Characterizing the Irreducible Representations of Compact Simple Lie Groups

Randy S

Abstract This article introduces the concept of a **weight lattice** associated with a compact simple Lie group G and reviews how the weight lattice is used to characterize finite-dimensional irreducible representations of G. The relationship between the sets of representations of different Lie groups with the same Lie algebra is also reviewed. The focus is on clarifying concepts and results. References to other sources are given instead of proofs.

Contents

1 Introduction and outline

Understanding finite-dimensional representations of compact Lie groups is often important in quantum field theory. Every finite-dimensional representations of a compact matrix Lie group can be written as a direct sum of irreducible representations.[1] This article reviews results about irreducible representations of compact simple Lie groups. Outline:

- Sections 3-8 review some conventions and background material.
- Sections 9-10 define *weights* and *roots* for compact Lie groups and their Lie algebras.
- Sections 11-17 introduce the concept of *complexification* of a Lie algebra and use it to give an alternative (equivalent) definition of *root*.
- Sections 18-19 define a positive-definite inner product on the real vector space spanned by the roots.
- Sections 20-21 use the concept of an *abstract root system* to state a key result about the roots of a Lie algebra.
- Sections 22-30 define the *weight lattice* and use it to characterize all finite-dimensional irreducible representations of a compact simple Lie group.
- Sections 31-33 mention more tools for working with that characterization.
- Section 34 uses the concept of a *coroot* to give an alternative (equivalent) definition of the weight lattice.

[1] Article 90757

2 Summary of conventions

These conventions are in effect throughout this article:

- Every Lie group G is understood to be given as a matrix group (section 4). One reason for this is to allow its Lie algebra to be defined as in section 6.
- The vector spaces on which representations act are always understood to be complex and finite-dimensional (section 4).
- All Lie groups are understood to be either connected or discrete. If neither of those adjectives is specified, then *connected* should be assumed (section 5).
- I made some effort to include the adjective *semisimple* (section 7) when citing results about Lie algebras that rely on that property, but when that adjective is omitted, the condition *semisimple* should probably still be assumed in case that condition was hiding somewhere in the cited source's context.[2,3]

[2] Fulton and Harris (1991), section 14.1, step 0

[3] My favorite way to expose such hidden conditions is to work through the proofs myself, but if I took the time to do that for all of the mathematical results I use, then I wouldn't have any time left to study physics.

3 Notation and vocabulary

- $\mathbb{Z}$ is the ring of integers, $\mathbb{R}$ is the field of real numbers, and $\mathbb{C}$ is the field of complex numbers.
- z^* is the complex conjugate of a complex number z
- V^* is the **dual** of a vector space V, defined as the vector space of linear functions from V to the appropriate field of scalars ($\mathbb{R}$ or $\mathbb{C}$).
- Article 29682 defines the concept of a **group**, the **center** of a group, the concept of a **homomorphism** from one group to another, the concept of an **isomorphism** between two groups, the concept of a **normal subgroup**, and the concept of a **quotient group**. Roughly: if G is a group and $\Gamma \subset G$ is a normal subgroup, then the **quotient group** G/Γ is "G modulo Γ."
- A generic Lie algebra will be denoted $\mathfrak{g}$, and the Lie algebra associated with a given Lie group G will be denoted $L(G)$.
- A vector space is called **real** if its field of scalars (coefficients) is $\mathbb{R}$. A vector space is called **complex** if its field of scalars (coefficients) is $\mathbb{C}$. Real and complex vector spaces are also called vector spaces **over** $\mathbb{R}$ and $\mathbb{C}$, respectively. The same language applies to Lie algebras (replace *vector space* with *Lie algebra*).
- R is the set of roots,[4] Λ_R is the root lattice, and Λ_W is the weight lattice. These will be defined later in this article.

[4]This notation is consistent with Hall (2015) (definition 8.1), Fulton and Harris (1991) (section 14.1), and Bröcker and tom Dieck (1985) (chapter 5, definition 3.3). In Knapp (2023) (text above proposition 2.17), the set of roots is denoted Δ. In Hall (2015) (definition 8.12), Δ denotes a base for R.

4 Representations of a group

If G is a group and V is a vector space, then a **representation** of G on V is a homomorphism ρ from G to the group of linear transformations of V. The **kernel** of a representation ρ is the set of elements of G for which $\rho(g)$ is the identity transformation of V. A representation of G is called **faithful** if its kernel contains nothing but the identity element of G.

In this article, every group G is understood to be given as a **matrix group** (also called a **linear group**),[5] which means that it is faithfully represented as a group of linear transformations of a finite-dimensional vector space. Every compact Lie group may be represented this way.[6]

Even though G is already given as a matrix group, we can still consider other representations of G.[7] In this article, the vector spaces on which representations act are always understood to be complex and finite-dimensional.[8]

A representation ρ of G on V is called **irreducible** if V does not have any subspace (other than the zero-dimensional subspace and V itself) that is self-contained under the action of ρ.[9] Two representations of G, one representation ρ on V and one representation ρ' on V', are said to be **equivalent** (or **isomorphic**) to each other if an invertible linear transformation $M : V \to V'$ exists for which $\rho'(g)M = M\rho(g)$ for all $g \in G$.[10]

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[5] Knapp (2023), abstract of introduction

[6] Knapp (2023), corollary 4.22

[7] Hall (2015), text below equation (4.1)

[8] This is consistent with Hall (2015) (stated at the beginning of section 12.1).

[9] Hall (2015), definition 4.2

[10] Knapp (2023), chapter 4, section 2, page 239

5 Compact Lie groups

A Lie group G is called **compact** if it is compact as a topological space, and it's called **simple** if it doesn't have any nontrivial connected normal subgroups other than G itself.[11] A Lie group Γ is called **discrete** if each element of Γ has a neighborhood that doesn't contain any other elements. If G is a connected matrix Lie group and Γ is a discrete normal subgroup of G, then Γ is in the center of G.[12] In this article, all Lie groups are understood to be either connected or discrete,[13] and if neither of those adjectives is specified, then *connected* should be understood.

Any compact connected Lie group has the form[11]

$$G = \frac{G_1 \times G_2 \times \cdots \times G_K}{\Gamma} \tag{1}$$

where Γ is a discrete subgroup of the center of $G_1 \times G_2 \times \cdots \times G_K$ and each factor G_k is either $U(1)$ or a simple 1-connected[14] Lie group. A Lie group G is called **semisimple** if it doesn't have any nontrivial connected abelian normal subgroups.[15] A semisimple Lie group has the form (1) in which all the factors G_k are simple and 1-connected.[11] A compact connected Lie group is semisimple if and only if its center is finite.[16] If G is a semisimple Lie group, then the center of the quotient group $G/Z(G)$ is trivial.[17,18]

Article 92035 lists all compact 1-connected simple Lie groups. All other compact simple Lie groups are quotients of those by discrete subgroups Γ of the center.

[11] Article 92035

[12] Hall (2015), chapter 1, exercise 11; Fulton and Harris (1991), exercise 7.11

[13] Examples: $SO(n)$ is connected, $\mathbb{Z}_n$ is discrete, and $O(n)$ is neither.

[14] **1-connected** means "connected and simply connected" (article 61813).

[15] Bröcker and tom Dieck (1985), chapter 5, definition 3.13

[16] Bröcker and tom Dieck (1985), chapter 5, remark 3.14

[17] Knapp (2023), proposition 6.30; Hall (2015), text below corollary 13.43 (assumes that G is compact)

[18] In Fulton and Harris (1991), exercise 7.11, the premise is *connected* instead of *semisimple*.

6 The Lie algebra of a Lie group

Every Lie group has a corresponding Lie algebra. The **Lie algebra** $L(G)$ of a matrix group G can be defined as the set of all matrixes X for which e^{sX} is in G for all real numbers s,[19] and then the **Lie bracket** of $X, Y \in L(G)$ can be defined as the commutator $[X, Y]$.[20]

If the Lie algebras of two 1-connected matrix Lie groups are isomorphic, then so are the Lie groups,[21] but the correspondence is not one-to-one: if Γ is a discrete subgroup of the center of G, then[22]

$$L(G/\Gamma) \text{ is isomorphic to } L(G), \tag{2}$$

even though G/Γ is typically not isomorphic to G. A homomorphism $G \to G/\Gamma$ with finite kernel Γ is called an **isogeny**,[23] so we can say that Lie groups with the same Lie algebra are in the same **isogeny class**. If $\mathfrak{g}$ is the Lie algebra of a compact connected simple Lie group, then two members of its isogeny class have special names: the 1-connected group G with Lie algebra $\mathfrak{g}$ is called the **simply connected form**, and the centerless group $G/Z(G)$ is called the **adjoint form** or **adjoint group**.[23,24] This article calls $G/Z(G)$ the **centerless group** to avoid confusion with the adjoint representation that will be introduced in section 8. The adjoint representation is one faithful representation of the centerless group, but it's not the only one.[25]

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[19]Hall (2015), definition 3.18 or 5.18

[20]These aren't the general definitions, but they're sufficient for *matrix* Lie groups/algebras.

[21]Hall (2015), corollary 5.7

[22]Fulton and Harris (1991), section 8.3, page 119

[23]Fulton and Harris (1991), section 7.3, text below exercise 7.11; Litterick (2018), end of section 2.1.1

[24]If G is already centerless (the group E_8 is one example), then of course G and $G/Z(G)$ are the same.

[25]Section 29

7 Simple and semisimple Lie algebras

A subspace $\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$ is called an **ideal** if $[X', X]$ is in $\mathfrak{g}'$ whenever $X' \in \mathfrak{g}'$ and $X \in \mathfrak{g}$,[26] and then $\mathfrak{g}'$ is automatically a subalgebra.[27] A Lie algebra is called **simple** if it has at least two dimensions (two linearly independent elements) and doesn't have any nontrivial ideals.[28] A Lie algebra is simple if and only if it doesn't have any nonzero abelian ideals.[29] The concept of an *ideal* of a Lie algebra corresponds to the concept of a (connected) *normal subgroup* of a Lie group: if H is a connected subgroup of a connected Lie group G and if $\mathfrak{h}$ and $\mathfrak{g}$ are their Lie algebras, then H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.[30]

A Lie algebra is called **semisimple** if it doesn't have any nonzero abelian ideals.[31] A Lie algebra is semisimple if and only if it is a direct sum of simple Lie algebras.[32] A connected Lie group, is semisimple if and only if its Lie algebra is $L(G)$ semisimple.[33]

[26] Fulton and Harris (1991), text above exercise 9.2; Hall (2015), definition 3.5

[27] Knapp (2023), text below equation (1.2)

[28] Fulton and Harris (1991), text below exercise 9.2; Hall (2015), proposition 7.32; Knapp (2023), text below proposition 1.12

[29] Fulton and Harris (1991), text below equation (9.4)

[30] Fulton and Harris (1991), exercise 9.2

[31] Fulton and Harris (1991), text below equation (9.4)

[32] Knapp (2023), theorem 1.54; Hall (2015), theorem 7.8 and the text below it

[33] Knapp (2023), text before corollary 1.126

8 Representations of Lie groups and Lie algebras

A representation of a matrix group G on a complex vector space V automatically gives a representation of $L(G)$ on V.[34] Representations of a 1-connected Lie group G are in one-to-one correspondence with representations of $L(G)$.[35] If Γ is a discrete normal subgroup of G, then representations of G/Γ are just those representations of G whose kernel includes Γ (those in which Γ acts trivially).[36]

If G is a connected matrix Lie group, then a representation of G is irreducible if and only if the associated representation of $L(G)$ is irreducible.[37]

Every Lie group has a representation called the **adjoint representation**. To define it for a matrix Lie group, first define an action of G on its Lie algebra $L(G)$ by

$$\rho(g) : X \mapsto gXg^{-1} \qquad \text{for all } X \in L(G) \text{ and } g \in G. \tag{3}$$

If we think of $L(G)$ as a vector space on which $\rho(g)$ acts, then ρ is a representation of G. This is the adjoint representation.[38] The kernel[39] of the adjoint representation ρ includes $Z(G)$. This should be clear from the definition (3). If G is a connected Lie group, then the kernel is equal to $Z(G)$,[40] so the adjoint representation is a faithful representation of $G/Z(G)$. Combine this with (2) to deduce that the adjoint representation of G gives a faithful representation of $L(G)$.[41]

[34]This follows from the way $L(G)$ was defined in section 6.

[35]Fulton and Harris (1991) pages 109-110

[36]The property $\rho(gz) = \rho(g)$ for all $z \in \Gamma$ allows us to define $\rho(g\Gamma) \equiv \rho(g)$, which makes ρ a representation of G/Γ because the elements of G/Γ are cosets $g\Gamma$ with $g \in G$.

[37]Hall (2015), proposition 4.5

[38]Hall (2015), definitions 3.32 and 4.9

[39]Section 4

[40]https://encyclopediaofmath.org/wiki/Adjoint_action; http://virtualmath1.stanford.edu/~conrad/249BW16Page/handouts/zgstr.pdf

[41]The adjoint representation of a semisimple Lie algebra is faithful (Fulton and Harris (1991), text below equation (9.4)).

9 Global weights and global roots for Lie groups

A Lie group isomorphic to the direct product of any number of $U(1)$ factors is called a **torus**. Even if a compact Lie group is semisimple (no $U(1)$ factors in (1)), it still has a subgroup isomorphic to a torus. If G is any connected Lie group, a subgroup that is isomorphic to a torus is called a **maximal torus** if it is not contained in any larger subgroup isomorphic to a torus.[42] If G is a compact connected Lie group, then

- The number of dimensions (number of $U(1)$ factors) of any maximal torus in G is called the **rank** of G.[43]
- The center of G is the intersection of all its maximal tori.[44] In particular, the center is contained in every maximal torus.[45,46]

Let ρ be a representation of a compact connected Lie group G on a complex vector space V, and let T be a fixed maximal torus in G. A **global weight** for (ρ, T) is a homomorphism $\omega : T \to U(1)$ for which the condition

$$\rho(t)v = \omega(t)v \qquad \text{for all } t \in T$$

is satisfied by at least one nonzero vector $v \in V$.[47] A nontrivial[48] global weight for the adjoint representation is also called a **global root**.[49,46]

[42] Bröcker and tom Dieck (1985), chapter 4, definition 1.1

[43] Article 92035; Bröcker and tom Dieck (1985), chapter 5, beginning of section 2; `https://ncatlab.org/nlab/show/maximal+torus`

[44] Bröcker and tom Dieck (1985), chapter 4, theorem 2.3

[45] Knapp (2023), corollary 4.47

[46] The adjective *global* is used to distinguish these weights from the ones that will be introduced in section 10.

[47] Bröcker and tom Dieck (1985), chapter 5, definition 1.1, combined with chapter 2, section 9

[48] Here, *nontrivial* means that it doesn't map all of T to 1.

[49] Bröcker and tom Dieck (1985), chapter 5, definition 1.3

10 Weights and roots for real Lie algebras

Let G be a compact connected semisimple Lie group and $L(G)$ its real Lie algebra. Let T be a maximal torus in G. The Lie algebra $L(T) \subset L(G)$ of T is called a **Cartan subalgebra** of $L(G)$,[50] and the number of dimensions of $L(T)$ is called the **rank** of $L(G)$.[51] Let ρ be a representation of G on a complex vector space V. This representation of G automatically gives a representation of $L(G)$ on V, and the same symbol ρ will be used for this representation of $L(G)$.

A **weight** for $(\rho, L(T))$ is a linear function $\alpha : L(T) \to \mathbb{C}$ for which the condition

$$\rho(H)v = \alpha(H)v \qquad \text{for all } H \in L(T) \tag{4}$$

is satisfied by at least one nonzero vector $v \in V$.[52] Nonzero weights for the adjoint representation are also called **roots**.[53] The set of roots will be denoted R.

The relationship $\rho\left(e^{2\pi H}\right) = e^{2\pi\rho(H)}$ gives this correspondence between weights α of $(\rho, L(T))$ and global weights ω of (ρ, T):

$$\omega\left(e^{2\pi H}\right) = e^{2\pi\alpha(H)}. \tag{5}$$

Every compact Lie group admits a faithful matrix representation in which each matrix is unitary.[54] In such a representation, each element of the real Lie algebra $L(G)$ is represented by an antihermitian matrix, which implies[55]

$$\big(\alpha(H)\big)^* = -\alpha(H), \tag{6}$$

so $\alpha(H)$ is i times a real number.

[50] Sepanski (2007), theorem 5.4

[51] Hall (2015), definition 7.12

[52] Bröcker and tom Dieck (1985), chapter 2, definition 9.3 and proposition 9.4, definition 9.5 and proposition 9.6, notation 9.7 and the text below it

[53] Bröcker and tom Dieck (1985), chapter 5, definition 1.3 calls them **infinitesimal roots**.

[54] Sepanski (2007), beginning of chapter 5

[55] To deduce this, apply $v^\dagger$ to both sides of (4) and use $(v^\dagger\rho(H)v)^\dagger = v^\dagger(\rho(H))^\dagger v$.

11 Complexification of a Lie algebra

The Lie algebra $L(G)$ of a compact Lie group G is a real Lie algebra. This means that the coefficients in the algebra must be real numbers, not complex numbers. This article is concerned with representations of Lie groups/algebras on complex vector spaces, so the components of a matrix M in a representation of $L(G)$ may be complex numbers, but the *coefficients* must still be in $\mathbb{R}$, so the matrix iM might not belong to the representation even if M does. If we want to allow arbitrary complex coefficients in the representation, then we must first allow them in the Lie algebra itself. This will be useful in section 16.

Given a real Lie algebra $\mathfrak{g}_0$, the complex Lie algebra $\mathfrak{g}$ obtained by allowing complex coefficients is called the **complexification** of $\mathfrak{g}_0$.[56] The relationship between $\mathfrak{g}_0$ and $\mathfrak{g}$ may be expressed symbolically like this:[57,58]

$$\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}.$$

In this article, the main reason for introducing the complexification of $L(G)$ is that most sources about the representation theory of Lie groups/algebras use complex Lie algebras as a tool (or even as an object of interest by itself), and many of the results cited in this article are drawn from those sources.

This article is ultimately concerned with irreducible representations of compact simple Lie groups G. The Lie algebra $L(G)$ is a real Lie algebra. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then the **compact form** of $\mathfrak{g}$ is the real Lie algebra $L(G)$ whose complexification $L(G) \otimes_{\mathbb{R}} \mathbb{C}$ is $\mathfrak{g}$, where G is a compact Lie group. Every semisimple complex Lie algebra has a unique compact form.[59]

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[56] Hall (2015), proposition 3.37

[57] Knapp (2023), text before proposition 1.17; Bröcker and tom Dieck (1985), chapter 2, text above definition 9.5, and chapter 5, text below definition 1.1

[58] If V and W are two vector spaces over $\mathbb{R}$, then $V \otimes_{\mathbb{R}} W$ is a vector space consisting of pairs (v, w) with $v \in V$ and $w \in W$ subject to the rules $(v, w) + (v', w) = (v + v', w)$ and $(v, w) + (v, w') = (v, w + w')$ and $(rv, w) = (v, rw)$ for all $r \in \mathbb{R}$. The subscript on $\otimes_{\mathbb{R}}$ means that real numbers (but not other complex numbers) may be passed from one side to the other.

[59] Fulton and Harris (1991), text after the proof of proposition 26.4, previewed in section 9.4, page 130

12 Complexification and (semi)simplicity

If $\mathfrak{g}_0$ is a real Lie algebra and $\mathfrak{g}$ its complexification, then:

- $\mathfrak{g}_0$ is semisimple if and only if $\mathfrak{g}$ is semisimple.[60]
- If $\mathfrak{g}$ is simple, then $\mathfrak{g}_0$ is simple,[61] but $\mathfrak{g}_0$ may be simple even if $\mathfrak{g}$ is not.[62]
- If $\mathfrak{g}_0$ is simple and $\mathfrak{g}_0 = L(G)$ for a compact matrix Lie group G, then $\mathfrak{g}$ is also simple.[63]

[60] Knapp (2023), corollary 1.53
[61] Hall (2015), proposition 7.31
[62] Hall (2015) gives an example in the text below proposition 7.31.
[63] Hall (2015), proposition 7.32

13 Complexification and representations

This section summarizes some relationships between representations of real Lie algebras and the corresponding complex Lie algebras.

- If $\mathfrak{g}_0$ and $\mathfrak{g}$ are both simple, then the irreducible representations of $\mathfrak{g}_0$ are the restrictions of irreducible representations of $\mathfrak{g}$.[64]

- If $\mathfrak{g}_0 = L(G)$ for a matrix Lie group G, then every finite-dimensional representation of $\mathfrak{g}_0$ over $\mathbb{C}$ extends to a representation of $\mathfrak{g}$, and the former is irreducible if and only if the latter is irreducible.[65]

- If $\mathfrak{g}_0$ is simple and $\mathfrak{g}_0 = L(G)$ for a compact matrix Lie group G, then every irreducible representation of $\mathfrak{g}$ gives an irreducible representation of $\mathfrak{g}_0$.[66]

- If G is a compact connected Lie group, then a finite-dimensional representation of G on a complex vector space gives a representation of $L(G)$ and of the complexification of $L(G)$.[67]

[64]Fulton and Harris (1991), exercise 26.14

[65]Hall (2015), proposition 4.6

[66]Fulton and Harris (1991), text above exercise 26.14, combined with the fact that in this case $\mathfrak{g}$ is simple (section 12)

[67]Knapp (2023), chapter 5, beginning of section 1

14 Weights and roots for complex Lie algebras

Let G be a compact simple Lie group and let $\mathfrak{g}$ be the complexification of its real Lie algebra $L(G)$. Let $\mathfrak{h}$ be a Cartan subaglebra of $\mathfrak{g}$, defined as in section 10.[68] The number of dimensions of $\mathfrak{h}$ (as a vector space over $\mathbb{C}$) is the **rank** of $\mathfrak{g}$,[69] so the rank of a real Lie algebra (section 10) and its complexification are the same.

Let ρ be a matrix representation of $\mathfrak{g}$ on a complex vector space V. If a nonzero vector $v \in V$ and a linear function $\alpha : \mathfrak{h} \to \mathbb{C}$ exist for which

$$\rho(H)v = \alpha(H)v \qquad \text{for all } H \in \mathfrak{h}, \tag{7}$$

then the function α is called a **weight** for $(\rho, \mathfrak{h})$.[70,71] As usual, a nonzero weight also called a **root** if ρ is the adjoint representation. Linearity implies

$$\alpha(iH) = i\alpha(H), \tag{8}$$

so this definition of weights and roots is consistent with the definition in section 10. Equations (6) and (8) together imply

$$\big(\alpha(H)\big)^* = \alpha(H^\dagger). \tag{9}$$

As in section 10, these definitions of roots and weights depend on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Any Cartan subalgebra of $\mathfrak{g}$ will work, but we need to choose one. The rest of this article uses the words *weight* and *root* without the qualifier "for $\mathfrak{h}$," but they are always understood to depend on which Cartan subalgebra $\mathfrak{h}$ is chosen.

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[68] The concept of a *Cartan subalgebra* $\mathfrak{h}$ of a complex semisimple Lie algebra $\mathfrak{g}$ can also be defined just in terms of $\mathfrak{g}$ without referring to an associated Lie group (Hall (2015), definition 7.10). Every complex semisimple Lie algebra has a Cartan subalgebra (Hall (2015), proposition 7.11).

[69] Knapp (2023), page 138

[70] Fulton and Harris (1991), section 14.1, text below equation (14.4); Hall (2015), definition 9.1

[71] This definition is valid for any complex semisimple Lie algebra.

15 The real vector space spanned by the weights

As before, let G be a compact simple Lie group, let $\mathfrak{g}$ be the complexification of its real Lie algebra $L(G)$, and let $\mathfrak{h}$ be a Cartan subaglebra of $\mathfrak{g}$. The vector space of linear functions from $\mathfrak{h}$ to $\mathbb{C}$ is called the **dual** of $\mathfrak{h}$, denoted $\mathfrak{h}^*$.[72] Weights are special elements of $\mathfrak{h}^*$,[73] and roots are special weights. The roots span $\mathfrak{h}^*$.[74]

If α is a weight for $\mathfrak{g}$, then $\alpha(H)$ is a complex number for most $H \in \mathfrak{h}$, but $i\alpha$ is not a weight.[75] The weights are elements of a complex vector space $\mathfrak{h}^*$, but since $i\alpha$ is not a weight, we can also treat the weights as elements of the *real* vector space V defined as the space spanned by the weights when only real coefficients are used. The complex vector space $\mathfrak{h}^*$ may be recovered from V by allowing complex coefficients, but allowing complex coefficients doesn't produce any new weights.

Define

$$\mathfrak{h}_{\text{antiherm}} \equiv \mathfrak{h} \cap L(G) \qquad\qquad \mathfrak{h}_{\text{herm}} \equiv i\mathfrak{h}_{\text{antiherm}}. \tag{10}$$

In a unitary representation of G, the subsets $\mathfrak{h}_{\text{antiherm}}$ and $\mathfrak{h}_{\text{herm}}$ consist of all the elements $\mathfrak{h}$ represented by antihermitian and hermitian matrixes, respectively. They are both real vector spaces: if v is a vector in one of them, then iv is a vector in the other one, so only real coefficients are allowed if we want to stay in the original space.

Section 14 defined weights to be special functions from $\mathfrak{h}$ to $\mathbb{C}$. When working in the real vector space spanned by the weights, we might as well restrict the domain (input) of each weight α to $\mathfrak{h}_{\text{herm}}$, because equation (9) says that a weight α is a function from $\mathfrak{h}_{\text{herm}}$ to $\mathbb{R}$. The **dual** $\mathfrak{h}^*_{\text{herm}}$ of the real vector space $\mathfrak{h}_{\text{herm}}$ is defined to be the vector space of linear functions from $\mathfrak{h}_{\text{herm}}$ to $\mathbb{R}$,[76] so weights may be regarded as elements of the real vector space $\mathfrak{h}^*_{\text{herm}}$.

[72]Section 3

[73]Fulton and Harris (1991), text below equation (12.1)

[74]Hall (2015), proposition 7.18

[75]Hall (2015) (definitions 11.34 and 12.1) calls $-i\alpha$ a **real weight** (or **real root** if it's a weight for the adjoint representation) because $-\alpha(H)$ is a real number when $H \in \mathfrak{h}_{\text{antiherm}} \subset L(G)$ (10). This article doesn't use "real weights" in that sense, but it does use the real vector space spanned by the weights, as explained in this section.

[76]Section 3

16 Direct definition of the roots of a Lie algebra

In the preceding sections, *roots* were defined as a special case of *weights*, namely the (nonzero) weights of the adjoint representation.[77] Roots have a special name because they play a special role, and they play a special role because the adjoint representation of a Lie algebra is defined by using the Lie algebra itself as the vector space on which the representation acts. This allows roots to be defined directly in terms of the Lie algebra itself, without explicitly invoking the concept of a representation.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. If a nonzero element $X \in \mathfrak{g}$ and a nonzero function $\alpha : \mathfrak{h} \to \mathbb{C}$ exist for which

$$[H, X] = \alpha(H)X \qquad \text{for all } H \in \mathfrak{h}, \tag{11}$$

then α is called a **root**.[78] The definition of the adjoint representation may be used to show that this agrees with the definition of *root* in section 14.

Even though roots (like all weights) may be regarded as elements of a real vector space,[79] using the condition $[H, X] \propto X$ to define them requires allowing arbitrary complex coefficients in the Lie algebra,[80,81] at least temporarily. To understand why, consider the Lie group $SU(2)$. Its real Lie algebra is generated by three elements A, B, H satisfying $[A, B] = H$, $[H, A] = B$, and $[H, B] = -A$. If we didn't allow complex coefficients, then the equation $[H, X] \propto X$ would not have any solutions with $X \neq 0$ and a nonzero proportionality factor,[82] but if we do allow complex coefficients, then it does: $[H, A \pm iB] = \mp i(A \pm iB)$.

[77] Fulton and Harris (1991), section 14.1, page 198

[78] Hall (2015), definition 7.13; Hall (2015), text below definition 8.1

[79] Section 15

[80] Bröcker and tom Dieck (1985), chapter 5, text below definition 1.1

[81] This is at least one reason why most sources about representations of Lie algebras use complex(ified) Lie algebras.

[82] To prove this, use a faithful matrix representation in which A, B, H are antihermitian. If $[H, X] = \alpha X$, then $\text{trace}\left(X^\dagger X\right)\alpha = \text{trace}\left(X^\dagger [H, X]\right) = \text{trace}\left([X^\dagger, H]X\right) = \text{trace}\left([H, X]X\right) = \text{trace}\left(X^2\right)\alpha = -\text{trace}\left(X^\dagger X\right)\alpha$, so $\text{trace}\left(X^\dagger X\right)\alpha$ equals its own negative, which implies either $\alpha = 0$ or $X = 0$.

17 Another derivation of (9) for roots

Consider the Lie algebra of a matrix group. Then each element of the Lie algebra is represented by a matrix X, and the hermitian conjugate $X^\dagger$ is defined. Equation (9) can be derived from (11) like this:

$$\begin{aligned}
\alpha(H)\text{trace}\big(X^\dagger X\big) &= \text{trace}\big(X^\dagger[H,X]\big) && \text{(using equation (11))}\\
&= \text{trace}\big([X^\dagger,H]X\big)\\
&= \text{trace}\big([H^\dagger,X]^\dagger X\big)\\
&= \text{trace}\big((\alpha(H^\dagger)X)^\dagger X\big) && \text{(using equation (11))}\\
&= \big(\alpha(H^\dagger)\big)^*\text{trace}\big(X^\dagger X\big)\,.
\end{aligned}$$

The quantity trace $\big(X^\dagger X\big)$ is nonzero whenever $X \neq 0$, so this implies (9).

18 An inner product on $\mathfrak{h}_{\text{herm}}$

Let R denote the set of roots. Let $\mathfrak{g}$ be a simple Lie algebra over either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. The bilinear form on $\mathfrak{h} \times \mathfrak{h}$ defined by

$$B(H, H') = \sum_{\alpha \in R} \alpha(H)\alpha(H') \tag{12}$$

is called the **Killing form**.[83,84] A Lie algebra is seimisimple if and only if its Killing form is nondegenerate.[85]

Let $\mathfrak{g}$ be a real semisimple Lie algebra, each element of which is represented by an antihermitian matrix. In this representation, the Killing form $B(H, H')$ is proportional to trace (HH') with a positive proportionality factor,[86] so[87]

$$B(H, H) \geq 0 \qquad \text{for all } H \in \mathfrak{h}_{\text{herm}}. \tag{13}$$

In words: the Killing form provides a (positive definite) inner product on the real vector space $\mathfrak{h}_{\text{herm}}$.

[83] Knapp (2023), corollary 2.24; Fulton and Harris (1991), text above equation (14.21)

[84] The Killing form can also be defined on all of $\mathfrak{g} \times \mathfrak{g}$, and equation (12) is a corollary of that definition. This article only uses the Killing form on $\mathfrak{h} \times \mathfrak{h}$, so equation (12) can be used as a definition.

[85] Fulton and Harris (1991), proposition C.10

[86] Sepanski (2007), theorem 6.16(i)

[87] This may also be expressed as $B(H, H) \leq 0$ for all $H \in \mathfrak{h}_{\text{antiherm}}$ (Bröcker and tom Dieck (1985), chapter 5, remark 7.13; Koch (2022), section 1.3). This is a special property of real Lie algebras of compact semisimple Lie groups (Fulton and Harris (1991), proposition 26.4; Bröcker and tom Dieck (1985), chapter 5, beginning of section 5).

19 An inner product on $\mathfrak{h}^*_{\text{herm}}$

Each root is an element of $\mathfrak{h}^*$. The bilinear form (12) can be used to select a corresponding element of $\mathfrak{h}$. For each root $\alpha \in R \subset \mathfrak{h}^*$, define an element $H_\alpha \in \mathfrak{h}$ by the condition[88]

$$B(H, H_\alpha) = \alpha(H) \qquad \text{for all } H \in \mathfrak{h}. \tag{14}$$

As in section 17, use a faithful matrix representation of the Lie algebra so that $H^\dagger$ is defined. Take the complex conjugate of both sides of (14) and use (9) to deduce $H_\alpha^\dagger = H_\alpha$. This implies that H_α is an element of $\mathfrak{h}_{\text{herm}}$.

Recall[89] that roots (like all weights) may be regarded as special elements of $\mathfrak{h}^*_{\text{herm}}$, which is a vector space over $\mathbb{R}$. Define an inner product on $\mathfrak{h}^*_{\text{herm}}$ like this:[90]

$$\langle \alpha, \beta \rangle \equiv B(H_\alpha, H_\beta) \qquad \text{for all } \alpha, \beta \in R, \tag{15}$$

using the definition (14) to relate each root $\alpha \in R \subset \mathfrak{h}^*$ to an element H_α of $\mathfrak{h}$. The quantities H_α are elements of $\mathfrak{h}_{\text{herm}}$, so the inequality (13) implies that (15) is a (positive definite) inner product on $\mathfrak{h}^*_{\text{herm}}$.[91]

A finite-dimensional real inner product space V is naturally isomorphic to its dual:[92] the inner product can be used to associate a unique element of V^* to each nonzero element of V in a way that respects linearity.[93] Even though they are naturally isomorphic, this article consistently distinguishes between them because this has some conceptual value.[94]

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[88]Knapp (2023), proposition 2.17(d); Sepanski (2007), definition 6.18

[89]Section 14

[90]Knapp (2023), equation (2.28); Fulton and Harris (1991), text below corollary 14.27

[91]Knapp (2023), corollary 2.38; Fulton and Harris (1991), theorem 14.30

[92]Even without an inner product, any finite-dimensional vector space V is isomorphic to its dual V^*, but they are not *naturally* isomorphic: without an inner product, no one isomorphism between V and V^* is more natural than any other. A subject called **category theory** provides a technical definition of *naturally isomorphic* (Awodey (2006), chapter 7).

[93]This is for *real* vector spaces. Section 1.1 in Selinger (2012) contrasts this with the case of complex inner product spaces.

[94]Koch (2022), section 17.7

20 The concept of an abstract root system

This section introduces the concept of an abstract root system. This is defined independently of Lie algebras, but it will be used in section 21 to state a key result about Lie algebras.

Let V be a finite-dimensional vector space over $\mathbb{R}$ with an inner product $\langle\cdot,\cdot\rangle$, and let R be a finite set of nonzero vectors in V. The pair (V, R) is called a **(reduced) abstract root system** if it has these properties:[95]

- Every vector in V is a linear combination of vectors in R.
- The quantity
$$2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \tag{16}$$
is an integer for all $\alpha, \beta \in R$.
- If $\alpha, \beta \in R$, then
$$\beta - 2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha \tag{17}$$
(the reflection of β along the direction α) is also in R. The case $\beta = \alpha$ implies $-\alpha \in R$.
- If $\alpha \in R$, then $-\alpha$ is the only other vector in R proportional to α.

The vectors in R are called **roots**, and the number of dimensions of V is the **rank** of the root system. The integers (16) are called **Cartan numbers**.[96]

Two abstract root systems (V, R) and (V', R') are said to be **isomorphic** to each other if a vector-space isomorphism $f : V \to V'$ exists that respects $f(R) = R'$ and preserves the Cartan numbers (16).[97]

[95] Hall (2015), definition 8.1; Koch (2022), definition 27; Knapp (2023), text above theorem 2.42 (explains what *reduced* means) and proposition 2.48

[96] Bröcker and tom Dieck (1985), chapter 5, definition 3.7

[97] Knapp (2023), text above proposition 2.44

21 Lie algebras and abstract root systems

Now a key result about the properties of the roots of a semisimple Lie algebra may be stated concisely: if $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra, and $V = \mathfrak{h}^*_{\text{herm}}$ is the real vector space of functions from $\mathfrak{h}_{\text{herm}}$ to $\mathbb{R}$, then the inner product $\langle\cdot,\cdot\rangle$ defined in section 19 makes (V, R) an abstract root system as defined in section 20.[98] Conversely, any reduced abstract root system is associated in this way with a complex semisimple Lie algebra, and this correspondence is one-to-one (up to isomorphism).[99]

A root system is called **irreducible** if it cannot be partitioned into two subsets so that all vectors in one subset are orthogonal to all vectors in the other subset.[100] The root system of a complex semisimple Lie algebra $\mathfrak{g}$ is irreducible if and only if $\mathfrak{g}$ is simple.[101]

A **Dynkin diagram** is a concise and graphic way to describe a root system. The root systems of all of the Lie algebras of compact simple Lie groups are shown in figure (1) using Dynkin diagrams. The caption of figure (1) explains how to decode the Dynkin diagrams. Figure 2 describes the root vectors more explicitly.

The roots of a simple Lie algebra either have two different lengths or are all the same length,[102] using the inner product $\langle\cdot,\cdot\rangle$ that was defined in section 19 to define *length*. Figure 3 specifies the lengths of the short and long roots in each root system. The lengths of the roots depend on the (arbitrary) normalization of the inner product, but the ratios (16) do not. The root lattice is not affected by the normalization of the inner product except for the (unimportant) overall scale.

[98] Hall (2015), theorem 7.30 and the text below theorem 7.30; Knapp (2023), theorem 2.42; Bröcker and tom Dieck (1985), chapter 5, theorem 3.12 (for groups: Bröcker and tom Dieck (1985), chapter 5, text below remark 3.13)

[99] Knapp (2023), chapter 2, abstract

[100] Knapp (2023), text above proposition 2.44

[101] Knapp (2023), proposition 2.44

[102] The cases with only one root-length (A, D, E) are called **simply laced** (Di Francesco *et al* (1997), text below equation (13.30)).

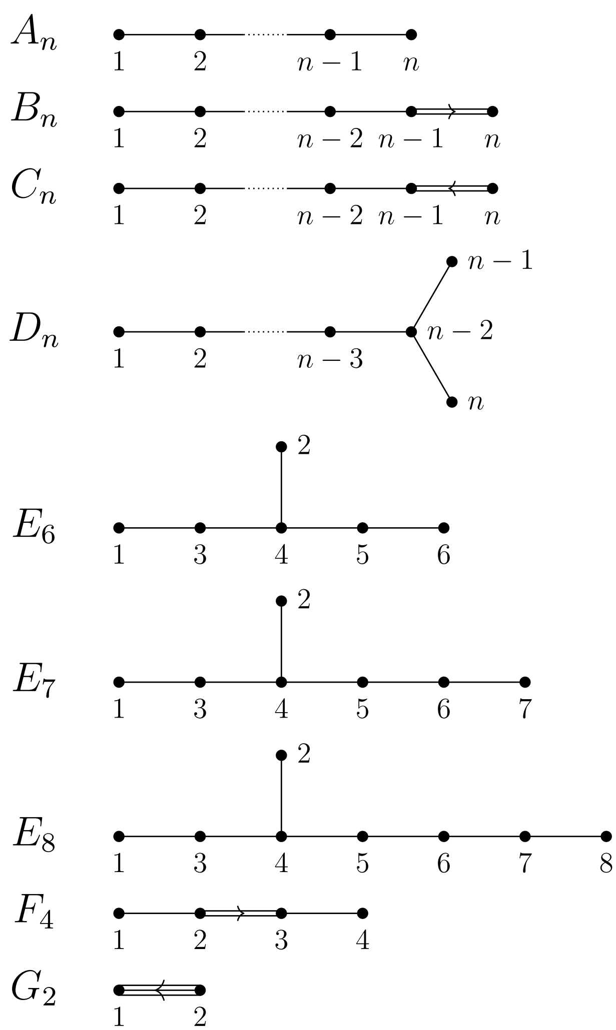

Figure 1 – Dynkin diagrams for the simple Lie algebras (Fulton and Harris (1991), theorem 21.11). The compact 1-connected Lie groups of types A_n, B_n, C_n, D_n are $SU(n+1)$, Spin$(2n+1)$, Sp(n), and Spin$(2n)$, respectively (article 92035). Each node represents a simple root (section 25). The integer labels specify an ordering of the simple roots according to the **Bourbaki convention**, which is also used in Sepanski (2007) (section 6.4.3), Figueroa-O'Farrill (1998) (table 6.2), and Litterick (2018) (section 2.1.1). The ordering convention is important when using a list of *Dynkin labels* to specify a representation (section 32). The angles between roots connected by 0, 1, 2, or 3 lines are $\pi/2$, $2\pi/3$, $3\pi/4$, or $5\pi/6$, respectively (Hall (2015), definition 8.31). In cases where roots have two different lengths, the diagram has an arrow pointing toward the side with the shorter root(s). Mnemonic: $x > y$ means x is longer than y.

	number of roots	root vectors (exclude the zero vector)	a basis of simple roots
A_n, $n \geq 1$	$n(n+1)$	$e_j - e_k \quad (1 \leq j, k \leq n+1)$	$e_j - e_{j+1} \quad (1 \leq j \leq n)$
B_n, $n \geq 2$	$2n^2$	$\pm e_j \pm e_k \quad (1 \leq j < k \leq n)$ $\pm e_j \quad (1 \leq j \leq n)$	$e_j - e_{j+1} \quad (1 \leq j \leq n-1)$ e_n
C_n, $n \geq 3$	$2n^2$	$\pm e_j \pm e_k \quad (1 \leq j < k \leq n)$ $\pm 2e_j \quad (1 \leq j \leq n)$	$e_j - e_{j+1} \quad (1 \leq j \leq n-1)$ $2e_n$
D_n, $n \geq 4$	$2n(n-1)$	$\pm e_j \pm e_k \quad (1 \leq j, k \leq n)$	$e_j - e_{j+1} \quad (1 \leq j \leq n-1)$ $e_{n-1} + e_n$
E_6	72	(omitted)	(omitted)
E_7	126	(omitted)	(omitted)
E_8	240	(omitted)	(omitted)
F_4	48	$\pm e_j \quad (1 \leq j \leq 4)$ $\pm e_j \pm e_k \quad (1 \leq j < k \leq 4)$ $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$	$e_2 - e_3,\ e_3 - e_4$ e_4 $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$
G_2	12	$\pm(e_2 - e_3),\ \pm(2e_1 - e_2 - e_3)$ $\pm(e_1 - e_3),\ \pm(2e_2 - e_1 - e_3)$ $\pm(e_1 - e_2),\ \pm(2e_3 - e_1 - e_2)$	$e_1 - e_2$ $-2e_1 + e_2 + e_3$

Figure 2 – Root vectors for the simple Lie algebras, expressed using a set of mutually orthogonal basis vectors e_k that all have the same length. The root vectors for E_6, E_7, E_8 are omitted to keep the table small, but they are shown in Geatti (2016) and Brouwer (2002). The root vectors for A_n and G_2 are expressed using more orthogonal basis vectors than the number of dimensions of the space spanned by the roots, because this allows using only rational coefficients. The last column shows a basis of simple roots. These correspond to the nodes in the Dynkin diagrams shown in figure 1. The vectors were compiled from Geatti (2016), section 2. The numbers of root are shown explicitly in Vogan (2017), table 1.

algebra	$1 / \langle \alpha^S, \alpha^S \rangle$	$\langle \alpha^L, \alpha^L \rangle / \langle \alpha^S, \alpha^S \rangle$
A_n	$n + 1$	1
B_n	$4n - 2$	2
C_n	$2n + 2$	2
D_n	$2n - 2$	1
E_6	12	1
E_7	18	1
E_8	30	1
F_4	18	2
G_2	12	3

Figure 3 – Values of $\langle \alpha, \alpha \rangle$ for the roots of a simple Lie algebras when $\langle \cdot, \cdot \rangle$ is the inner product defined in section 19. Short roots and long roots are denoted α^S and α^L, respectively. If the roots are all the same length, then α^L and α^S are equal. Source: Broughton (2014), table 2

22 The concept of a lattice

Let V be an n-dimensional real vector space with a (positive definite) inner product $\langle\cdot,\cdot\rangle$, and let $v_1,...,v_r$ be a finite list of vectors in V, possibly with $r > n$. Let Λ be the set of vectors that can be written $k_1v_1 + \cdots + k_rv_r$ with integer coefficients $k_1,...,k_r$. If Λ does not include any nonzero vectors with norm[103] less than ϵ for some finite $\epsilon > 0$, then Λ is called the **lattice** generated by $v_1,...,v_r$.

That definition allows $r > n$, but it doesn't need to: even if $r > n$, the lattice Λ automatically includes a **basis** of n or fewer vectors that generate Λ.[104]

Many sources define a lattice to consist of all integer linear combinations of a set of linearly independent vectors.[105] The result cited above says that this definition is equivalent to the preceding one. A virtue of the second definition is that it doesn't rely on any inner product.[106] In this article, though, a natural inner product is already given,[107,108] and then the definition in the first paragraph has the virtue of not explicitly constraining Λ to be generated by a basis of linearly independent vectors. The existence of such a basis is a (far from obvious) consequence of the definition, but it's not explicitly imposed as a constraint.

A lattice Λ is called **integral** if the inner product between each pair of vectors in Λ is an integer. The root and weight lattices that will be introduced in section 23 can be made integral by rescaling the inner product, but that won't be necessary in this article because the scale factor cancels in the ratio (16).

[103] The *norm* of a vector is defined using the inner product.

[104] Cassels (1997), section III.4, theorem IV; Barvinok (2020), theorem 3.1; Dadush (2018), theorem 7; Basu (2010), theorem 8

[105] The same word *lattice* is also commonly used with an unrelated meaning, namely a partially ordered set in which any two elements have a least upper bound and a greatest lower bound (Jacobson (1985), definition 8.1).

[106] The inner product is essential in the first definition because of the minimum-norm condition. That condition excludes cases like the set of all linear integer combinations of the vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(\sqrt{2},0,0)$.

[107] Section 19

[108] The overall scale of the inner product is arbitrary, but that doesn't affect the point being made here.

23 The root lattice and the weight lattice

Let G be a compact 1-connected simple Lie group, let $\mathfrak{g}$ be its complexified Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. The **root lattice** Λ_R is the lattice generated by the roots. The root lattice lives in the real vector space $\mathfrak{h}^*_{\text{herm}}$.

The result cited in section 21 says, among other things, that the quantity

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \tag{18}$$

is an integer whenever α and β are roots.[109] This implies that (18) is also an integer whenever α is a root and β is an element of the root lattice Λ_R. The **weight lattice** Λ_W is the lattice in $\mathfrak{h}^*_{\text{herm}}$ consisting of all vectors $\beta \in \mathfrak{h}^*_{\text{herm}}$ for which (18) is an integer for all roots α.[110] This is clearly a lattice, and it clearly includes Λ_R, but in most cases (namely all cases for which G has a nontrivial center), the weight lattice Λ_W includes more than just the root lattice Λ_R.[111]

The weight lattice Λ_W is important because of this result: it is precisely the union of all the weights of all the finite-dimensional irreducible representations of $\mathfrak{g}$ (and therefore of the 1-connected group G). Sections 26-27 will describe this relationship in more detail, and section 29 will explain how this relates to the other groups G/Γ that have the same Lie algebra as G.

[109] Fulton and Harris (1991), corollary 14.29

[110] Koch (2022), definition 26; Koch (2022), theorem 70; Sepanski (2007), definition 6.23; Fulton and Harris (1991), section 14.1, page 200; Fulton and Harris (1991), lecture 12, top of page 173 (example)

[111] Section 26

24 Explicit descriptions of root and weight lattices

These examples of weight and root lattices are shown on the next few pages:[112]

- Figure 4 shows the lattices for the 8-dimensional Lie group $SU(3)$.
- Figure 5 shows the lattices for the 10-dimensional Lie group Spin(5).
- Figure 6 shows the lattices for the 14-dimensional Lie group G_2.

Even though these three groups have different numbers of dimensions, their maximal tori are all two-dimensional (isomorphic to $U(1) \times U(1)$), so their weight and root lattices are also two-dimensional.

The root and weight lattices of all the classical (A_n, B_n, C_n, D_n) simple Lie algebras are described in figures 7-8.

[112] Hall (2015), figure 8.3 (for the roots) and figure 8.11 (weight lattices for Spin(5) and G_2)

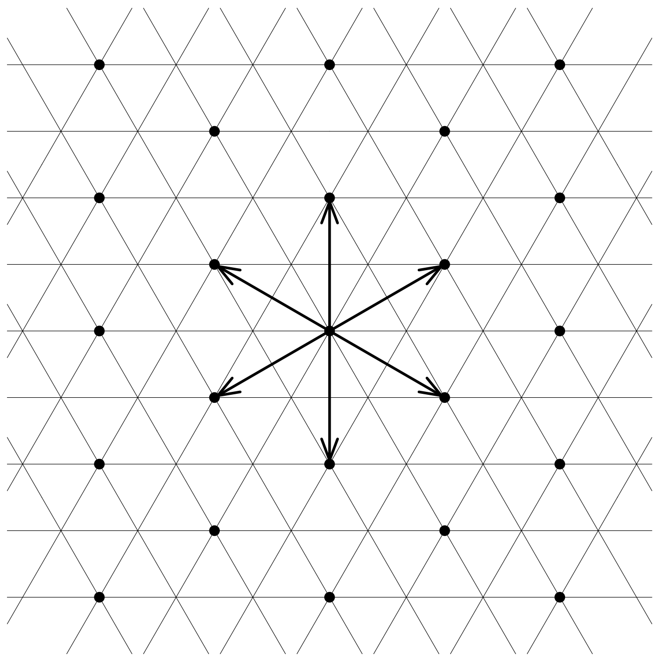

Figure 4 – Weight lattice for $SU(3)$. Intersections of lines represent elements of the weight lattice: the intersection point is the tip of a vector (not shown) from the origin, and that vector is an element of the weight lattice. Each arrow is a root. Dots represent elements of the root lattice. Notice that the weight lattice is three copies of the root lattice offset relative to each other. This corresponds to the fact that the center of $SU(3)$ is the three-element group $\mathbb{Z}_3$. The root and weight lattices for the centerless group $SU(3)/\mathbb{Z}_3$ are both the same as $SU(3)$'s root lattice.

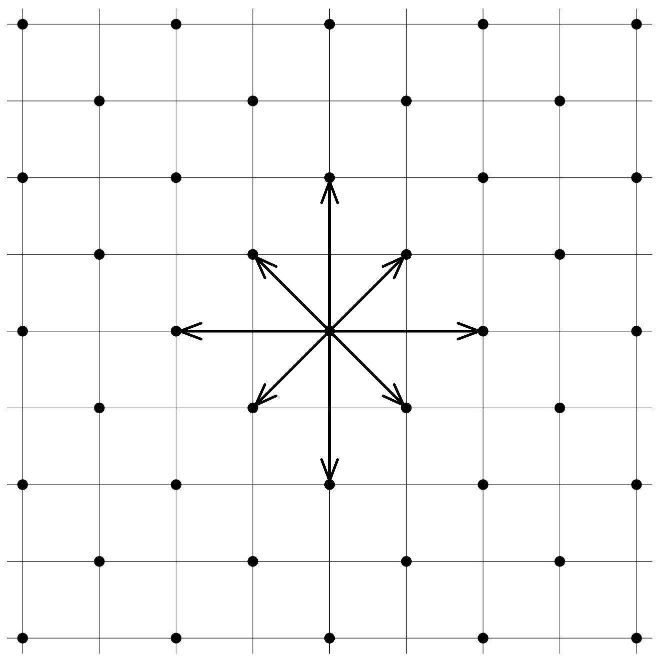

Figure 5 – Weight lattice for Spin(5). As in figure 4, intersections of lines represent elements of the weight lattice. Each arrow is a root. Dots represent elements of the root lattice. Notice that the weight lattice is two copies of the root lattice offset relative to each other. This corresponds to the fact that the center of Spin(5) is the two-element group $\mathbb{Z}_2$. The root and weight lattices for the centerless group $\text{Spin}(5)/\mathbb{Z}_2 \simeq SO(5)$ are both the same as Spin(5)'s root lattice.

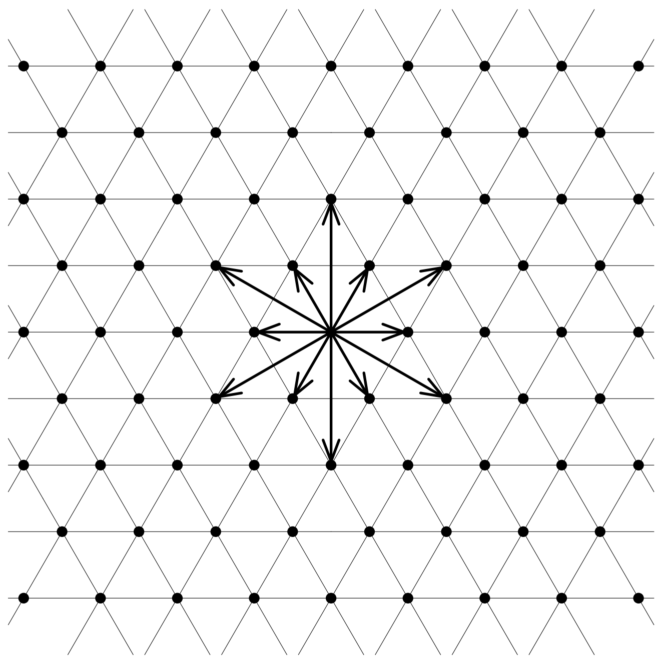

Figure 6 – Weight lattice for the compact simple Lie group of type G_2. As in figures 4 and 5, intersections of lines represent elements of the weight lattice. Each arrow is a root. The center of the group G_2 is trivial, so the root lattice is the same as the weight lattice.

	vectors in the root lattice	
A_n	$k_1e_1 + \cdots + k_{n+1}e_{n+1}$	$(k_j \in \mathbb{Z},\ k_1 + \cdots + k_{n+1} = 0)$
B_n	$k_1e_1 + \cdots + k_ne_n$	$(k_j \in \mathbb{Z})$
C_n	$k_1e_1 + \cdots + k_ne_n$	$(k_j \in \mathbb{Z},\ k_1 + \cdots + k_n \in 2\mathbb{Z})$
D_n	$k_1e_1 + \cdots + k_ne_n$	$(k_j \in \mathbb{Z},\ k_1 + \cdots + k_n \in 2\mathbb{Z})$

Figure 7 – Root lattices for the classical simple Lie algebras, using the same conventions as in figure 2. Sources: Sepanski (2007), exercise 6.25(2) with definition 6.23(a); Kac (2010)

	vectors in the weight lattice	
A_n	$k_1e_1 + \cdots + k_{n+1}e_{n+1} + k_0v$	$(k_j \in \mathbb{Z},\ k_0 + \cdots + k_{n+1} = 0)$
	with $v \equiv \frac{1}{n+1}(e_1 + \cdots + e_{n+1})$	
B_n	$k_1e_1 + \cdots + k_ne_n + k_0v$	$(k_j \in \mathbb{Z})$
	with $v \equiv \frac{1}{2}(e_1 + \cdots + e_n)$	
C_n	$k_1e_1 + \cdots + k_ne_n$	$(k_j \in \mathbb{Z})$
D_n	$k_1e_1 + \cdots + k_ne_n + k_0v$	$(k_j \in \mathbb{Z})$
	with $v \equiv \frac{1}{2}(e_1 + \cdots + e_n)$	

Figure 8 – Weight lattices for the classical simple Lie algebras, using the same conventions as in figure 2. Source: Sepanski (2007), exercise 6.25(2) with definition 6.23(b)

25 Positive roots, simple roots, and dominant weights

Let G be a compact 1-connected simple Lie group, and let (V, R) be the corresponding abstract root system. Choose a vector $v \in V$ that is not orthogonal to any roots, and define the set $R^+ \subset R$ of **positive roots** to consist of the roots α for which $\langle v, \alpha \rangle > 0$.[113] A positive root is called **simple** if it cannot be written as a sum of any two positive roots.[114] The simple roots are a basis for the root lattice.[115] The other positive roots are sums of simple roots.

The **fundamental Weyl chamber** $C \subset V$ is the set of vectors whose inner products with the positive roots are all non-negative.[116] Any subset of V obtained from C by reflections reflections along the roots is called a **Weyl chamber**.[117] If V is r-dimensional, then each $(r-1)$-dimensional boundary of a Weyl chamber is orthogonal to a root.[118] Distinct Weyl chambers don't intersect each other except on their boundaries, and their union is all of V.[118]

An element β of the weight lattice Λ_W is called **dominant** if it is in the fundamental Weyl chamber – in other words, if $\langle \alpha, \beta \rangle \geq 0$ for all positive roots $\alpha \in R^+$. Irreducible representations of the 1-connected group G are in one-to-one correspondence with dominant elements of Λ_W.[119] Section 26 will define that correspondence, and section 27 will specify which weights each irreducible representation includes.

Figures 9-11 show the simple roots (for a particular choice of the vector v that is used to define which roots are *positive*) and corresponding fundamental Weyl chamber and dominant weights for a few groups G whose weight lattices are two-dimensional.

[113] Hall (2015), theorem 8.16 and the text below it; Fulton and Harris (1991), section 14.1, text around equation (14.12)

[114] Hall (2015), definition 8.15, theorem 8.16 and the text below it; Fulton and Harris (1991), text above observation 14.16

[115] Hall (2015), definition 8.12; Koch (2022), definition 25; Fulton and Harris (1991), section 14.1, page 198; Fulton and Harris (1991), lecture 12, top of page 166 (example)

[116] Hall (2015), proposition 8.21

[117] Which Weyl chamber we call *fundamental* depends on which vector we used to define the *positive* roots.

[118] Hall (2015), definition 8.20

[119] Hall (2015), theorems 9.4 and 9.5

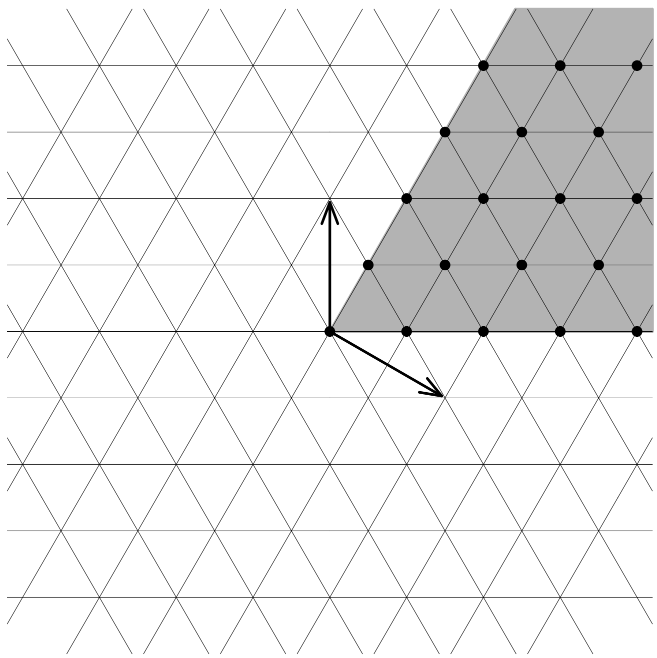

Figure 9 – Simple roots (arrows), fundamental Weyl chamber (shaded), and dominant weights (dots) for $SU(3)$.

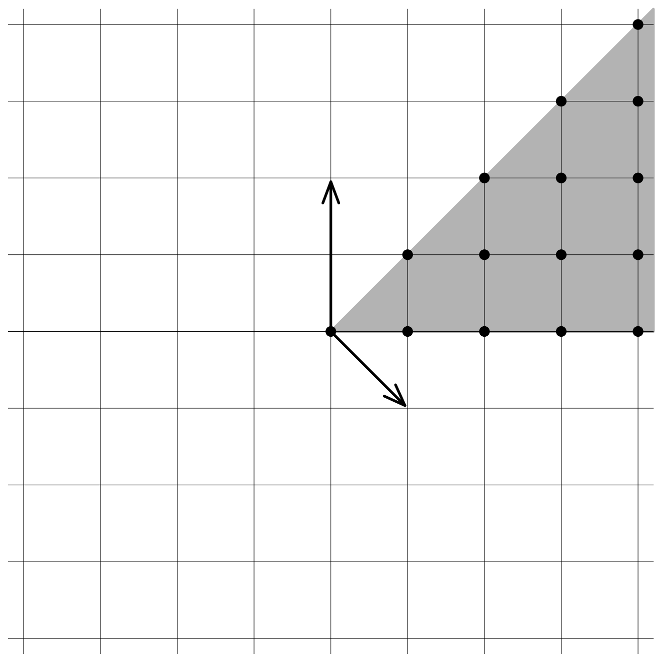

Figure 10 – Simple roots (arrows), fundamental Weyl chamber (shaded), and dominant weights (dots) for $SO(5)$.

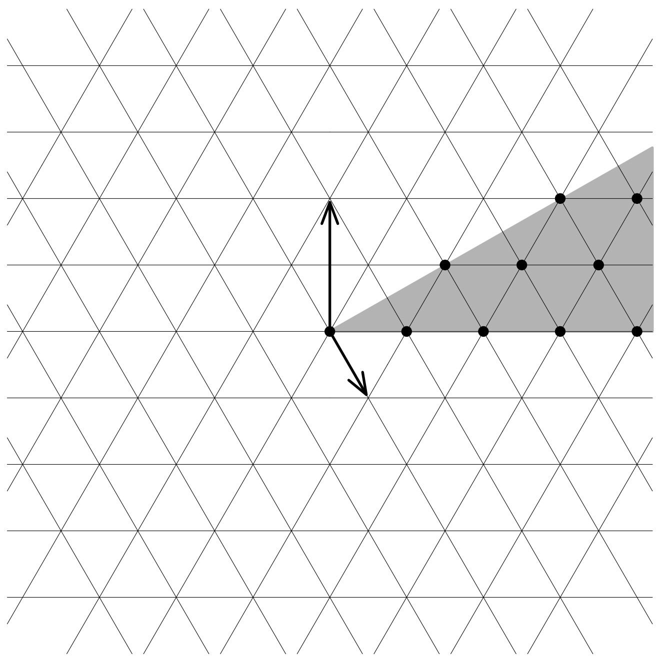

Figure 11 – Simple roots (arrows), fundamental Weyl chamber (shaded), and dominant weights (dots) for G_2.

26 Characterizing representations, part 1

If α and β are two elements of the weight lattice, then α is called **higher** than β if $\alpha - \beta$ may be written as a linear combination of simple positive roots with non-negative coefficients.[120]

Let G be a compact 1-connected simple Lie group. If ρ is a finite-dimensional irreducible representation of G, then every weight of ρ is an element of the weight lattice,[121] ρ has a unique highest weight,[122] the highest weight is dominant (as defined in section 25),[123] and any other irreducible representation with the same highest weight is equivalent to ρ.[123] Conversely, every dominant element of the weight lattice is the highest weight of some irreducible representation.[124] Altogether, this establishes the one-to-one correspondence that was previewed in section 25.[125]

[120]Hall (2015), definition 8.39
[121]Hall (2015), definition 8.34 and proposition 9.2; Fulton and Harris (1991), section 14.1, page 200
[122]Hall (2015), theorem 9.4 (existence); Fulton and Harris (1991), proposition 14.13 (existence and uniqueness)
[123]Hall (2015), theorem 9.4
[124]Hall (2015), theorem 9.5
[125]Knapp (2023), theorem 5.110; Sepanski (2007), theorem 7.34 (and theorem 7.3); Hall (2015), theorem 12.6

27 Characterizing representations, part 2

This section describes the set of weights that belong to a given finite-dimensional irreducible representation.[126]

First define the **Weyl group** W to be the group of linear transformations of the weight lattice given by reflections along the roots.[127,128] The Weyl group is finite[129] and maps roots to roots.[130] Let G be a compact simple Lie group, and let ρ an irreducible representation of G with highest weight α. (If G is 1-connected, then the highest weight α could be any dominant weight.[131] Section 29 will determine which weights can be used when G is not 1-connected.) The **orbit** of α under W, denoted $W\alpha$, is the set of elements in the weight lattice that can be obtained by applying transformations in W to α. The **convex hull** of $W\alpha$ is the set of vectors that can be written as linear combinations of vectors in $W\alpha$ using only non-negative coefficients whose sum is 1.[132] Intuitively, it's a convex polytope that has the elements of $W\alpha$ as vertexes. A weight β belongs to the representation ρ with highest weight α if and only if it satisfies these conditions:[133]

- β is in the convex hull of $W\alpha$,
- $\alpha - \beta$ is a vector in the root lattice.

The Weyl group preserves the set of weights of any finite-dimensional irreducible representation of a complex semisimple Lie algebra.[134]

[126] Section 9.2 in Hall (2015) describes a method for constructing the representations. This article only characterizes representations in terms of their weights without explicitly constructing the representations.

[127] Knapp (2023), text above equation (2.59); Fulton and Harris (1991), text below equation (14.8)

[128] Some sources about this subject describe reflections as being *through a hyperplane*, but reflections are more simply described (and understood) as being *along a direction*. The quantity (17) refers to the direction α, not to the orthogonal hyperplane.

[129] Hall (2015), corollary 7.27

[130] Hall (2015), theorem 7.26

[131] In this case, representations of G are in one-to-one correspondence with elements of Λ_W/W, the weight lattice modulo the Weyl group (Tong (2018), section 2.6.2).

[132] Hall (2015), definition 6.23

[133] Hall (2015), theorem 10.1; Fulton and Harris (1991), theorem 14.18

[134] Hall (2015), theorem 9.3; Fulton and Harris (1991), text below equation (14.9)

28 The dual of a lattice

Section 3 defined the dual of a vector space over $\mathbb{R}$ or $\mathbb{C}$. This section introduces two different concepts of the dual of a lattice. Both are widely used, and they are related to each other, but they're not the same:

- First concept (dual lattice in the dual vector space): If Λ is a lattice in a real vector space V, then the **dual lattice** Λ^* is defined to be the set of dual vectors $\alpha^* \in V^*$ for which $\alpha^*(\beta)$ is an integer for all $\beta \in \Lambda$.[135] The lattice Λ^* lives in the dual vector space V^* and is defined without using an inner product.

- Second concept (dual lattice in the original vector space): If Λ is a lattice in a real vector space V with an inner product $\langle\cdot,\cdot\rangle$, then the **dual lattice** $\Lambda^{\langle*\rangle}$ is defined to be the set of vectors $\alpha \in V$ for which $\langle\alpha,\beta\rangle$ is an integer for all $\beta \in \Lambda$. The lattice $\Lambda^{\langle*\rangle}$ lives in the original vector space V and is only defined with respect to a given inner product.

Even though $\Lambda^{\langle*\rangle}$ lives in the original vector space V, it is typically not the same as the original lattice Λ.[136] The lattices Λ^* and $\Lambda^{\langle*\rangle}$ live in different spaces but are related to each other like this: the vectors $\alpha^* \in \Lambda^*$ and $\alpha \in \Lambda^{\langle*\rangle}$ correspond to each other if $\alpha^*(\beta) = \langle\alpha,\beta\rangle$ for all $\beta \in V$.

Sections 29-30 will use the first concept. Basic properties include:[137,138]

- If Λ is a lattice in V^*, then Λ^* is a lattice in V.
- $(\Lambda^*)^* = \Lambda$.
- If Λ_1 and Λ_2 are lattices with $\Lambda_1 \subset \Lambda_2$, then $\Lambda_2/\Lambda_1 \simeq \Lambda_1^*/\Lambda_2^*$.

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[135] Elkies (2019)

[136] Example: suppose that V is two-dimensional with the standard inner product $\big\langle(a,b),(a',b')\big\rangle = aa' + bb'$. If Λ is the lattice generated by the two vectors $(1,\pm 1)$, then $(1,0)$ is in $\Lambda^{\langle*\rangle}$ but not in Λ.

[137] Sepanski (2007), exercise 6.24

[138] This assumes that each lattice has **full rank**, which means that its vectors would span V if arbitrary real coefficients were allowed.

29 Isogeny and the weight lattice, part 1

Let G be a compact 1-connected simple Lie group, and let Γ be a finite subgroup of its center $Z(G)$. This section explains which elements of the weight lattice can occur as weights of representations of G/Γ.

Let $T \subset G$ be a maximal torus. Suppose $H \in L(T)$, so $e^{2\pi H} \in T$.[139] A representation ρ of G is also a representation of G/Γ if and only if Γ is in the kernel of ρ.[140] The condition for $e^{2\pi H} \in \Gamma$ to be in the kernel of ρ is $e^{2\pi\rho(H)}v = v$ for all vectors v in the complex vector space on which ρ acts. This implies

$$e^{2\pi\alpha(H)}v = v \qquad \text{for all weights } \alpha \text{ of the representation } \rho. \tag{19}$$

The converse also holds because the set of eigenvectors of all the representation's weights span the vector space on which ρ acts. The condition (19) is equivalent to $\alpha(iH)$ being an integer for all weights α of the representation ρ, so it's equivalent to $iH \in \Lambda_\rho^*$ where Λ_ρ is the lattice generated by those weights. This shows that the set of Hs for which $e^{2\pi H}$ is in the kernel of ρ is the set of Hs with $iH \in \Lambda_\rho^*$.

Now consider *all* of the representations ρ whose kernel includes Γ, including representations whose kernel is precisely Γ. The preceding result implies that the set of Hs for which $e^{2\pi H} \in \Gamma$ is the set of Hs with $iH \in \Lambda^*$, where Λ is the lattice consisting of all weights of all finite-dimensional irreducible representations ρ of G/Γ. With the help of the identity $(\Lambda^*)^* = \Lambda$, we can infer Λ from Γ or conversely.

When Γ is trivial ($G/\Gamma = G$), the result in section 26 says that this correspondence gives $\Lambda = \Lambda_W$, with Λ_W defined as in section 23. When $\Gamma = Z(G)$, the set of representations whose kernel is Γ includes the adjoint representation, whose weights are roots (by definition). The kernel of a nontrivial representation cannot contain more than $Z(G)$, so in this case $\Lambda = \Lambda_R$. All other choices of Γ correspond to lattices Λ between these two extremes. This correspondence between subgroups Γ of $Z(G)$ and lattices Λ with $\Lambda_R \subset \Lambda \subset \Lambda_W$ is one-to-one.[141]

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[139] Section 10

[140] Section 8

[141] Fulton and Harris (1991), theorem 23.16

30 Isogeny and the weight lattice, part 2

The captions of figures 4-6 mentioned that the center $Z(G)$ of the 1-connected group G is isomorphic to the quotient group Λ_W/Λ_R.[142,143] The same relationship may also be written[144,145]

$$\Lambda_R^*/\Lambda_W^* \simeq Z(G) \tag{20}$$

where Λ^* is the dual of Λ (section 28). This section describes a one-to-one correspondence between subgroups of Λ_R^*/Λ_W^* and sublattices $\Lambda_R \subset \Lambda \subset \Lambda_W$, which in turn correspond to subgroups $\Gamma \subset Z(G)$ as explained in section 29.

If A is a group and $B \subset A$ is a normal subgroup, then elements of the quotient group A/B are subsets $a \circ B \subset A$ with $b \in A$, where $\circ$ is the group operation.[146] These subsets are called **cosets**. For lattices, the group operation is addition, so an element σ of Λ_R^*/Λ_W^* is a coset $\sigma = \Lambda_R + \Lambda_W^*$ with $\Lambda_R \in \Lambda_R^*$. In words: an element of Λ_R^*/Λ_W^* is a set of linear transformations from Λ_R to $\mathbb{Z}$ such that the difference between any two of them is a linear transformation from Λ_W to $\mathbb{Z}$. Let $\Lambda(\sigma)$ denote the set of weights $\beta \in \Lambda_W$ that satisfy the condition $\beta(\sigma) \in \mathbb{Z}$. For any given σ, $\Lambda(\sigma)$ is clearly a lattice that satisfies $\Lambda_R \subset \Lambda(\sigma) \subset \Lambda_W$.

Now consider a subgroup $\Sigma \subset \Lambda_R^*/\Lambda_W^*$, and let $\Lambda(\Sigma)$ be the intersection of all the lattices $\Lambda(\sigma)$ with $\sigma \in \Sigma$. Given a lattice Λ with $\Lambda_R \subset \Lambda \subset \Lambda_W$, we can use the definition to construct a subgroup Σ for which $\Lambda(\Sigma) = \Lambda$. If $\sigma' \notin \Sigma$, then $\Lambda(\Sigma)$ has an element that is not in $\Lambda(\sigma')$, so the condition $\Sigma' \neq \Sigma$ implies $\Lambda(\Sigma') \neq \Lambda(\Sigma)$. Altogether, this establishes the one-to-one correspondence promised at the beginning of this section.

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[142] Fulton and Harris (1991), text above lemma 23.15

[143] A lattice is an abelian group with respect to addition of vectors, so the quotient of a lattice by a sublattice is also an abelian group.

[144] Koch (2022), corollary 11; Hall (2015), corollary 13.43; Sepanski (2007), theorem 6.30

[145] The quotient Λ_R^*/Λ_W^* is naturally isomorphic to $Z(G)$ (with the technical meaning of *naturally*), but Λ_W/Λ_R is merely isomorphic to $Z(G)$ (`https://math.stackexchange.com/questions/1749386`).

[146] Article 29682

31 Some uses for weights

The set of weights of a representation constitute a kind of encoding of that representation. Some things can be calculated directly from this encoding without constructing the representation explicitly. This includes the number of dimensions of the representation[147] and the numeric value of the quadratic Casimir invariant.[148] Chapter 13 in Di Francesco *et al* (1997) reviews more examples.

[147] Fulton and Harris (1991), corollary 24.6; Koch (2022), definition 44 and theorem 116 (alternatively theorem 117), but beware that the remark after definition 44 is incorrect (the root/weight diagram of $SO(5)$ is a counterexample, which can be checked using figure 10.1 in Hall (2015) where arrows denote roots and grid-points denote weights).

[148] Hall (2015), proposition 10.6; Di Francesco *et al* (1997), section 13.2.3

32 Dynkin labels for representations

Let G be a compact 1-connected simple Lie group with rank r, and let $\alpha_1, ..., \alpha_r$ be a set of simple roots. They are a basis for the root lattice of G. The vectors $\eta_1, ..., \eta_r$ defined by the condition

$$2\frac{\langle \eta_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

are called **fundamental weights**.[149] They are a basis for the weight lattice of G. Each fundamental weight is a dominant integral element of the weight lattice, so linear combinations of fundamental weights with non-negative integer coefficients correspond to highest weight vectors of finite-dimensional irreducible representations.[150] Figure 12 describes a complete set of fundamental weights for the classical simple Lie algebras.

For each element β of the weight lattice, the quantities $\langle \beta, \eta_j \rangle$ are integers,[151] and equation (21) implies that these integers are the coefficients when β is written as a linear combination of fundamental weights:

$$\beta = \sum_{j=1}^{r} \langle \beta, \eta_j \rangle \, \eta_j. \tag{22}$$

If β is the highest weight of a given representation, then this ordered list of r integers is commonly used as a label (name) for the representation. These integers are called **Dynkin labels**.[152] Each list of r non-negative integers specifies a representation of G by taking (22) to be its highest weight, and representations with different lists of Dynkin labels are inequivalent.[153]

[149] Knapp (2023), chapter 5, text above problem 28; Hall (2015), definition 8.36; Foster (2019), section 8.1.1

[150] Section 26

[151] Sections 20-21

[152] Foster (2019), text below equation (8.1.3)

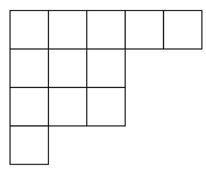
[153] This correspondence assumes a fixed ordering of the fundamental weights. Figure 1 shows a standard ordering.

	a complete set of fundamental weights	
A_n, $n \geq 1$	$e_1 + \cdots + e_j - v$	$(1 \leq j \leq n)$
	with $v \equiv \frac{1}{n+1}(e_1 + \cdots + e_{n+1})$	
B_n, $n \geq 2$	$e_1 + \cdots + e_j$	$(1 \leq j < n)$
	$\frac{1}{2}(e_1 + \cdots + e_n)$	
C_n, $n \geq 3$	$e_1 + \cdots + e_j$	$(1 \leq j \leq n)$
D_n, $n \geq 4$	$e_1 + \cdots + e_j$	$(1 \leq j < n-1)$
	$\frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n)$	
	$\frac{1}{2}(e_1 + \cdots + e_{n-1} + e_n)$	

Figure 12 – A complete set of fundamental weights for each of the classical simple Lie algebras, using the same conventions as in figure 2. Source: Seelinger (2017), section 11, after correcting an error in the A_n case (the subtraction of v was missing)

33 Young diagrams for representations of $SU(n)$

A **Young diagram**[154] is a finite list of positive integers in non-increasing order, like $(5, 3, 3, 1)$, represented graphically by an array of boxes: the kth integer is the number of boxes in the kth row, and the rows are left-justified. The graphic representation of the Young diagram $(5, 3, 3, 1)$ is



Each Young diagram with fewer than n rows corresponds to an (equivalence class of) irreducible representation of $SU(n)$, and this exhausts all of the finite-dimensional irreducible representations of $SU(n)$.[155] Examples: the defining representation of $SU(n)$ corresponds to the Young diagram with one box, and its dual representation[156] corresponds to the Young diagram with a single column of $n-1$ boxes.

Now consider the group $SU(n)$, and order the simple roots as indicated in figure 1. With that ordering convention,[157] if $(m_1, ..., m_r)$ are the Dynkin labels for the highest weight of a given irreducible representation ρ of $SU(n)$, then the corresponding Young diagram has m_j columns of length j.[158] The Dynkin labels corresponding to the Young diagram shown above are $(2, 0, 2, 1)$.

An easy (but not obvious) formula gives the number of dimensions of an irreducible representation from its Young diagram, and another easy (but not obvious) recipe expresses the tensor product of two irreducible representations as a direct sum of irreducible representations.[159]

[154] A Young diagram is also called a **Young tableau** (Foster (2019), section 8.5.1), but sometimes the name *Young tableau* is reserved for a Young diagram whose boxes are filled with labels.

[155] Koch (2022), theorem 145

[156] Article 90757 defines *dual representation*.

[157] An ordering convention for the simple roots implies an ordering convention for the fundamental weights defined by equation (21).

[158] Foster (2019), text above equation (8.5.1)

[159] Eichmann (2020), appendix A.3

34 Coroots

This section reviews another (equivalent) common way to define the weight lattice. Section 19 defined an element H_α of $\mathfrak{h}_{\text{herm}}$ for each root $\alpha \in R \subset \mathfrak{h}^*_{\text{herm}}$. The **coroot** corresponding to α is[160]

$$\alpha^\vee \equiv \frac{2H_\alpha}{B(H_\alpha, H_\alpha)}. \tag{23}$$

This is an element of $\mathfrak{h}_{\text{herm}}$, and it clearly satisfies $\alpha(\alpha^\vee) = 2$.

Here's an equivalent[161] definition of *coroot* that doesn't use the Killing form, but it still uses the structure of the Lie algebra. Recall that roots satisfy equation (11). If α is a root, then $-\alpha$ is also a root,[162] so for any given α, the existence of an $X_\alpha \in \mathfrak{h}$ that satisfies

$$[H, X_\alpha] = \alpha(H)X_\alpha$$

implies the existence of an $X_{-\alpha} \in \mathfrak{h}$ that satisfies

$$[H, X_{-\alpha}] = -\alpha(H)X_{-\alpha}.$$

Now the coroot $\alpha^\vee \in \mathfrak{h}$ is uniquely determined by the conditions[163]

$$\alpha^\vee \propto [X_\alpha, X_{-\alpha}] \qquad\qquad \alpha(\alpha^\vee) = 2. \tag{24}$$

The **coroot lattice** is the lattice generated by the coroots.[164] The weight lattice is the dual of the coroot lattice,[165] and the definition of the weight lattice is often expressed this way.

[160] The notation $\alpha^\vee$ is mentioned in Sepanski (2007), text below definition 6.18, and in Bröcker and tom Dieck (1985), chapter 5, text below equation (2.14) and proposition 5.13. The name *coroot* is used in Koch (2022) (one of the remarks after definition 27) and in Hall (2015) (definition 7.28). Both Hall (2015) and Fulton and Harris (1991) use the notation H_α for $\alpha^\vee$.

[161] The equivalence is not obvious, but it may be inferred from the text below corollary 2.25 in Knapp (2023) using the definition of H_α in Knapp (2023), proposition 2.17(d), or from Sepanski (2007), theorem 6.20.

[162] Knapp (2023), proposition 2.17(c)

[163] Fulton and Harris (1991), text below facts 14.6

[164] Exercise 13.9 in Hall (2015) describes the coroot lattices for $SU(n)$, $SO(n)$, and $Sp(n)$.

[165] This should be clear from the definition of *weight lattice* in section 23.

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