B_2, B_3 is a real symmetric matrix, so $\dim \mathrm{Sp}(n) = n(n-1)/2 + 3n(n+1)/2 = (2n+1)n$.

[78] Salamon (2022), end of section 11

[79] The dimensions of the exceptional Lie groups are also shown in Adams (1996), theorems 5.5 and 6.1.

[80] The groups $\mathrm{Spin}(n)$, $O(n)$, and $SO(n)$ all have the same number of dimensions, because $\mathrm{Spin}(n)$ and $SO(n)$ are both covering spaces of $SO(n)$.

[81] The number of dimensions of $U(n)$ is one more than the number of dimensions of $SU(n)$.

17 Centers

For each of the Lie groups that was listed in section 15, the next table lists a finite abelian group that is isomorphic to ($\simeq$) the center of that Lie group:[82,83,84]

group		other name	center
A_n	$(n \geq 1)$	$SU(n+1)$	$\simeq \mathbb{Z}_{n+1}$
B_n	$(n \geq 2)$	$\mathrm{Spin}(2n+1)$	$\simeq \mathbb{Z}_2$
C_n	$(n \geq 3)$	$\mathrm{Sp}(n)$	$\simeq \mathbb{Z}_2$
D_{2k}	$(k \geq 2)$	$\mathrm{Spin}(4k)$	$\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$
D_{2k+1}	$(k \geq 2)$	$\mathrm{Spin}(4k+2)$	$\simeq \mathbb{Z}_4$
E_6			$\simeq \mathbb{Z}_3$
E_7			$\simeq \mathbb{Z}_2$
E_8			trivial
F_4			trivial
G_2			trivial

In this table, the symbols $E_\bullet$, $F_\bullet$, and $G_\bullet$ refer to the unique simply-connected compact group with the corresponding Lie algebra. The non-simply-connected groups $E_6/\mathbb{Z}_3$ and $E_7/\mathbb{Z}_2$ have the same Lie algebras as the simply-connected versions E_6 and E_7, respectively,[85] and may sometimes be denoted by the same symbols.

[82]Goto and Kabayashi (1969), section 3, page 255; Bredon (1972), chapter 0, end of section 6

[83]The results for the Spin groups are also shown in Mimura and Toda (1991), chapter 2, theorem 4.4.

[84]$\mathbb{Z}_k$ denotes the cyclic group of order k, and "trivial" means the trivial group with only one element.

[85]Mimura and Toda (1991), chapter 6, text above lemma 7.17

18 Some non-simply-connected Lie groups

Let G be any of the simply-connected Lie groups listed in section 15, and let Γ be any subgroup of the center of G. The table in section 17 shows that for two families of classical Lie groups, $B_n = \text{Spin}(2n+1)$ and $C_n = \text{Sp}(n)$, we have only one nontrivial choice for Γ. The resulting non-simply-connected Lie groups are:

- $\text{PSp}(n) \equiv \text{Sp}(n)/\mathbb{Z}_2$,
- $\text{Spin}(2n+1)/\mathbb{Z}_2 \simeq SO(2n+1)$.

For the groups A_n and D_n, we have more than one nontrivial choice for Γ. For A_n and D_{2k+1}, all of the distinct subgroups of the center have different numbers of elements, so they must all give different quotient groups. In particular, the center of $D_{2k+1} = \text{Spin}(4k+2)$ has one subgroup with two elements and one with four elements (the whole center). The corresponding quotients are[86]

- $\text{Spin}(4k+2)/\mathbb{Z}_2 \simeq SO(4k+2)$,
- $\text{Spin}(4k+2)/\mathbb{Z}_4 \simeq SO(4k+2)/\mathbb{Z}_2$.

Section 19 will address the case $D_{2k} = \text{Spin}(4k)$, whose center has distinct subgroups with the same number of elements.

[86] If $n \geq 3$, then the center of the group $SO(n)$ is trivial when n is odd and is isomorphic to $\mathbb{Z}_2$ when n is even (Mimura and Toda (1991), chapter 2, theorem 4.10).

19 The case Spin($4k$)

The center of the Lie group Spin($4k$) is $\mathbb{Z}_2 \times \mathbb{Z}_2$. The quotient by the whole center is $SO(4k)/\mathbb{Z}_2$.

The center $\mathbb{Z}_2 \times \mathbb{Z}_2$ has three distinct two-element subgroups. In that situation, this result is important:[87] if X and Y are two subgroups of the center of a connected simply-connected Lie group G, then G/X and G/Y are isomorphic to each other if and only if $X = \sigma Y$ for some automorphism σ of G.[88] The (non-obvious) results are:[89]

- If $k \neq 2$, then the quotient of Spin($4k$) by a two-element subgroup of the center may be either $SO(4k)$ or something called the **semispinor group**, depending on which two-element subgroup of the center is used. These two possible outcomes are not isomorphic to each other.

- If $k = 2$, so that the simply-connected group is Spin(8), then each quotient by a two-element subgroup of the center is isomorphic to $SO(8)$.

[87] Goto and Kabayashi (1969), section 0

[88] Such an automorphism must be what is called an **outer automorphism**: it can't have the form $\sigma x = gxg^{-1}$ for any $g \in G$, because $gxg^{-1} = x$ for all $x \in Z$ (by the definition of the *center*).

[89] Mimura and Toda (1991), chapter 2, theorem 4.15

20 The fundamental group of a Lie group

The **fundamental group** $\pi_1(X)$ of a topological space X is the first in a series of topological invariants called **homotopy groups** $\pi_k(X)$.[90] If G is a connected Lie group, then its fundamental group $\pi_1(G)$ is a finitely generated abelian group.[91] Example:[92]

$$\begin{aligned}\pi_1\big(U(1)\times\cdots\times U(1)\big) &\simeq \mathbb{Z}\times\cdots\times\mathbb{Z}\\ &\simeq \mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\end{aligned}$$

with equal numbers of $U(1)$ factors and $\mathbb{Z}$ factors. The number of generators is the number of factors. If G is a compact connected Lie group, then $\pi_1(G)$ is finite if and only if G is semisimple,[93] which essentially means that $U(1)$ factors are absent.[94]

Each of the groups listed in section 17 is simply-connected, which means that its fundamental group is trivial. By taking quotients of those groups by discrete subgroups of their centers, we can generate Lie groups whose fundamental groups are not trivial. If G is a simply-connected topological group and Γ is a discrete subgroup of its center, then the fundamental group of G/Γ is isomorphic to Γ:[95]

$$\pi_1(G/\Gamma)\simeq\Gamma \qquad\qquad \text{if } \pi_1(G)=0.$$

Examples:[96,97]

$$\begin{aligned}\pi_1\big(SO(n)\big)=\pi_1\big(\mathrm{Spin}(n)/\mathbb{Z}_2\big) &\simeq \mathbb{Z}_2 \qquad \text{if } n\geq 3\\ \pi_1\big(SU(n)/\mathbb{Z}_n\big) &\simeq \mathbb{Z}_n.\end{aligned}$$

[90] Article 61813

[91] Hilgert and Neeb (2012), corollary 14.2.10

[92] Article 61813

[93] Mimura and Toda (1991), chapter 5, theorem 5.29

[94] Section 12 defined **semisimple**.

[95] Mimura and Toda (1991), chapter 2, theorem 4.8

[96] Mimura and Toda (1991), chapter 2, theorem 4.12 (for $SO(n)$)

[97] The $SU(n)/\mathbb{Z}_n$ example is used in Nash (1983), section 3.

21 Higher homotopy groups

For $k \geq 2$, the homotopy groups $\pi_k(\cdot)$ of $SU(n)$, $SO(n)$, and $\mathrm{Spin}(n)$ are determined by those of $U(n)$ and $O(n)$ through these isomorphisms:[98,99]

$$\begin{aligned} \pi_k(U(n)) &\simeq \pi_k(SU(n)) && \text{for } k \geq 2 \\ \pi_k(O(n)) &\simeq \pi_k(SO(n)) && \text{for } k \geq 1 \\ \pi_k(\mathrm{Spin}(n)) &\simeq \pi_k(SO(n)) && \text{for } k \geq 2. \end{aligned} \tag{3}$$

The homotopy groups $\pi_k(\cdot)$ with $k = 2$ and $k = 3$ are easy to summarize:[100]

- If G is a compact connected Lie group, then $\pi_2(G) = 0$.
- If G is a compact connected simple Lie group, then $\pi_3(G) \simeq \mathbb{Z}$.

The rest of this section lists some results for $k \geq 4$.

This table lists the first few homotopy groups of some classical Lie groups, excluding some cases covered by the isomorphisms listed in section 15:[101]

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$\pi_k(\mathrm{Sp}(1))$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_k(\mathrm{Sp}(n))$, $n \geq 2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$\pi_k(U(3))$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$	0	$\mathbb{Z}_{12}$
$\pi_k(U(4))$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_{24}$
$\pi_k(U(n))$, $n \geq 5$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$\pi_k(O(7))$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\pi_k(O(8))$	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\pi_k(O(9))$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\pi_k(O(n))$, $n \geq 10$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

[98] Mimura and Toda (1991), section 4.6, pages 216, 218, and 219

[99] If G is a Lie group and Z is a discrete subgroup of its center, then $\pi_k(G/Z) \simeq \pi_k(G)$ for all $k \geq 2$ (article 61813).

[100] Mimura and Toda (1991), chapter 6, theorem 4.17:

[101] Mimura and Toda (1991), section 4.6, table 4.2, after correcting a presumed typographical error (in the second row, the book says "$\pi_k(\mathrm{Sp}(2)),\ n \geq 2$" instead of "$\pi_k(\mathrm{Sp}(n)),\ n \geq 2$")

Here are some results for the exceptional Lie groups:[102]

$$
\begin{array}{ll}
\pi_k(G_2) = 0 \text{ for } 3 < k < 6 & \qquad \pi_6(G_2) \simeq \mathbb{Z}_3 \\
\pi_k(F_4) = 0 \text{ for } 3 < k < 8 & \qquad \pi_8(F_4) \simeq \mathbb{Z}_2 \\
\pi_k(E_6) = 0 \text{ for } 3 < k < 9 & \qquad \pi_9(E_6) \simeq \mathbb{Z} \\
\pi_k(E_7) = 0 \text{ for } 3 < k < 11 & \qquad \pi_{11}(E_7) \simeq \mathbb{Z} \\
\pi_k(E_8) = 0 \text{ for } 3 < k < 15 & \qquad \pi_{15}(E_8) \simeq \mathbb{Z}.
\end{array}
$$

The homotopy groups $\pi_k(X)$ consist of homotopy classes of maps from S^k into X, and the rotation group $O(n)$ may be viewed as a group of symmetries of the unit sphere S^{n-1} in n-dimensional euclidean space, so these results may be of interest:[103]

$$
\begin{array}{ll}
\pi_{n-1}(O(n)) \simeq \mathbb{Z} \oplus \mathbb{Z} & \qquad n \bmod 8 \ \in \{0, 4\} \\
\pi_{n-1}(O(n)) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 & \qquad n \bmod 8 \ = 2 \text{ but } n \neq 2 \\
\pi_{n-1}(O(n)) \simeq \mathbb{Z} & \qquad n \bmod 8 \ = 6 \\
\pi_{n-1}(O(n)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \qquad n \bmod 8 \ = 1 \text{ but } n \neq 1 \\
\pi_{n-1}(O(n)) \simeq \mathbb{Z}_2 & \qquad n \bmod 8 \ \in \{3, 5, 7\} \text{ but } n \notin \{3, 7\} \\
\quad \pi_2(O(3)) = \pi_6(O(7)) = 0. &
\end{array}
$$

[102] Mimura and Toda (1991), chapter 6, theorem 7.12 and remark 7.12′ (for G_2 and F_4), theorem 7.19 (for E_6 and E_7), and theorem 7.15 (for E_8).

[103] Mimura and Toda (1991), section 4.6, corollary 6.14, using equation (3)

22 Stable homotopy groups

The homotopy groups $\pi_k(O(n))$, $\pi_k(U(n))$, and $\pi_k(\mathrm{Sp}(n))$ become independent of n when n is large enough compared to k, specifically when

$$k < d(n+1) - 2$$

with $d = 1, 2, 4$ for $O(n)$, $U(n)$, and $\mathrm{Sp}(n)$, respectively.[104] The same statement is expressed in symbols like this:[105]

$$\begin{aligned}
\pi_k\big(O(n)\big) &\simeq \pi_k\big(O(n+1)\big) && \text{for } k < n-1 \\
\pi_k\big(U(n)\big) &\simeq \pi_k\big(U(n+1)\big) && \text{for } k < 2n \\
\pi_k\big(\mathrm{Sp}(n)\big) &\simeq \pi_k\big(\mathrm{Sp}(n+1)\big) && \text{for } k < 4n+2.
\end{aligned}$$

These are called **stable** homotopy groups. Explicitly:[106]

k mod 8	$\pi_k\big(O(n)\big)$ $1 \le k < n-1$	$\pi_k\big(U(n)\big)$ $1 \le k < 2n$	$\pi_k\big(\mathrm{Sp}(n)\big)$ $1 \le k < 4n+2$
0	$\mathbb{Z}_2$	0	0
1	$\mathbb{Z}_2$	$\mathbb{Z}$	0
2	0	0	0
3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
4	0	0	$\mathbb{Z}_2$
5	0	$\mathbb{Z}$	$\mathbb{Z}_2$
6	0	0	0
7	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

The fact that the pattern repeats as a function of k is called **Bott periodicity**.[107]

[104] Mimura and Toda (1991), section 4.6, page 216
[105] Mimura and Toda (1991), section 2.3, corollary 3.17
[106] Mimura and Toda (1991), section 4.6, table 4.1 and theorem 6.2
[107] Hatcher (2001), example 4.55

23 Real homology groups

The real homology groups[108] of a compact connected Lie group G are same as those of a product of odd-dimensional spheres:[109]

$$H_j(G;\mathbb{R}) \simeq H_j(S^{n_1}\times\cdots\times S^{n_r};\mathbb{R}) \qquad \text{for all } j, \tag{4}$$

where r is the rank of the Lie group. The sphere-dimensions $n_1, ..., n_r$ for the Lie groups listed in section 15 are summarized in these tables:[110,111,112]

	$n_1, ..., n_r$
$SU(n)$	$3, 5, 7, ..., 2n-1$
$\mathrm{Sp}(n)$	$3, 7, 11, ..., 4n-1$
$SO(2k+1)$	$3, 7, 11, ..., 4k-1$
$SO(2k)$	$3, 7, 11, ..., 4k-5, 2k-1$

	$n_1, ..., n_r$
G_2	$3, 11$
F_4	$3, 11, 15, 23$
E_6	$3, 9, 11, 15, 17, 23$
E_7	$3, 11, 15, 19, 23, 27, 35$
E_8	$3, 15, 23, 27, 35, 39, 47, 59$

In each case, the number of integers in the list is the rank of the group, and the sum of the integers in the list is the number of dimensions of the group. In the case $SO(2k)$, the last integer in the list, namely $2k-1$, doesn't follow the pattern of the preceding integers. Examples (for all j):[113]

$$\begin{aligned} H_j\big(SO(4);\mathbb{R}\big) &\simeq H_j\big(S^3\times S^3;\mathbb{R}\big) \\ H_j\big(SO(6);\mathbb{R}\big) &\simeq H_j\big(S^3\times S^7\times S^5;\mathbb{R}\big). \end{aligned}$$

[108] Article 28539 reviews the concept of a **homology group**.

[109] Boya (2002), section 2

[110] Boya (1989), equations I.1, I.2, I.3, I.4, IV.1; Coleman (1958), page 354. The $SO(\cdot)$ cases are also in Hatcher (2001), §3.D, p 300

[111] These results also apply when the compact simply connected Lie group G is replaced by G/Γ, where Γ is any discrete subgroup of the center of G (theorem 6 in Pontrjagin (1939), using the definition of *Betti number* reviewed in article 28539).

[112] The cases F_4 and $E_{6,7,8}$ each have a symmetry in the differences between consecutive n_ks: the first difference is the same as the last difference, the second difference is the same as the second-to-last difference, and so on. This is sometimes called a **capicua** symmetry (Boya (2002), section 3.1).

[113] Boya (1989), equation III.11

24 Limitations of real homology groups

Section 23 listed some results for homology groups with coefficients in the field $\mathbb{R}$ of real numbers. Most compact connected Lie groups G are not homeomorphic to a cartesian product of spheres, even though they have the same real homology groups. Examples:

- if $n \geq 3$, then $SU(n)$ is not homeomorphic to a cartesian product of spheres.[114]
- $SO(3)$ and $SO(4)$ are homeomorphic to $\mathbb{R}\mathrm{P}^3$ and $S^3 \times \mathbb{R}\mathrm{P}^3$, respectively,[115] and $\mathbb{R}\mathrm{P}^3$ is not homeomorphic to a sphere.
- $SO(5)$ is not homeomorphic to the cartesian product of any two compact manifolds with dimensions ≥ 1.[116]

Homology with coefficients in the group $\mathbb{Z}$ of integers carries additional information about the Lie group's topology, called *torsion*.[117] Sections 25-26 will review some results about torsion for compact connected Lie groups. Article 28539 shows that if M is a cartesian product of spheres, then $H_k(M;\mathbb{Z})$ does not have torsion, so Lie groups whose homology groups have torsion cannot be homeomorphic to a cartesian product of spheres. In particular, each of the $SO(n)$ examples listed above has torsion. The homology groups of $SU(n)$ don't have torsion, though, so in that case even the homology groups with coefficients in $\mathbb{Z}$ fail to detect the difference between the topology of $SU(n)$ and the topology of $S^3 \times S^5 \times \cdots \times S^{2n-1}$ when $n \geq 3$.

[114] Borel (1955), section 18, page 426

[115] Hatcher (2001), section 3.D, page 294

[116] Hatcher (2001), section 3.E, page 309

[117] Article 28539

25 Torsion

Here are some results about torsion for compact Lie groups:[118,119]

- $SU(n)$ does not have p-torsion for any $p \geq 2$.
- $\mathrm{Sp}(n)$ does not have p-torsion for any $p \geq 2$.
- If $n \geq 3$, then $SO(n)$ has p-torsion if $p = 2$ but not for any $p \geq 3$.[120]
- If $n \leq 6$, then $\mathrm{Spin}(n)$ does not have p-torsion for any $p \geq 2$.[121]
- If $n \geq 7$, then $\mathrm{Spin}(n)$ has p-torsion if $p = 2$ but not for any $p \geq 3$.
- G_2 has p-torsion for $p = 2$ but not for any $p \geq 3$.
- F_4, E_6, and E_7 each have p-torsion for $p = 2$ and $p = 3$ but not for any $p \geq 5$.
- E_8 has p-torsion for $p = 2$, $p = 3$, and $p = 5$, but not for any $p \geq 7$.

[118] Kumpel (1965), page 1351 (for the exceptional groups only); Borel (1955), section 11 (except for a discrepancy in the case F_4)

[119] Most of the cases in the list are covered by Mimura and Toda (1991), chapter 7, theorems 5.11 and 5.12 and the intervening text, and the other cases have their own footnotes. The results in Mimura and Toda (1991) actually refer to p-torsion in the *cohomology* groups, but they also apply to the *homology* groups because of a relationship that will be highlighted in section 29 (equation (17)).

[120] The lack of p-torsion for any $p \geq 3$ is proposition 3D.3 in Hatcher (2001). Sections 26-28 will cover $p = 2$.

[121] This follows from the isomorphisms listed at the end of section 15 together with the fat that $SU(n)$ and $\mathrm{Sp}(n)$ don't have 2-torsion for any n.

26 Torsion in $SO(n)$: inputs

Section 25 mentioned that $SO(n)$ has 2-torsion for all $n \geq 3$. This section gathers inputs that may be used to determine which specific homology groups of $SO(n)$ have 2-torsion. Sections 27-28 will work through two examples.

One input is the complete result[122] for $H_k(M;\mathbb{Z})$ whenever M is a cartesian product of spheres. In particular, that result shows that $H_k(M;\mathbb{Z})$ does not have torsion for any k. Another input is a relationship between $H_k(M;\mathbb{Z})$ and $H_k(M;\mathbb{Z}_2)$ that holds whenever $H_k(M;\mathbb{Z})$ does not have 2-torsion for any k, namely[122]

$$H_k(M;\mathbb{Z}_2) \simeq H_k(M;\mathbb{Z}) \otimes \mathbb{Z}_2. \tag{5}$$

Another input is the identities[122]

$$\mathbb{Z} \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2 \qquad\qquad \mathbb{Z}_2 \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2. \tag{6}$$

Another input is this result for the homology groups with coefficients in $\mathbb{Z}_2$:[123]

$$H_k\big(SO(n);\mathbb{Z}_2\big) \simeq H_k(M(n);\mathbb{Z}_2) \tag{7}$$

with $M(n) \equiv S^1 \times S^2 \times \cdots \times S^{n-1}$. The universal coefficient theorem[122] implies that $H_k(SO(n);\mathbb{R})$ becomes the non-torsion part of $H_k(SO(n);\mathbb{Z})$ after replacing each $\mathbb{R}$ with $\mathbb{Z}$. The homology groups of a cartesian product of spheres don't have torsion,[122] so the results in section 23 give

$$H_k\big(SO(n);\mathbb{Z}\big) \simeq H_k\big(M'(n);\mathbb{Z}\big) \oplus T\left(H_k\big(SO(n);\mathbb{Z}\big)\right), \tag{8}$$

where $M'(n)$ is the product of spheres specified in section 23 and $T(G)$ is the torsion part of G. Another special case of the universal coefficient theorem says[122]

$$H_k\big(SO(n);\mathbb{Z}_2\big) \simeq \Big(H_k\big(SO(n);\mathbb{Z}\big) \otimes \mathbb{Z}_2\Big) \oplus \Big(T\left(H_{k-1}\big(SO(n);\mathbb{Z}\big)\right) \otimes T(\mathbb{Z}_2)\Big). \tag{9}$$

Sections 27-28 will use these inputs to determine $H_2(SO(n))$ and $H_3(SO(n))$.

[122] Article 28539

[123] Hatcher (2001), text above theorem 3D.2

27 Example: from $H_1(SO(n);\mathbb{Z})$ to $H_2(SO(n);\mathbb{Z})$

This section starts with a result for $H_1(SO(n);\mathbb{Z})$ and then uses it to derive a result for $H_2(SO(n);\mathbb{Z})$.

For $n \geq 3$, section 20 says $\pi_1(SO(n)) \simeq \mathbb{Z}_2$. Use this in the Hurwicz isomorphism theorem[124] to get the result for $H_1(SO(n);\mathbb{Z})$:

$$H_1\big(SO(n);\mathbb{Z}\big) \simeq \mathbb{Z}_2 \qquad\qquad \text{for } n \geq 3. \tag{10}$$

With $M(n)$ defined as in section 26, a result from article 28539 gives

$$H_2\big(M(n);\mathbb{Z}\big) \simeq \mathbb{Z} \qquad\qquad \text{for } n \geq 3.$$

Use this in equation (5) together with the first identity in (6) to get

$$H_2\big(M(n);\mathbb{Z}_2\big) \simeq \mathbb{Z}_2 \qquad\qquad \text{for } n \geq 3.$$

Use this in (7) to get

$$H_2\big(SO(n);\mathbb{Z}_2\big) \simeq \mathbb{Z}_2 \qquad\qquad \text{for } n \geq 3.$$

Use this and equations (6) and (10) in (9) to get

$$\mathbb{Z}_2 \simeq \Big(H_2\big(SO(n);\mathbb{Z}\big) \otimes \mathbb{Z}_2\Big) \oplus \mathbb{Z}_2 \qquad\qquad \text{for } n \geq 3,$$

which implies

$$H_2\big(SO(n);\mathbb{Z}\big) \otimes \mathbb{Z}_2 = 0 \qquad\qquad \text{for } n \geq 3. \tag{11}$$

We already know that $SO(n)$ doesn't have p-torsion for any prime $p \neq 2$ (section 25), so (6) and (11) give the final result[125]

$$H_2\big(SO(n);\mathbb{Z}\big) = 0 \qquad\qquad \text{for } n \geq 3. \tag{12}$$

[124] Article 28539

[125] This is consistent with (8).

28 Example: from $H_2(SO(n);\mathbb{Z})$ to $H_3(SO(n);\mathbb{Z})$

This section uses the result for $H_2(SO(n);\mathbb{Z})$ from section 27 to derive a result for $H_3(SO(n);\mathbb{Z})$.

With $M(n)$ and $M'(n)$ defined as in section 26, a result from article 28539 gives

$$H_3\big(M(n);\mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & \text{if } n=3, \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } n\geq 4 \end{cases} \qquad H_3\big(M'(n);\mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } n=4, \\ \mathbb{Z} & \text{if } n\neq 4. \end{cases} \tag{13}$$

Use the first of these in equation (5) together with the first identity in (6) to get

$$H_3\big(M(n);\mathbb{Z}_2\big) \simeq \begin{cases} \mathbb{Z}_2 & \text{if } n=3, \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{if } n\geq 4. \end{cases}$$

Use this in (7) to get

$$H_3\big(SO(n);\mathbb{Z}_2\big) \simeq \begin{cases} \mathbb{Z}_2 & \text{if } n=3, \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 & \text{if } n\geq 4. \end{cases}$$

Use this and equations (6) and (12) in (9) to get

$$\begin{gathered} \mathbb{Z}_2 \simeq H_3\big(SO(n);\mathbb{Z}\big)\otimes\mathbb{Z}_2 \text{ if } n=3, \\ \mathbb{Z}_2\oplus\mathbb{Z}_2 \simeq H_3\big(SO(n);\mathbb{Z}\big)\otimes\mathbb{Z}_2 \text{ if } n\geq 4, \end{gathered} \tag{14}$$

which implies that $H_3(SO(n);\mathbb{Z})$ must be isomorphic to one of these:

$$\begin{gathered} \mathbb{Z} \text{ or } \mathbb{Z}_2 \text{ if } n=3, \\ \mathbb{Z}\oplus\mathbb{Z} \text{ or } \mathbb{Z}\oplus\mathbb{Z}_2 \text{ or } \mathbb{Z}_2\oplus\mathbb{Z}_2 \text{ if } n\geq 4. \end{gathered}$$

To determine which one it is, we can use the second of equations (13) in (8) to get

$$H_3\big(SO(n);\mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z}\oplus\mathbb{Z}\oplus(\text{torsion, if any}) & \text{if } n=4, \\ \mathbb{Z}\oplus(\text{torsion, if any}) & \text{if } n\neq 4. \end{cases}$$

Combine this with (14) to get the final result[126]

$$H_3\big(SO(n);\mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & \text{if } n=3, \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } n=4, \\ \mathbb{Z}\oplus\mathbb{Z}_2 & \text{if } n\geq 5. \end{cases} \tag{15}$$

This shows that $H_1(SO(3);\mathbb{Z})$ is not the only one of $SO(n)$'s homology groups that has torsion, at least when $n \geq 5$.[127]

[126]The cases $n=3$ and $n=4$ may be checked using the homeomorphisms $SO(3)\simeq \mathbb{R}\mathrm{P}^3$ and $SO(4)\simeq S^3\times\mathbb{R}\mathrm{P}^3$ (section 24), the homology groups of S^n and $\mathbb{R}\mathrm{P}^n$ (article 28539), and the Künneth formula (article 28539).

[127]I haven't found an independent check of the result for $n \geq 5$, so beware of possible mistakes.

29 Cohomology rings

The information that homology groups $H_k(M;\mathbb{Z})$ convey about the topology of a manifold M can be repackaged into **cohomology groups**,[128] denoted $H^k(M;\mathbb{Z})$. These can be combined and promoted to a **cohomology ring**,[128] denoted $H^*(M;\mathbb{Z})$, or more generally $H^*(M;R)$ where R is the ring of coefficients. Cohomology rings can convey more information than cohomology groups, but here the cohomology rings will be regarded only as a way of packaging the collection of cohomology groups. This section lists some cohomology rings of classical compact Lie groups and then reviews how to extract the corresponding cohomology groups.

Using some new notation that will be deciphered below,[129] here are a few examples of cohomology rings:[130,131,132]

$$\begin{aligned}
H^*(U(n);\mathbb{Z}) &= \Lambda(e_1, e_3, ..., e_{2n-1}) \\
H^*(SU(n);\mathbb{Z}) &= \Lambda(e_3, e_5, ..., e_{2n-1}) \\
H^*(SO(n);\mathbb{R}) &= \begin{cases} \Lambda(e_3, e_7, ..., e_{2n-3}) & \text{for } n \text{ odd,} \\ \Lambda(e_3, e_7, ..., e_{2n-5}, e_{n-1}) & \text{for } n \text{ even} \end{cases}
\end{aligned} \tag{16}$$

Each e_j denotes an element with **grade** j. Each cohomology ring is generated by a list of elements with the specified grades. This means that every element of $H^*(M;R)$ is a linear combination (with coefficients in R) of products of the specified elements e_j. The symbol Λ says something about which products are nonzero (more about this below). The grade of a product is the sum of the grades of the factors. The kth cohomology group $H^k(M;R)$ consists of the elements with

[128] Article 28539

[129] I'll give just enough information about the notation to explain how (16) encodes the collection of cohomology groups, without trying to be thorough.

[130] Mimura and Toda (1991), chapter 3, corollary 3.11 (for U, SU, and Sp) and corollary 3.15 (for SO)

[131] The corresponding results for the exceptional groups with coefficients in $\mathbb{R}$ are given by Mimura and Toda (1991), chapter 6, theorem 5.10 with the help of equation 5.1.

[132] The analog of footnote 111 in section 23 holds for cohomology rings, too: if two compact connected Lie groups are locally isomorphic, then they have isomorphic cohomology rings with coefficients in $\mathbb{R}$ (Chevalley and Eilenberg (1948), theorem 15.3; Mimura and Toda (1991), chapter 6, lemma 5.2).

grade k. If no nonzero products of the e_js have grade k, then $H^k(M;R)=0$. If one of the products of the e_js is such that all elements with grade k are proportional to it, then $H^k(M;R)\simeq R$. If two products of the e_js are such that all elements with grade k are linear combinations of them, then $H^k(M;R)\simeq R\oplus R$, and so on.

$\Lambda(a,b,c,...)$ is standard notation for the **exterior algebra** generated by $a,b,c,...$, with coefficients in whatever ring R is specified on the left-hand side of equations (16).[133] Here, the important things to know about $\Lambda(a,b,c,...)$ are that the only nonzero products of the generators $a,b,c,...$ are those with no repeated factors, like a and ab and abc, and that changing the order of the factors only changes the product's overall sign (or doesn't change anything at all).

Using that information, the cohomology groups $H^k(M;R)$ may be extracted from the cohomology rings listed above.

The sequences of subscripts in equations (16) match the sequences of sphere-dimensions in section 23. That can be explained by this relationship between homology groups and cohomology groups, which holds whenever the homology groups are finitely generated, as they are for any compact manifold M:[134]

$$\begin{aligned} H^k(M;\mathbb{Z}) \simeq\ & (\text{non-torsion part of } H_k(M;\mathbb{Z})) \\ & \oplus (\text{torsion part of } H_{k-1}(M;\mathbb{Z})). \end{aligned} \tag{17}$$

When $\mathbb{R}$ is used for the coefficients instead of $\mathbb{Z}$, this reduces to[135]

$$H^k(M;\mathbb{R}) \simeq H_k(M;\mathbb{R}).$$

Article 28539 reviews a general result that gives the homology groups $H_k(M;\mathbb{Z})$ when M is any cartesian product of spheres. Those homology groups don't have torsion, so equation (17) says that they're isomorphic to the cohomology groups $H^k(M;\mathbb{Z})$. This can be confirmed by comparing the cohomology groups extracted from equations (16) to the homology groups that were described in section 23, using the result reviewed in article 28539 for the homology groups of a product of spheres.

[133] Hatcher (2001), example 3.13

[134] Article 28539

[135] Mimura and Toda (1991), chapter 3, equation 1.8

30 A brief note about Lie algebras

Every Lie group has an associated Lie algebra. Conversely, every finite-dimensional Lie algebra is the Lie algebra of exactly one simply-connected Lie group G, and every other Lie group with this Lie algebra is a quotient of G by a discrete subgroup of its center.[136]

If G is a Lie group, then its Lie algebra $L(G)$ is the space of vectors tangent to G at the identity element of G, equipped with an algebraic structure related to the group structure of G. Knowing the tangent space at one point of a smooth manifold doesn't tell us anything about the manifold's topology except for the number of dimensions, but a Lie algebra tells us much more than this, thanks to its algebraic structure. The result quoted in the first paragraph above shows that the Lie algebra $L(G)$ doesn't (quite) know everything about G's topology[137] but that it does know much more than just the number of dimensions. This is possible because every connected topological group is generated by a neighborhood of the identity element.[138] In particular, every Lie group – whether simply-connected or not – is generated by a neighborhood of the identity element.[139]

Representations of a connected simply-connected Lie group are in one-to-one correspondence with representations of its Lie algebra.[140] Other Lie groups with the same Lie algebra lack some of those representations.[141] This difference in the set of representations can sometimes be used to distinguish between different Lie groups with the same Lie algebra in contexts where the global structure of the Lie group is not directly visible.[142,143]

[136] Fulton and Harris (1991), section 8.3, page 119

[137] Examples: the Lie groups $U(1)$ and $\mathbb{R}$ have the same Lie algebra but are topologically distinct, and the Lie groups $SO(3)$ and Spin(3) have the same Lie algebra but are topologically distinct.

[138] Mimura and Toda (1991), chapter 1, theorem 1.12

[139] Fulton and Harris (1991), exercise 8.1

[140] Fulton and Harris (1991), section 8.1, page 109

[141] One famous example of this is the fact that Spin(3) has a "spin 1/2" representation and $SO(3)$ does not.

[142] Tong (2017)

[143] A compact topological group can be reconstructed from its category of representations. This is called **Tannaka-Krein duality** (Bröcker and tom Dieck (1985), chapter 3, section 7).

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