Introduction to Hilbert Space

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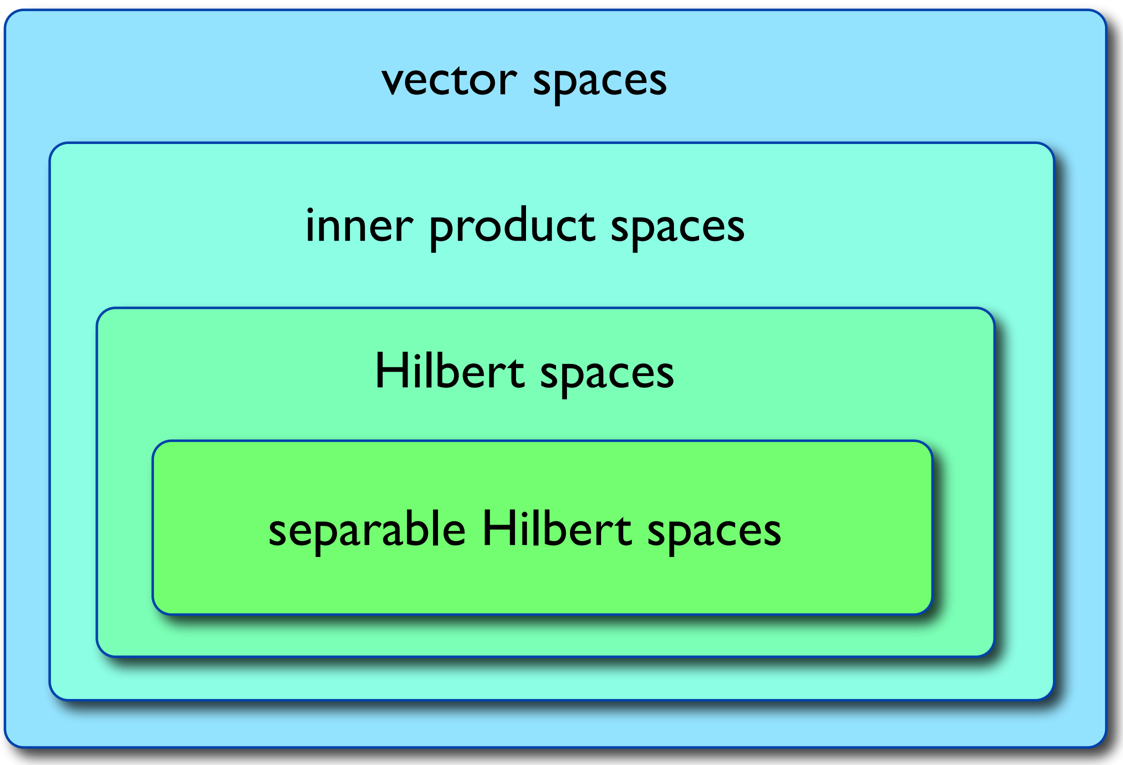
Abstract The general principles of quantum theory are expressed in terms of observables (things that can be measured), which are represented by linear operators on a Hilbert space. This article gives a brief introduction to Hilbert space.

Contents

1 Introduction

The general principles of quantum theory can be expressed in terms of a mathematical structure called a **Hilbert space**, specifically a Hilbert space over the field $\mathbb{C}$ of complex numbers. A Hilbert space a special kind of vector space. This article starts with the general idea of a vector space over $\mathbb{C}$ and then walks through a series of specializations to arrive at the idea of a Hilbert space.[1] Quantum theory uses only **separable** Hilbert spaces, which satisfy an extra condition that is closely related to the origin of the name "quantum." The sequence of specializations is shown in this Venn diagram:



[1]For a more thorough introduction, classic texts include Debnath and Mikusiński (2005) and the two-volume set by Kadison and Ringrose (1997). Online resources include chapter 8 in the beautiful book by Axler (2021).

2 Two notations

A Hilbert space is a special kind of vector space. The vectors in a Hilbert space are abstract vectors.[2] In addition to being a vector space, a Hilbert space also has an inner product. The inner product takes any two vectors as input and returns a single complex number as output.

Two different notations for the inner product are commonly used. Each has its own advantages. This article uses the notation that is most common in the physics literature:

- A vector is written as $|a\rangle$, or as $|b\rangle$, and so on. This notation facilitates using more elaborate names for the vectors. We can use a whole word as the name of a vector, as in $|\text{dog}\rangle$. We can even use a crazy combination of symbols as the name of a single vector, as in $|☺ \div ★\rangle$.

- The inner product of two vectors $|a\rangle$ and $|b\rangle$ is written as $\langle a|b\rangle$.

A different notation is common in the mathematics literature. To switch to the mathematician's notation:

- Instead of writing a vector as $|a\rangle$, write it as a.

- Instead of writing the inner product of two vectors as $\langle a|b\rangle$, write it as $\langle a, b\rangle$ or $\langle b, a\rangle$.[3] Sometimes the inner product is written using round brackets instead of angled brackets.

This article uses the physicist's notation, even though the mathematician's notation makes some things easier to express.

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[2] These vectors do not represent directions in three-dimensional physical space. The next two sections review the abstract definition of a vector space.

[3] When the $\langle a|b\rangle$ notation is used, the inner product is linear in the second vector and is conjugate-linear in the first vector (section 5). The opposite convention is common when the comma-notation is used, as in Axler (2021), definition 8.1.

3 Addition

A Hilbert space is, among other things, a vector space. A **vector space** consists of these ingredients:[4]

- A set V of things called **vectors**.
- An operation called **addition**: any two vectors $|a\rangle$ and $|b\rangle$ can be added to each other, and the result is another vector $|a\rangle + |b\rangle$. Addition is subject to these conditions:
    - It is **associative**: $\big(|a\rangle + |b\rangle\big) + |c\rangle = |a\rangle + \big(|b\rangle + |c\rangle\big)$.
    - It is **commutative**: $|a\rangle + |b\rangle = |b\rangle + |a\rangle$.
    - There is a **zero vector** 0 satisfying $|a\rangle + 0 = |a\rangle$ for all vectors.[5]
    - Each vector $|a\rangle$ has a **negative** $-|a\rangle$ satisfying[6]

    $$|a\rangle + \big(-|a\rangle\big) = 0.$$

    The sum of $|a\rangle$ and $-|b\rangle$ is denoted $|a\rangle - |b\rangle$. This is called **subtraction**.
- An operation called **scalar multiplication**: any vector can be multiplied by a complex number, and the result is another vector in V. Scalar multiplication is subject to the conditions shown in the next section.

[4] Axler (2021), definition 6.27

[5] The zero vector is usually *not* denoted $|0\rangle$, because that symbol is often used for a nonzero vector that represents the lowest-energy state (also called the ground state in condensed matter physics or the vacuum state in particle physics).

[6] The negative of a vector $|a\rangle$ is usually *not* denoted $|-a\rangle$.

4 Scalar multiplication

Let $\mathbb{C}$ denote the field of complex numbers. The type of Hilbert space used in quantum theory is a vector space over $\mathbb{C}$. We could also define a vector space over a different field, such as the field $\mathbb{R}$ of real numbers, but this article only considers vector spaces over $\mathbb{C}$.

In this section, the letters z and w denote complex numbers. Multiplication of a vector by a complex number is called **scalar multiplication**. For any vector $|a\rangle$ and any complex number z, scalar multiplication produces another vector $z|a\rangle$, subject to these conditions:

$$
\begin{gathered}
(w+z)|a\rangle = w|a\rangle + z|a\rangle \\
z\big(|a\rangle + |b\rangle\big) = z|a\rangle + z|b\rangle \\
(zw)|a\rangle = z\big(w|a\rangle\big) \\
1|a\rangle = |a\rangle.
\end{gathered}
$$

Together with the conditions shown in the previous section, these imply

$$
(-1)|a\rangle = -|a\rangle \qquad\qquad 0|a\rangle = 0.
$$

In the last equation, the symbol "0" is used for two different things: one is a number and the other is a vector.

The vectors in a list $|1\rangle$, $|2\rangle$, ..., $|N\rangle$ are called **linearly independent** if the condition

$$
z_1|1\rangle + z_2|2\rangle + \cdots + z_N|N\rangle = 0
$$

cannot be satisfied for any finite N unless all of the coefficients z_n are zero. A vector space is called N**-dimensional** if it contains a set of N linearly independent vectors but does not contain any set of $N+1$ linearly independent vectors. A vector space is called **infinite-dimensional** if it is not N-dimensional for any finite N.[7]

[7] Axler (2021), definition 6.54 and Debnath and Mikusiński (2005), section 1.2, page 8

5 Inner product

An **inner product space** is a vector space equipped with an **inner product**. For any two vectors $|a\rangle$ and $|b\rangle$, the inner product $\langle a|b\rangle$ is a complex number. More notation:

- The complex conjugate of $\langle a|b\rangle$ is denoted $\langle a|b\rangle^*$.
- If z is another complex number, the product of z with the complex number $\langle a|b\rangle$ is denoted $z\langle a|b\rangle$.

The inner product is subject to these conditions:[8,9]

- If $|a\rangle \neq 0$, then $\langle a|a\rangle > 0$ (the **positive-definite** condition).
- $\langle a|b\rangle^* = \langle b|a\rangle$.
- If $|a+b\rangle = |a\rangle + |b\rangle$,[10] then $\langle ☺|a+b\rangle = \langle ☺|a\rangle + \langle ☺|b\rangle$ for all vectors $|☺\rangle$.
- If $|zb\rangle = z|b\rangle$,[11] then $\langle ☺|zb\rangle = z\langle ☺|b\rangle$ for all vectors $|☺\rangle$.

The second and fourth conditions imply:

- If $|zb\rangle = z|b\rangle$, then $\langle zb|☺\rangle = z^*\langle b|☺\rangle$ for all vectors $|☺\rangle$,

where z^* denotes the complex conjugate of z. Altogether, the inner product is **linear** in the second vector (the one on the right-hand side), and it is **conjugate linear** in the first vector (the one on the left-hand side).

Two vectors $|a\rangle$ and $|b\rangle$ are called **orthogonal** to each other if $\langle a|b\rangle = 0$.

[8] Axler (2021), definition 8.1; Debnath and Mikusiński (2005), definition 3.2.1

[9] Sometimes this is called a *positive definite inner product* so that the shorter name *inner product* can be used for one that doesn't necessarily satisfy the first condition.

[10] Warning: the sum of $|a\rangle$ and $|b\rangle$ is *not* usually denoted $|a+b\rangle$, at least not in the physics literature. I'm using that here only as a temporary abbreviation. That's why the sentence starts with "If $|a+b\rangle = |a\rangle + |b\rangle$, then ..."

[11] Warning: the sum of z and $|b\rangle$ is not usually denoted $|zb\rangle$, at least not in the physics literature. Again, I'm using that here only as a temporary abbreviation.

6 Some basic consequences

The condition in the previous section imply

- The zero vector is orthogonal to itself.
- If $\langle☺|a\rangle = \langle☺|b\rangle$ for all vectors $|☺\rangle$, then $|a\rangle = |b\rangle$.
  - To prove this, consider the case $|☺\rangle = |a\rangle - |b\rangle$ and use the positive-definite condition.
- The **Cauchy-Schwarz inequality**: $\big|\langle b|a\rangle\big|^2 \le \langle a|a\rangle\,\langle b|b\rangle$.
  - To prove this, consider the vector $|☺\rangle = |a\rangle - z|b\rangle$ with $z = \langle b|a\rangle/\langle b|b\rangle$, and use the positive-definite condition $\langle☺|☺\rangle \ge 0$.

7 Hilbert space

A **Hilbert space** is a vector space that has a positive-definite inner product and that is also **complete**. This section explains what complete means. A finite-dimensional vector space with a positive-definite inner product is automatically complete, so it is automatically a Hilbert space.

Define the **norm** of a vector $|a\rangle$ to be the real number[12,13]

$$\big\|\,|a\rangle\,\big\| \equiv \sqrt{\langle a|a\rangle}.$$

Complete means[14] that every sequence of vectors $|a_1\rangle$, $|a_2\rangle$, ... satisfying

$$\lim_{n,m\to\infty} \big\|\,|a_n\rangle - |a_m\rangle\,\big\| = 0$$

has a **limit**, which is a vector $|a\rangle$ satisfying

$$\lim_{n\to\infty} \big\|\,|a_n\rangle - |a\rangle\,\big\| = 0.$$

Loosely speaking, saying that a Hilbert space is complete means that it contains all of its limits.

[12] Axler (2021), definition 8.4 and Debnath and Mikusiński (2005), definition 3.2.11

[13] In some contexts, the word "norm" is used for the result of multiplying two copies of something *without* taking the square root (Hogben (2007), section 69.3, example 1).

[14] Riesz and Sz.-Nagy (1990), section 83

8 Different concepts of basis

A vector $|a\rangle$ is called a **unit vector** if $\langle a|a\rangle = 1$. A set $\mathcal{S} \subset \mathcal{H}$ of vectors in a Hilbert space $\mathcal{H}$ is called **orthonormal** if they are all orthogonal to each other and each one is a unit vector. An orthonormal set of vectors is called **maximal** if it is not contained in any larger orthonormal set.

In a finite-dimensional Hilbert space, any basis[15] of linearly independent vectors has the same (finite) number of members, whether or not they are orthogonal. In the infinite-dimensional case, a *linear basis* (also called a **Hamel basis**)[16] may have different cardinality than a maximal orthonormal set, which is called an **orthogonal basis**[17] or a **Conway basis**.[18] An orthogonal basis may be countable or uncountable.

A **Schauder basis**[19] is a countable sequence[20] of vectors $|1\rangle$, $|2\rangle$, ..., such that every vector $|a\rangle$ in the Hilbert space can be uniquely written in the form

$$|a\rangle = \sum_{n=1}^{\infty} z_n |n\rangle.$$

Every Hilbert space has an orthogonal basis,[21] but it's not necessarily countable. Many Hilbert spaces don't have a Schauder basis, which is countable by definition.

Even if a Hilbert space has a countable orthogonal basis, it may still have an uncountable linear basis.[22] Unless specified otherwise, the **dimension** of a Hilbert space means the cardinality of an *orthogonal* basis[23] (which is the same as the cardinarlity of a Schauder basis, if one exists).

[15] The plural of *basis* is *bases*.

[16] Halmos (1982), page 6; and Heil (2006), definition 4.1

[17] Halmos (1982), page 6

[18] Heil (2006), definition 4.4

[19] Heil (2006), definition 4.9

[20] Usually the word *sequence* implies *countable*, but I'm including the word *countable* here for emphasis.

[21] Heil (2006), exercise 4.5

[22] Halmos (1982), pages 6 and 170-171

[23] Halmos (1982), page 170

9 Separable Hilbert space

Let $\mathcal{H}$ be a Hilbert space. A countable sequence of orthonormal vectors $|1\rangle$, $|2\rangle$, ... in $\mathcal{H}$ is called **complete** if every vector $|a\rangle \in \mathcal{H}$ can be written[24]

$$|a\rangle = \sum_{n=1}^{\infty} |n\rangle \langle n|a\rangle.$$

The word *sequence* is important here, because this equation really means

$$\lim_{N\to\infty} \left\| |a\rangle - \sum_{n=1}^{N} |n\rangle \langle n|a\rangle \right\| = 0.$$

A Hilbert space is called **separable** if it has a complete countable sequence of mutually orthonormal vectors.[25,26] In a separable Hilbert space, such a sequence is a Schauder basis.[27] A finite-dimensional Hilbert space is automatically separable.[28]

Quantum theory uses only separable Hilbert spaces. The name *quantum* in quantum theory is related to the fact that in a separable Hilbert space, any set of mutually orthonormal vectors is countable.

[24] Debnath and Mikusiński (2005), definition 3.4.12

[25] Debnath and Mikusiński (2005), definition 3.4.20 and Heil (2006), exercise 4.19

[26] In the context of Hilbert space, this definition of *separable* is consistent with the one used in topology, where it refers to the existence of a countable dense subset (Axler (2021), definition 8.64)

[27] Heil (2006), exercise 4.11

[28] According to remark 2.2.14 in Kadison and Ringrose (1997a), separable does not imply infinite-dimensional. In Debnath and Mikusiński (2005), definition 3.4.20 says, “Finite dimensional Hilbert spaces are considered separable.”

10 Closed subspaces of a Hilbert space

Let $\mathcal{H}$ be a Hilbert space. A **closed subspace** of $\mathcal{H}$ is a subset of $\mathcal{H}$ that qualifies as a Hilbert space all by itself.[29] Basic facts:

- For any subset $\mathcal{S} \subset \mathcal{H}$ (not necessarily a subspace), let $\mathcal{S}^\perp$ denote the set of all vectors in $\mathcal{H}$ that are orthogonal to all vectors in $\mathcal{S}$. Then $\mathcal{S}^\perp$ is a closed subspace of $\mathcal{H}$.[30]

- If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, then every vector $|a\rangle \in \mathcal{H}$ can be written $|a\rangle = |a_\parallel\rangle + |a_\perp\rangle$, with $|a_\parallel\rangle \in \mathcal{S}$ and $|a_\perp\rangle \in \mathcal{S}^\perp$, in exactly one way.[31]

- Every closed subspace of a separable Hilbert space is a separable Hilbert space.[32]

- Every finite-dimensional subspace is closed.

A one-dimensional subspace is also called a **ray**.

[29] Axler (2021), example 8.22
[30] Axler (2021), theorem 8.40a and Debnath and Mikusiński (2005), theorem 3.6.2
[31] Axler (2021), theorem 8.43
[32] Axler (2021), exercise 8C-14

11 Relationships between two Hilbert spaces

Isomorphism is the appropriate notion of equivalence between Hilbert spaces. If two Hilbert spaces are isomorphic to each other, then they are the same as far as their Hilbert-space structure is concerned. Here's the precise definition:

- A map σ from one Hilbert space $\mathcal{H}_1$ to another one $\mathcal{H}_2$ is called **linear** if it respects linear combinations:

$$\sigma\big(\,|a\rangle + |b\rangle\big) = \sigma|a\rangle + \sigma|b\rangle \qquad\qquad \sigma\big(z|a\rangle\big) = z\big(\sigma|a\rangle\big) \tag{1}$$

  for all vectors $|a\rangle$ and $|b\rangle$ in $\mathcal{H}_1$ and all complex numbers z.

- A linear map $\sigma : \mathcal{H}_1 \to \mathcal{H}_2$ is called **unitary** if it also respects inner products:

$$\text{If } |a'\rangle = \sigma|a\rangle \text{ and } |b'\rangle = \sigma|b\rangle\text{, then } \langle a'|b'\rangle = \langle a|b\rangle. \tag{2}$$

- A unitary map $\sigma : \mathcal{H}_1 \to \mathcal{H}_2$ is called an **isomorphism** if it is bijective.[33] In this case, $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be **isomorphic** to each other.

- An isomorphism from a Hilbert space to itself ($\sigma : \mathcal{H} \to \mathcal{H}$) is called an **automorphism**.

All N-dimensional Hilbert spaces over $\mathbb{C}$ are isomorphic to each other, and all infinite-dimensional separable Hilbert spaces over $\mathbb{C}$ are isomorphic to each other.[34] In this sense, there is only one infinite-dimensional separable Hilbert space over $\mathbb{C}$, even though it may be constructed in many different-looking ways. In particular, the Hilbert spaces in examples 16 and 17 are isomorphic to each other, even though they look different.[35]

[33] A unitary map σ is called **bijective** if another unitary map σ^{-1} exists for which the compositions $\sigma\sigma^{-1}$ and $\sigma^{-1}\sigma$ are both the identity map.

[34] Debnath and Mikusiński (2005), theorem 3.4.27

[35] In quantum physics, completely different models often use isomorphic Hilbert spaces. Different models are distinguished from each other by their different patterns of observables, not just by their (abstract) Hilbert spaces.

12 Antilinear relationships

The maps defined above are linear. An **antilinear** map[36] is one for which the conditions (1) are replaced by

$$\sigma\big(\,|a\rangle+|b\rangle\big)=\sigma|a\rangle+\sigma|b\rangle \qquad\qquad \sigma\big(z|a\rangle\big)=z^*\big(\sigma|a\rangle\big). \tag{3}$$

An antilinear map is called **antiunitary** if it also satisfies

$$\text{If } |a'\rangle=\sigma|a\rangle \text{ and } |b'\rangle=\sigma|b\rangle,\text{ then } \langle a'|b'\rangle=\langle b|a\rangle.$$

This is like (2), but the right-hand side is complex-conjugated, as it must be for consistency with (3).

For an example of an antiunitary map, let $|a_1\rangle$, $|a_2\rangle$, ... be an orthonormal basis for the Hilbert space. Define the effect of the map σ on these basis vectors to be

$$\sigma\big(\,|a_k\rangle\big)=|a_k\rangle,$$

and use (3) to define effect of σ on all other vectors. Then σ is antiunitary.

[36] Uhlmann (2015) reviews antilinear maps in more detail.

13 Wigner's theorem

If an invertible map σ from one Hilbert space to another preserves the values of all quantities of the form

$$\frac{\langle a|b\rangle\,\langle b|a\rangle}{\langle a|a\rangle\,\langle b|b\rangle}, \tag{4}$$

then either σ is a unitary transformation (which means it is linear and preserves inner products), or else σ is antilinear and antiunitary.[37] This is called **Wigner's theorem**.

The quantity (4) is not affected if $|a\rangle$ and $|b\rangle$ are multiplied by arbitrary nonzero complex numbers, so it depends only on the corresponding rays (section 10). Like any closed subspace, a ray can be represented by a projection operator (article 74088). For projection operators, this relative of Wigner's theorem holds:[38] if $\mathcal{H}$ is a Hilbert space with at least 3 dimensions, then any automorphism of its projection lattice[39] has the form $P \to U^*PU$, where U is either a unitary (and therefore linear) or antiunitary (and antilinear) operator on $\mathcal{H}$.

[37] Weinberg (1995), section 2.2 and appendix A in chapter 2

[38] Hamhalter (2003), theorem 8.1.3

[39] This is defined in Hamhalter (2003).

14 A finite-dimensional example

Here's an example of an N-dimensional Hilbert space. Regard each sequence of N complex numbers $(a_1, a_2, ..., a_N)$ as a vector:

$$|a\rangle = (a_1, a_2, ..., a_N).$$

Define addition and scalar multiplication by

$$\begin{aligned} |a\rangle + |b\rangle &= (a_1 + b_1, a_2 + b_2, ..., a_N + b_N) \\ z\,|a\rangle &= (za_1, za_2, ..., za_N), \end{aligned}$$

and define the inner product by

$$\langle b|a\rangle = b_1^* a_1 + b_2^* a_2 + \cdots + b_N^* a_N.$$

This is a Hilbert space, because any finite-dimensional inner product space is automatically a Hilbert space.

Every N-dimensional Hilbert space is isomorphic to this one.

15 An infinite-dimensional example

For an example of an infinite-dimensional Hilbert space, take each vector in to be an endless list of complex numbers,

$$|a\rangle = (z_1,\, z_2,\, z_3,\, ...), \tag{5}$$

subject to the condition that[40]

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots < \infty.$$

Multiplication by a complex number w is defined componentwise, like this:

$$w\,|a\rangle = (wz_1,\, wz_2,\, wz_3,\, ...).$$

Given two vectors

$$|a\rangle = (z_1,\, z_2,\, z_3,\, ...) \qquad\qquad |b\rangle = (w_1,\, w_2,\, w_3,\, ...),$$

their sum and inner product are also defined componentwise, like this:

$$\begin{aligned} |a\rangle + |b\rangle &= (z_1 + w_1,\, z_2 + w_2,\, z_3 + w_3,\, ...). \\ \langle a|b\rangle &= z_1^* w_1 + z_2^* w_2 + z_3^* w_3 + \cdots \end{aligned}$$

This is a *separable* Hilbert space, because any vector can be arbitrarily well-approximated using linear combinations of the vectors in this countable list of orthogonal vectors:[41]

$$\begin{aligned} |1\rangle &= (1, 0, 0, 0, ...) \\ |2\rangle &= (0, 1, 0, 0, ...) \\ |3\rangle &= (0, 0, 1, 0, ...) \\ &\vdots \end{aligned}$$

[40] The inequality "$r < \infty$" means "r is finite."

[41] By definition, a **linear combination** involves only a finite number of terms, but the number can be arbitrarily large (https://math.stackexchange.com/questions/672791).

16 An infinite-dimensional example using functions

Consider the set of all complex-valued functions $a(x)$ of a single real variable x such that the norm $\|a\|$ defined by

$$\|a\|^2 \equiv \int_{-\infty}^{\infty} dx\ |a(x)|^2$$

is finite. To each such function $a(x)$, associate a vector $|a\rangle$, with the understanding that two functions correspond to the *same* vector if the norm of their difference is zero.[42] Define the inner product by[43]

$$\langle b|a\rangle \equiv \int_{-\infty}^{\infty} dx\ b^*(x)a(x) \tag{6}$$

where $b^*(x)$ denotes the complex conjugate of $b(x)$. This vector space, when equipped with this inner product, is an example of an infinite-dimensional Hilbert space.

The Hilbert space in this example is separable. To prove this, start by observing that the set of functions

$$x^n e^{-x^2} \qquad n \geq 0$$

is countable, and that the set $\mathcal{D}$ of all linear combinations of such functions with rational coefficients is still countable.[44] This set $\mathcal{D}$ is a dense subset of the Hilbert space. To illustrate the fact that this set of functions is dense, consider the function $e^{-(x-a)^2}$, which clearly belongs to the Hilbert space. Although it cannot be written as a linear combination of any finite number of the functions $x^n e^{-x^2}$, it can be written as a limit, namely

$$e^{-(x-a)^2} \propto e^{2ax}e^{-x^2} = \lim_{N\to\infty} f_N(x) \qquad \text{with} \qquad f_N(x) = \sum_{n=0}^{N} \frac{(2ax)^n}{n!} e^{-x^2}.$$

[42] Debnath and Mikusiński (2005), sections 2.6, 2.7, and 2.13. Example: if $a(x) = b(x)$ except at a finite list of values of x, then the norm of their difference is zero, so they both represent the same vector in the Hilbert space.

[43] The integral here is a Lebesgue integral. Debnath and Mikusiński (2005) comments on the significance of this.

[44] Recall that any given linear combination has only a finite number of terms, by definition.

17 More infinite-dimensional examples

The preceding example can be generalized to complex-valued functions $a(x,y)$ of two real variables x, y with inner product

$$\langle b|a\rangle \equiv \int dx\, dy\ b^*(x,y)a(x,y).$$

The generalization to functions of more than two variables should be obvious. This Hilbert space is separable, because the set of functions

$$x^n y^m e^{-x^2-y^2} \qquad n,m \geq 0$$

is countable and spans a dense subset of the Hilbert space.

The generalization to functions of $N > 2$ variables should be clear. The construction *looks* different for different values of N, and all of these *look* different than the example in section 15, but remember: all infinite-dimensional separable Hilbert spaces over $\mathbb{C}$ are isomorphic to each other (section 11), even though they may look different.

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