Quantum Gauge Fields on a Spacetime Lattice

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Abstract This article constructs some of the simplest quantum models that are believed to have nontrivial continuum limits with Lorentz symmetry, even though they are initially defined by treating spacetime as a lattice. The only field used in the construction is a gauge field. When the gauged group is nonabelian, these are called **Yang-Mills theories**. The models are constructed using the path integral formulation, paying special attention to how Wick rotation is used to ensure that time evolution is unitary without compromising the intuitive reason to expect that the continuum limit has Lorentz symmetry. The relationship between the path integral and hamiltonian formulations is reviewed, including an explanation of how they handle observables that are extended in time.

Contents

1 Introduction

This article introduces a family of quantum models whose observables are expressed exclusively in terms of *gauge fields*. Each model is based on a compact Lie group G, which I'll call the **gauged group**.[1] Different choices of G give models with different properties. The case $G = \mathbb{Z}_2$ is sometimes called the **Ising gauge model**.[2] The case $G = U(1)$ is quantum electrodynamics without matter.[3] The case $G = SU(N_c)$ is quantum chromodynamics without quarks, also called **Yang-Mills theory**.[4]

The only known ways to construct most of these models involve treating space or spacetime as discrete. This is called **lattice gauge theory**. Even though its construction treats spacetime as discrete, the $SU(N_c)$ Yang-Mills theory is believed to have a nontrivial Lorentz-symmetric continuum limit[5] with a rich spectrum of interacting particles called **glueballs**, at least when the number of spacetime dimensions is 3 or 4.

Calculations in models like these tend to be difficult because the equations of motion are not linear. This article focuses on the easy part – constructing the models without any mathematical ambiguity, so that calculations and intuition both have a solid place to start.

[1] In the physics literature, the group G is often called the **gauge group**, but in the math literature, that name is used for the group of gauge transformations, which is much larger than G (article 76708). G is often called the **structure group** in the math literature, but that can also be ambiguous (article 70621). The name **gauged group** is not standard, but it avoids those ambiguities and is consistent with the important idea of *gauging* a symmetry (using the word *gauge* as a verb).

[2] In the ordinary **Ising model** (article 51033), the $\mathbb{Z}_2$-valued variables in the path integral formulation are associated with points in the lattice. In the Ising gauge model included here, the $\mathbb{Z}_2$-valued variables in the path integral formulation will be associated with links instead, and the action is invariant under gauge transformations.

[3] Article 51376 formulates the $U(1)$ model using the hamiltonian formulation, which has some advantages but obscures Lorentz symmetry (section 2).

[4] Sometimes it's called **pure QCD**, where *pure* means that the $SU(N_c)$ gauge field is the only field (no quarks). The integer N_c is the number of **colors** that each quark species would have, which is 3 in the Standard Model.

[5] Article 07611 reviews part of the evidence for a nontrivial continuum limit.

2 Lorentz symmetry, lattice models, and unitarity

The models mentioned in section 1 can be constructed in either of two ways. One is the hamiltonian formulation. The hamiltonian formulation makes the model's consistency with the general principles of quantum theory[6] clear by inspection, including the unitarity of time evolution. Time remains continuous even though space is treated as a lattice, and time translations are implemented by unitary operators e^{-iHt}, where H is the hamiltonian.[7] In the Schrödinger picture, this means that time evolution preserves the inner product between time-dependent state vectors, as required by **Wigner's theorem**.[8]

One disadvantage of the hamiltonian formulation is that it obscures Lorentz symmetry. Treating space as a lattice is not the issue here, because we can reasonably expect deviations from continuous space to be negligible at resolutions much coarser than the lattice scale. The issue is that the hamiltonian formulation obscures boost symmetry, regardless of resolution. In the hamiltonian formulation, intuitively anticipating the presence of boost symmetry – even only at coarse resolution – is difficult without checking the commutation relations of the operators that allegedly generate those symmetries. Those calculations are routine, but the outcomes are usually not easy to anticipate by inspection.

This article starts with the path integral formulation instead, because it makes Lorentz symmetry easier to anticipate intuitively. To keep the number of integration variables finite, this formulation treats spacetime as a lattice. This shouldn't cause any perceptible deviations from Lorentz symmetry at resolutions much coarser than the lattice scale. We need to be careful, though: in a naïve path integral formulation, time evolution may fail to be unitary, even in the continuous-time limit. Deviations from unitary time evolution would be in conflict with the general principles of quantum theory. The formulation used in this article ensures that time evolution is unitary.[9]

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[6] Article 03431

[7] Article 22871

[8] Article 90771

[9] Section 18

3 Notation and conventions

- The system of units is such that Planck's constant $\hbar$ and the speed of light are both equal to 1.
- d is the number of dimensions of spacetime.
- D is the number of dimensions of space.
- x and y denote points in the lattice.
- ℓ is a link in the lattice.
- $\square$ is a plaquette in the lattice.
- $u(x,y)$ or $u(\ell)$ is the field variable associated with a link ℓ whose endpoints are x and y. The letter u is a reminder that a unitary representation is being used.
- A state is represented by an element of the Hilbert space, and each element of the Hilbert space is represented by a complex-valued function $\Psi[u]$ of the link variables.
- G is the gauged group.
- $r(g)$ is the matrix representing an element g in a matrix representation r of the gauged group G.
- $\mathbb{Z}_n$ is the subgroup of $U(1)$ consisting of complex numbers $e^{2\pi ik/n}$ with $k \in \{0,1,2,...,n-1\}$.
- ϵ is the lattice spacing.
- The lattice spacing in the time direction will be denoted dt when it needs to be distinguished the lattice spacing in the space directions.
- Section 15 will define a quantity $\nu \in \{1, 1/2\}$ that will be used to account for two different standard normalization conventions, one for abelian G and one for nonabelian G.

4 Matrix representations

A **matrix representation** r of a group G represents each element $u \in G$ as a square matrix $r(u)$ and represents the group operation as the usual matrix product, subject to the condiction

$$r(u)r(u') = r(uu').$$

A matrix representation is called **faithful** if no two elements G are represented by the same matrix.[10] A representation is called **unitary** if every matrix $r(u)$ satisfies

$$r^{-1}(u) = r^{\dagger}(u),$$

where the superscript † denotes the adjoint (transpose and complex conjugate) of the matrix. Every compact Lie group has a faithful unitary representation.[11] In this article, such a representation will always be given, and for that representation, $r(u)$ will be denoted simply as u.[12] This is the only representation that will be used to construct the action, but other representations are still useful for constructing a variety of observables.[13]

Since a faithful unitary representation will always be given anyway, we can use that representation as the definition of the group G itself. This is valid because the representation is faithful. In particular, each element of $SU(N_c)$ may be represented as a unitary matrix with size $N_c \times N_c$ and with determinant equal to 1, using matrix multiplication as the group operation. Similarly, the groups $U(1)$ and $\mathbb{Z}_n$ may be defined and represented using unitary "matrices" of size 1×1.

[10] Every group has a **trivial representation** in which every element of G is represented by the identity matrix. Most representations are somewhere between these two extremes, neither trivial nor faithful.

[11] Taylor (2021), proposition 2.8.8

[12] Elements of G in this faithful representation will always appear inside a matrix trace. The trace could also be defined by using a quadratic form on the Lie algebra instead (Witten (1991), beginning of section 2), and then we wouldn't need to specify any particular faithful matrix representation.

[13] Section 27

5 Outline

Here is an outline of the rest of the article:

- Sections 6 through 8 will introduce the spacetime lattice, the field variables (link variables), and gauge transformations.
- Sections 9-11 will introduce the Hilbert space.
- Sections 12-19 will introduce the path integral, giving special attention to some technical issues related to unitary time evolution.
- Sections 20 and 21 will show that the action becomes Lorentz invariant after taking the continuum limit and applying Wick rotation.
- Sections 22-24 will use the small-dt approximation to derive an expression for the hamiltonian H (the generator of unitary time evolution), referring to article 51033 for some details that are already covered there.
- Sections 25-26 will provide some perspectives related to gauge invariance.
- Sections 27-32 will characterize the model's observables, giving special attention to some issues related to observables that are extended in time.

Article 07611 reviews some insights about the continuum limit of the quantum model.[14]

[14]This is more challenging than merely taking the continuum limit of the action, which is done in sections 20-21.

6 The lattice

Treat d-dimensional flat spacetime as a lattice generated by d mutually orthogonal basis vectors, all with the same magnitude[15] ϵ. In this basis, the coordinates of a point in the lattice (also called a **site**) are integers.

Two points x and y in the lattice are called **nearest neighbors** if they have the same coordinates except for one coordinate in which they differ by ± 1. An ordered pair (x, y) of nearest neighbors will be called a **directed link**, and an unordered pair $\{x, y\}$ of nearest neighbors will be called an **undirected link**. The two directed links (x, y) and (y, x) will be called **oppositely directed** compared to each other. The points x and y are the link's **endpoints**.

The set of points that all have a given value of the time coordinate will be called a **spatial lattice**. The spacetime lattice is a sequence of identical spatial lattices, one for each integer value of the time coordinate. The number of field variables will be kept finite in either of two ways:[16]

- To define the **truncated** version of the spatial lattice, think of the lattice as a special set of points in smooth space, and choose a very large spatial region $\mathcal{O}$ (the same region at every time) with no points of the lattice exactly on its boundary. Each point inside $\mathcal{O}$ will be called an **interior point**, and any other point connected to an interior point by a single link will be called a **boundary point**. Only links with at least one **interior endpoint** (at least one endpoint inside $\mathcal{O}$) will have associated link variables.

- To define the **wrapped** version of the spatial lattice, choose a very large integer K, and declare two points in the spatial lattice to be equivalent if their spatial coordinates are equal modulo K. In this version, every point is an interior point.

[15] We're using units where the speed of light is equal to 1, so the magnitudes of spacelike and timelike intervals are comparable. Later, to facilitate taking a continuous-time limit, we'll allow the timelike basis vector to have a different magnitude that the spacelike basis vectors.

[16] Article 51376 describes these two **long-distance cutoffs** in more detail. More generally, we could allow the lattice to be truncated in some dimensions and wrapped in others.

7 The gauged group and the link variables

When spacetime is treated as a lattice, a quantum field is represented by a large number of variables called **field variables**, each associated with a particular point, link, or other element of the lattice. In this article, each field variable takes values in a group[17] G that will be called the **gauged group**.[18] A group G is called **abelian** if all its elements commute with each other: $uu' = u'u$ for all $u, u' \in G$. Otherwise, it's called **nonabelian**. A group G is called **finite** if it has only a finite number of elements. In this article, the gauged group G will be a compact Lie group.[19] It may be connected, or finite, or neither, but it will always be compact.[20] The model's properties may depend on which group G we choose, but the model's construction works the same way for any G.[21] Special attention will be given to the cases $G = SU(N_c)$, $G = U(1)$, and $G = \mathbb{Z}_n$.

The field consists of one **link variable** $u(x, y)$ for each directed link (x, y) with at least one interior endpoint. A **value** of the link variable $u(x, y)$ is an element of the gauged group G. All of these variables are independent of each other except for this constraint:

$$u(x, y)u(y, x) = 1. \tag{1}$$

Again, only links with at least one interior endpoint have associated link variables. If a link (x, y) doesn't have an associated link variable, then $u(x, y) \equiv 1$.

The field consisting of these link variables will be called the **gauge field**, and any assignment of specific values to all the link variables (one value per link variable) will be called a **configuration** of the gauge field.

[17] Article 29682 reviews the definition of **group**.

[18] Footnote 1 in section 1

[19] Article 92035

[20] Any finite group qualifies as a compact Lie group (Harlow and Ooguri (2021), end of section 1.1), one whose elements are all disconnected from each other.

[21] If the group is not *simple* (as defined in article 92035), then using an action with different coefficients for different parts of the group may be allowed. This article ignores that option.

8 Gauge transformations

Let h be a map that assigns an element $h(x)$ of the gauged group G to each point x. In this article, any transformation of the link variables that replaces the value of each link variable $u(x,y)$ with the new value

$$u^{(h)}(x,y) \equiv h(x)u(x,y)h^{-1}(y) \tag{2}$$

will be called a **gauge transformation**.[22] Let I denote the identity element of the gauged group G. A gauge transformation for which $h(x) = I$ whenever x is not an interior point will be called an **interior gauge transformation**.[23,24] The group of all interior gauge transformations will be denoted $\mathcal{G}$, and a function of the link variables will be called **$\mathcal{G}$-invariant** if it is invariant under all such transformations.

Let $\bar{\mathcal{G}}$ denote the group of all gauge transformations. Then $\mathcal{G}$ is a subgroup of $\bar{\mathcal{G}}$, but its complement (the set of transformations in $\bar{\mathcal{G}}$ but not in $\mathcal{G}$) is not a group. The appropriate complementary concept is the quotient group $\bar{\mathcal{G}}/\mathcal{G}$, which consists of transformations in $\bar{\mathcal{G}}$ modulo transformations in $\mathcal{G}$.[25,26]

[22] The same name is sometimes used for any transformation that leaves all observables invariant (Avery and Schwab (2016), section 2.1). In this article, observables will be invariant under some of the transformations (2) (the ones that act trivially on boundary points) but not necessarily under all them.

[23] If the spatial lattice doesn't have any boundary points, then every gauge transformation is an interior gauge transformation.

[24] This name is not standard. In continuous spacetime, interior gauge transformations have been called **small gauge transformations**, and gauge transformations that have $h(x) \neq I$ for one or more boundary points have been called **large gauge transformations** (example: Miller (2021), text around equation (1.1)). This article doesn't use those names because they are often used differently, namely for gauge transformations that are/aren't continuously connected to the identity element of the gauge group (example: Fradkin (2022), section 22.2). The difference amounts to using the words *small* and *large* to describe either the transformation's support in space or the transformation's support in the gauge group. The last two sentences deliberately say *gauge group*, not *gauged group* (footnote 1 in section 1).

[25] Article 29682

[26] This quotient group is a lattice version of the group of **asymptotic symmetries** in continuous spacetime (Strominger (2017), equation (2.10.1)).

9 Haar measure: definition

The Haar measure generalizes the Lebesgue measure to locally compact groups. We'll only need it for compact groups, a special case of locally compact groups. This section reviews the definition, specialized to compact groups.[27]

Let G be a compact Lie group, and let $C(G)$ be the space of continuous real-valued functions from G to $\mathbb{R}$. As usual, write $f(g)$ for the real number that a function $f \in C(G)$ assigns to $g \in G$. The **Haar measure** defines an integral with these properties:

- The integral $\int dg\ f(g)$ is a real number.
- The integral is *linear*, which means

$$\int dg\ \big(r_1 f_1(g) + r_2 f_2(g)\big) = r_1 \int dg\ f_1(g) + r_2 \int dg\ f_2(g)$$

  for all real numbers r_1, r_2 and all $f_1, f_2 \in C(G)$.

- For any given $h \in G$,

$$\int dg\ f(g) = \int dg\ f(hg) = \int dg\ f(gh) = \int dg\ f(g^{-1}).$$

- $\int dg\ 1 = 1$.

The Haar measure is uniquely determined by (a subset of) these properties. The definition extends to complex-valued functions in the obvious way:

$$\int dg\ \big(f_R(g) + i f_I(g)\big) \equiv \int dg\ f_R(g) + i \int dg\ f_I(g),$$

where f_R and f_I are the real and imaginary parts of a complex-valued function f.

[27] This is based on theorem 4.1 in Salamon (2022).

10 Haar measure: examples

When G is a finite group with n elements, the Haar measure is given by

$$\int dg\ f(g) \equiv \frac{1}{n}\sum_{g\in G} f(g).$$

When $G = U(1)$, the Haar measure is given by

$$\int dg\ f(g) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta\ f(e^{i\theta}).$$

11 The Hilbert space

This section introduces the Hilbert space that will be used for both the path integral and hamiltonian formulations of models with gauge fields. States are represented by elements of the Hilbert space, and observables are represented by linear operators on the Hilbert space.[28,29]

Let $[u]$ be the set of link variables associated with a given time t. A state[30] is represented by a $\mathcal{G}$-invariant[31,32] complex-valued function $\Psi[u]$. Given two states $\Psi_1[u]$ and $\Psi_2[u]$, their inner product is

$$\begin{aligned}\langle\Psi_1|\Psi_2\rangle &\equiv \int [du]\;\Psi_1^*[u]\Psi_2[u] \\ &\equiv \int \left(\prod_\ell du(\ell)\right)\;\Psi_1^*[u]\Psi_2[u].\end{aligned} \tag{3}$$

For each link variable $u(\ell)$, $du(\ell)$ is the Haar measure for the gauged group G.

[28] Article 03431

[29] Section 28 will specify which operators represent observables in these models.

[30] Most of this article uses the word *state* to mean a state-vector in the Hilbert space.

[31] Section 8

[32] The Hilbert space would be well-defined even without requiring the functions $\Psi[u]$ to be $\mathcal{G}$-invariant, but this family of models only uses $\mathcal{G}$-invariant functions to represent states. Section 25 will explain why.

12 Preview of the path integral

In the Schrödinger picture, time evolution is a linear transformation $\Psi \to \Psi'$ from a state Ψ at time t to a state Ψ' at a later time $t' > t$. In the path integral formulation, this linear transformation has the form

$$\Psi'[u]_{t'} \propto \int_{<t'} [du]\; e^{-S_\omega[u]} \Psi[u]_t, \tag{4}$$

where:

- $[u]_t$ denotes the set of link variables whose endpoints are both at time t,
- the **action** $S_\omega[u]$ is a $\mathcal{G}$-invariant function of all the link variables whose endpoints are in the range $\geq t$ and $\leq t'$,
- the integral is over of the link variables that have at least one endpoint in the range $\geq t$ and $< t'$ (with no more than one endpoint at time t').

The action also depends on a parameter $0 \leq \omega \leq \pi/2$ whose significance will be explained in section 18, but here's a preview. Time evolution should be unitary, and we want the model defined by (4) to have a continuum limit with Lorentz symmetry. When $\omega = 0$, the action has a Lorentz-invariant continuum limit, but then the path integral (4) is not unitary (not even in the continuum limit). To achieve both unitarity and Lorentz symmetry, we must keep $\omega > 0$ until after the integrals are evaluated, and then we can extend the result to $\omega = 0$ to get Lorentz symmetry. Changing the value of ω is called **Wick rotation**. Section 18 will explain this in more detail.

13 Plaquette variables

Section 17 will express the action $S_\omega[u]$ in terms of plaquette variables. This section defines the plaquette variables.

A sequence of four directed links that traces around the perimeter of a square is called a **plaquette**. Let x_1, x_2, x_3, x_4 be the points at the corners of a plaquette, oriented sequentially around the perimeter. The quantity

$$W(x_1, x_2, x_3, x_4) \equiv \text{trace}\big(u(x_1, x_2)u(x_2, x_3)u(x_3, x_4)u(x_4, x_1)\big) \tag{5}$$

is called a **plaquette variable**. The trace is defined using the defining unitary representation of the gauged group G. The fact that the representation is unitary implies

$$W(x_4, x_3, x_2, x_1) = W^*(x_1, x_2, x_3, x_4). \tag{6}$$

In words: reversing the plaquette's orientation has the same effect on the plaquette variable as complex conjugation does. When the corners of the plaquette don't need to be specified, the abbreviation

$$W(\square) \equiv W(x_1, x_2, x_3, x_4) \tag{7}$$

will also be used.

If the four corners of a plaquette are not all interior points,[33] then the four links in a plaquette variable might not all have associated link variables. Example: suppose that x_2 is an interior point, x_1 and x_3 are boundary points, and x_4 is neither. Then the links (x_1, x_2) and (x_2, x_3) have associated link variables, but the links (x_3, x_4) and (x_4, x_1) do not. In this case, $u(x_3, x_4) \equiv 1$ and $u(x_4, x_1) \equiv 1$,[34] so the plaquette variable reduces to $W(x_1, x_2, x_3, x_4) = \text{trace}\big(u(x_1, x_2)u(x_2, x_3)\big)$.

[33] Section 6

[34] Section 7

14 Properties of plaquette variables

Plaquette variables are $\mathcal{G}$-invariant (invariant under interior gauge transformations).[35] They are the smallest examples of *Wilson loops*,[36] which are all are $\mathcal{G}$-invariant.

Let N denote the trace of the identity matrix in the defining unitary representation of the gauged group G (section 4). The value of N is determined by the representation, not just by the abstract group G. Example: the groups $U(1)$ and $SO(2)$ are isomorphic to each other, but their defining representations use $N = 1$ and $N = 2$, respectively.

If U is a unitary matrix of size $N \times N$, then[37]

$$\left|\frac{\text{trace}(U)}{N}\right| \leq 1 \quad \text{for all } U \qquad\qquad \left|\frac{\text{trace}(U)}{N}\right| = 1 \quad \text{only if } U \propto I \tag{8}$$

This implies

$$\left|\frac{W(\square)}{N}\right| \leq 1. \tag{9}$$

[35]Section 8

[36]Section 27

[37]More generally, if U and V are unitary, then $f(UV) \leq f(U) + f(V)$ with $f(\cdots) \equiv \sqrt{1 - |\text{trace}(\cdots)/N|^2}$, and equality holds only if U or V is proportional to I (Wang and Zhang (1994)). Set $V = U^\dagger$ to get (8).

15 Two normalization conventions

When G is connected, a matrix U representing an element of the Lie group G may be written as the exponential of a matrix representing an element of the Lie algebra. The matrix U is unitary, so we can write

$$U = \exp\left(\sum_k \theta_k T_k\right) \tag{10}$$

using a set of real variables θ_k, where $T_1, T_2, ...$ is a set of linearly independent antihermitian generators of the Lie algebra. The normalization of the generators T_k is a matter of convention. Two different conventions are prevalent in the literature. Both have the form[38,39,40]

$$\text{trace}(T_j T_k) = -\nu\delta_{jk}, \tag{11}$$

with typical values[41]

$$\nu = \begin{cases} 1 & \text{if } G \text{ is abelian,} \\ 1/2 & \text{if } G \text{ is nonabelian.} \end{cases}$$

To accommodate both conventions, this article leaves the value of ν unspecified.

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[38]Mnemonic: the Greek letter ν is transliterated to the letter "n" in english, and "n" stands for "normalization."

[39]Article 90757 explains why we can choose a basis for the Lie algebra in which $\text{trace}(T_j T_k) = 0$ for all $j \neq k$.

[40]Many sources use hermitian generators instead, which introduces a factor of i in the exponent of (10) and eliminates the negative sign in (11).

[41]Sources that use the $\nu = 1/2$ convention include Peskin and Schroeder (1995), equation (15.96); Montvay and Münster (1997), equation (3.25); Creutz (1983), equation (6.7)

16 Notation for the coefficients in the action

Every plaquette is one of two types: it is either a **time-space plaquette**, which is made from both timelike links and spacelike links, or it's a **space-space plaquette**, which is made from only spacelike links. To accommodate Wick rotation and a small-dt approximation, we need to use two different coefficients in the action: one coefficient $\beta_{\text{t-s}}$ that multiplies terms involving time-space plaquettes, and one coefficient $\beta_{\text{s-s}}$ for terms involving space-space plaquettes. The notation

$$\beta(\square) \equiv \begin{cases} \beta_{\text{t-s}} & \text{if } \square \text{ is a time-space plaquette,} \\ \beta_{\text{s-s}} & \text{if } \square \text{ is a space-space plaquette} \end{cases}$$

will also be used. The values of the coefficients are

$$\beta_{\text{t-s}} = \frac{2N}{g^2\nu} \times \frac{\epsilon^{d-3}}{dt} \times \frac{1}{ie^{-i\omega}} \qquad\qquad \beta_{\text{s-s}} = \frac{2N}{g^2\nu} \times \epsilon^{d-5} dt \times ie^{-i\omega} \tag{12}$$

where

- dt and ϵ are the lattice spacings in the time and space directions, respectively,
- d is the number of dimensions of spacetime,
- N is the trace of the identity matrix (section 14),
- $\nu \in \{1, 1/2\}$ is defined in section 15,
- ω is the Wick rotation parameter that was previewed in section 12,
- g is a positive real number that would be called the **coupling constant** in the context of quantum chromodynamics.

The coefficients $\beta(\square)$ must be dimensionless for every d, so the units of g^2 must be $[g^2] = [\epsilon^{d-4}]$.

17 The action

The path integral will be expressed in terms of an **action**, which is a function of all the link variables. The action is[42]

$$S_\omega[u] = \sum_{\square} \frac{\beta(\square)}{2}\left(1 - \frac{W(\square)}{N}\right) \tag{13}$$

where the sum is over all oriented plaquettes, so that each of the two possible orientations contributes its own term to the sum. The sum includes plaquettes involving fewer than four link variables,[43] which ensures that the action depends on all the link variables, including those with only one interior endpoint.[44] Square brackets are used to indicate that the action is a function of a number of variables that diverges in the continuum limit, namely all the link variables. The quantities $\beta(\square)$ and N were defined in section 16.

The action is $\mathcal{G}$-invariant because the plaquette variables $W(\square)$ are $\mathcal{G}$-invariant. This property is not affected by Wick rotation, which only affects the values of the coefficients $\beta(\square)$.

When the Wick rotation parameter ω is $\pi/2$, the coefficients $\beta(\square)$ are real and positive, so the **euclidean action** $S_{\pi/2}[u]$ is real-valued and non-negative. The fact that it is real-valued follows from equation (6). The fact that it is non-negative follows from the inequality (9).

[42]This particular choice is the **Wilson action**. Many other choices give the same continuum limit, and some of them make numerical calculations more efficient, but the Wilson action is simpler.

[43]Section 13

[44]Harlow and Ooguri (2021), text below equation (3.26)

18 Path integrals, unitarity, and Lorentz symmetry

When $\omega = 0$, the path integral (4) becomes the **lorentzian path integral**

$$\Psi'[u]_{t'} \propto \int_{<t'} [du]\; e^{iS_L[u]}\Psi[u]_t, \tag{14}$$

where the real-valued function $S_L[u]$ is defined by

$$iS_L[u] = -S_\omega[u]\Big|_{\omega=0}.$$

The subscript L stands for **lorentzian**. The function S_L has a Lorentz-invariant continuum limit, which gives us an intuitive reason to anticipate that the model defined by (14) has Lorentz symmetry at resolutions much coarser than the lattice scale, if (14) defines a quantum model at all. According to the general principles of quantum theory, time evolution should be unitary. The linear transformation defined by equation (14) is unitary for some models, but for most models it isn't. It is unitary for the models constructed in article 63548, but it's not unitary for the models constructed in this article, not even in the continuous-time limit.[45,46] The rest of this section explains how to restore unitarity without losing the intuitive reason to expect that the model has a Lorentz-symmetric continuum limit.

Figure 1 summarizes the idea. Start by setting $\omega = \pi/2$ in equation (4), which gives the **euclidean path integral**.[47] The transformation $\Psi \to \Psi'$ defined by the euclidean path integral has the form $\Psi' = M\Psi$ for some positive definite operator M.[48] The fact that M is positive definite implies that is can be written as $M = e^{-X}$ for some hermitian operator X. Replacing e^{-X} with $e^{-iX\,dt}$ would make time evolution unitary.[49] That replacement is not equivalent to Wick rotation, but it

[45]Matsumoto (2022)

[46]Article 51033 uses a simple example to explain why it's not unitary.

[47]Instead of starting with the euclidean path integral, we could start with ω close to zero (still positive), but starting with $\omega = \pi/2$ makes the reasoning easier to articulate.

[48]When the interval $t' - t$ is a single time step dt, this operator is called the **transfer matrix**. Article 43634 shows that the transfer matrix is positive definite.

[49]Kanwar and Wagman (2021), page 2

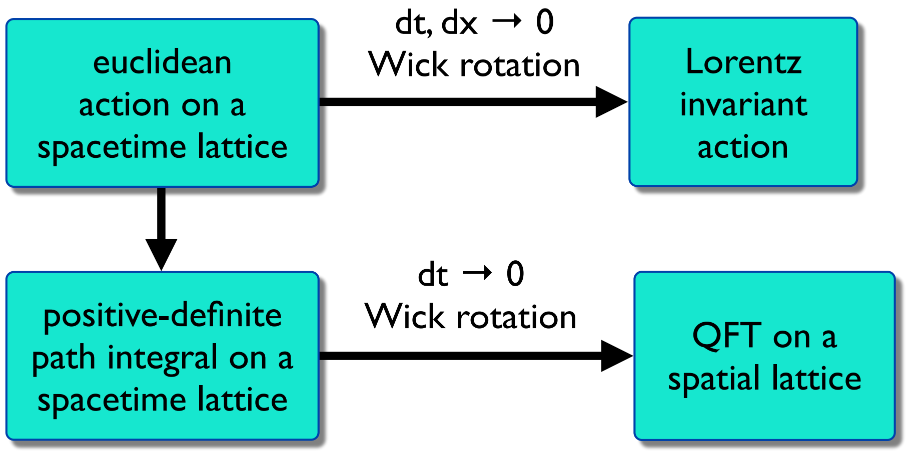


Figure 1 – Graphic depiction of how Wick rotation is used to construct a quantum model with unitary time evolution and a Lorentz-symmetric continuum limit. QFT stands for *quantum field theory*. Along the bottom row, the limit $dt \to 0$ is taken before Wick rotation to ensure that time evolution ends up being unitary. The fact that the top and bottom rows both use essentially the same combination of ideas ($dt \to 0$ and Wick rotation) gives us an intuitive reason to expect that the resulting quantum model should have a Lorentz-symmetric continuum limit.

becomes equivalent in the limit $dt \to 0$. This is important because the intuition about the resulting model's Lorentz symmetry comes from using the essentially the same ideas in the top and bottom rows of figure (1).

In the small-dt approximation, the evolution equation (4) for a single time step becomes

$$\Psi'[u]_{t+dt} = \exp\left(-iH\, e^{-i\omega} dt\right) \Psi[u]_t \tag{15}$$

for an operator H that can be determined explicitly.[50] The operator H is hermitian and is independent of the parameters ω and dt, so taking $\omega \to 0$ in equation (15) makes the transformation $\Psi \to \Psi'$ unitary, as desired, and we end up in the hamiltonian formulation with an explicit expression for the hamiltonian H.

If unitarity were the only thing we cared about, then we would not need to go through all this. We could start with the hamiltonian formulation instead, which

[50] Sections 23-24

is manifestly unitary. Starting with the path-integral formuluation has advantages, though, including these:

- It gives us an intuitive reason to expect that the resulting quantum model has Lorentz symmetry.

- It gives us a geometric understanding of why quantum field models with Lorentz symmetry also have **CPT symmetry**.[51]

- The path integral with $\omega > 0$ leads to a concise expression for correlation functions in the vacuum state that doesn't require having any explicit expression for the vacuum state itself.[52]

The first reason is the one emphasized in this article.

[51] Goodhew *et al* (2024), section 2.2

[52] Article 63548

19 Wick rotation and terminology

The lorentzian and euclidean versions of the path integral are sometimes called **real-time** and **imaginary-time path integrals**, respectively. Those names come from equation (15), in which all dependence on ω and dt occurs in the combination $e^{-i\omega}dt$. When ω is rotated from $\omega = 0$ to $\omega = \pi/2$, that combination changes from dt to $i\,dt$, as though we were merely replacing all occurrences of dt with $i\,dt$ in the path integral.

Those names can be misleading, though, because Wick rotation from $\omega = 0$ to $\omega = \pi/2$ does not always replace all occurrences of dt with $i\,dt$ in the path integral. It does in models whose field variables are all associated with individual points of the lattice,[53] but models involving gauge fields have field variables associated with links of the lattice. In that case, recovering the correct Lorentz-invariant continuum limit of the action $S_L[u]$ uses dt in the relationship $u(\ell) = e^{dt\,A(\ell)}$, where $u(\ell)$ is the link variable associated with a timelike link ℓ and $A(\ell)$ becomes the timelike component of a local potential in the limit $dt \to 0$.[54] The link variable $u(\ell)$ is an element of the gauged group G, and $A(\ell)$ is an element of the corresponding Lie algebra. When G is represented as a subgroup of a unitary group,[55] $u(\ell)$ is a unitary matrix and $A(\ell)$ is an antihermitian matrix, so we can't replace $dt \to i\,dt$ (or conversely) without ruining that essential relationship.[56] For that reason, this article doesn't use the names *real-time* or *imaginary-time* when referring to the path integral.

The names *lorentzian* and *euclidean* may be slightly better, but they're still not perfect. They're not perfect because they emphasize a side-effect of Wick rotation instead of emphasizing the reason for using Wick rotation. For the models considered in this article and in articles 63548 and 51033, a side-effect of Wick rotation

[53]This includes the models constructed in articles 63548 and 51033.

[54]Section 20

[55]Section 4

[56]We could avoid that issue by working in the temporal gauge (in which all timelike components of the local potential are zero), which sections 23-24 will do anyway when deriving the hamiltonian, but that doesn't change this paragraph's message.

is to change the signature of the spacetime metric from lorentzian to euclidean (or conversely), but that way of thinking about Wick rotation becomes tricky in models with spinor fields. Various signature-based definitions of Wick rotation have been proposed, and they're not all equivalent to each other when spinor fields are involved.[57] We should remember, though, that changing the signature of the spacetime metric is not the purpose of Wick rotation. The purpose of Wick rotation is to ensure that time evolution is unitary when a model is constructed using a path-integral formulation. The fact that it often amounts to changing the signature of spacetime is interesting and maybe even important,[58] but it's not the effect that matters in this article.

[57]Examples include Kontsevich and Segal (2021) and Nieuwenhuizen and Waldron (1997). A brief review of different approaches is given in section 1 of Mountain (2000)

[58]Witten (2021) reviews and elaborates on the role of euclidean path integrals in the context of quantum gravity research.

20 Continuum limit of the action

Suppose that the gauged group G is connected. This section shows that in that case, the euclidean action (13) has a continuum limit that respects the symmetries of d-dimensional euclidean spacetime, which becomes lorentzian after changing the Wick rotation parameter from $\omega = \pi/2$ to $\omega = 0$. This gives us a reason to anticipate that the quantum model might also have a Lorentz-symmetric continuum limit.[59]

The action $S_\omega[u]$ is a function of all the link variables u. We're assuming that G is connected, so we can write a link variable $u(x, y)$ as[60]

$$u(x, y) = e^{\theta(x,y)}, \tag{16}$$

where $\theta(x, y)$ belongs to the Lie algebra of the gauged group G. For a given oriented plaquette $\square$, write its four corners as

$$x \qquad x + \delta_1 x \qquad x + \delta_1 x + \delta_2 x \qquad x + \delta_2 x$$

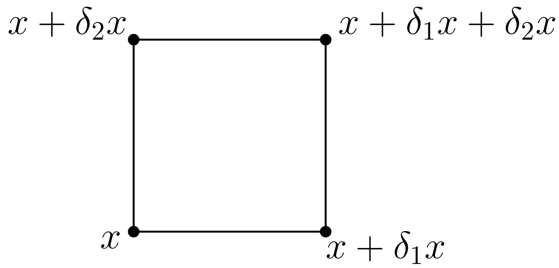
in cyclic order around the plaquette. The displacements $\delta_1 x$ and $\delta_2 x$ are orthogonal to each other. Denote their magnitudes by ϵ_1 and ϵ_2, which may be either dt or ϵ according to whether the displacement is timelike or spacelike. Define quantities A_1 and A_2 by writing

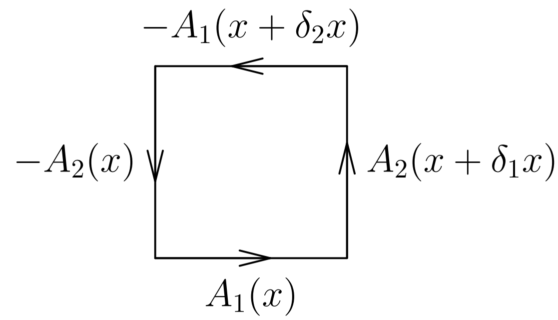
$$\begin{aligned} \theta(x,\, x + \delta_1 x) &= \epsilon_1 A_1(x) \qquad & \theta(x + \delta_2 x,\, x + \delta_2 x + \delta_1 x) &= \epsilon_1 A_1(x + \delta_2 x) \\ \theta(x,\, x + \delta_2 x) &= \epsilon_2 A_2(x) \qquad & \theta(x + \delta_1 x,\, x + \delta_1 x + \delta_2 x) &= \epsilon_2 A_2(x + \delta_1 x). \end{aligned}$$

[59] More importantly, the model is believed to have a *nontrivial* continuum limit when $d \in \{3, 4\}$. The case $d = 1$ is empty, because a one-dimensional lattice doesn't have any plaquettes, and Lorentz symmetry is trivial in one-dimensional spacetime anyway. The case $d = 2$ has a trivial continuum limit with Lorentz symmetry (article 07611). The interesting cases are $d \in \{3, 4\}$, and some of the evidence that these cases have a nontrivial Lorentz-symmetric continuum limit when $G = SU(N_c)$ are reviewed in article 07611.

[60] This follows from the fact that every element of a compact connected Lie group is contained in a torus (Hall (2015), theorem 11.9). It is not necessarily true for other connected Lie groups (Hall (2015), example 3.41 and the text below corollary 3.47).

The setup is depicted here:[61]





In the continuum limit, the quantities A_k will become the components of a local potential.[62] Use equations (5)-(7) to get

$$W(\square) = \text{trace}\big(u(\square)\big) \tag{17}$$

with[63]

$$u(\square) \equiv e^{-\epsilon_2 A_2(x)} e^{\epsilon_1 A_1(x)} e^{\epsilon_2 A_2(x+\delta_1 x)} e^{-\epsilon_1 A_1(x+\delta_2 x)}. \tag{18}$$

To determine the continuum limit of the action 13, we should consider what happens to (18) in the limit where the quantities $A_k(x)$ change arbitrarily little from one point to the next, which we can describe formally as a limit of arbitrarily small ϵ_k. The factors of ϵ_k that are implicit in the displacements $\delta_k x$ can be made explicit by defining

$$\partial_j A_k(x) \equiv \frac{A_k(x+\delta_j x) - A_k(x)}{\epsilon_j}, \tag{19}$$

which implies

$$A_k(x+\delta_j) = A_k(x) + \epsilon_j \partial_j A_k(x).$$

Use this in (18) to get

$$u(\square) = e^{-\epsilon_2 A_2(x)} e^{\epsilon_1 A_1(x)} e^{\epsilon_2 A_2(x) + \epsilon_1\epsilon_2 \partial_1 A_2(x)} e^{-\epsilon_1 A_1(x) - \epsilon_1\epsilon_2\partial_2 A_1(x)}. \tag{20}$$

[61] The signs come from equation (1).

[62] Article 11617

[63] The identity trace(XY) = trace(YX) was used to order the factors in a way that will be convenient later.

As $\epsilon_k \to 0$, the right-hand side of (19) becomes a derivative, as suggested by the notation on the left-hand side. The small-ϵ_k limit of $W(\Box)$ should be defined so that these derivatives remain finite, because the small-ϵ_k limit is really meant to be the limit where the quantities $A_k(x)$ change arbitrarily little from one point to the next. By taking the limit with the derivatives held fixed, section 21 will derive the identity

$$u(\Box) + u^{-1}(\Box) = 2 + (\epsilon_1\epsilon_2 F_{12})^2 + O(\epsilon^5) \tag{21}$$

with

$$F_{ab} \equiv \partial_a A_b - \partial_b A_a + [A_a, A_b]. \tag{22}$$

Use this in equations (17) and (13) to get

$$S_\omega[u] = \frac{-\epsilon_a^2\epsilon_b^2}{2N} \sum_{a<b} \beta_{ab}\,\text{trace}(F_{ab}^2) + O(\epsilon^5), \tag{23}$$

using β_{ab} as another way to write $\beta(\Box)$ when $\Box$ is in the a-b plane. The quantity $-\text{trace}(F_{ab}^2)$ is positive because equation (16) does not have an i in the exponent, so F_{ab} is antihermitian (instead of hermitian) in a unitary representation of the Lie group. Using the values of β that were given in section 16, we can use equation (23) to get the continuum limit[64]

$$S_{\pi/2}[u] = \frac{-1}{2\nu g^2}\int d^dx\ \sum_{a<b}\text{trace}(F_{ab}^2) = \frac{-1}{4\nu g^2}\int d^dx\ \sum_{a,b}\text{trace}(F_{ab}^2) \tag{24}$$

when the Wick rotation parameter ω is $\pi/2$. When $\omega = 0$ in equation (23), the corresponding integral is Lorentz invariant, as promised in section 18.

[64] Many calculations (like the ones cited in article 07611) use a small-A expansion, which can be recast as a small-g expansion by rescaling $A \mapsto gA$. To apply that computational method to the model defined on a lattice, we may want to expand the action in powers of A without taking $\epsilon \to 0$. The details of that expansion won't be worked out here, but one feature is worth mentioning: in equation (23), the terms of order ϵ^5 (and higher) are also of order A^3 (and higher), so the quadratic-in-A part of the result is the same in both the small-A and small-ϵ expansions.

21 Derivation of (21)

This section derives (21) from (13) and (20) by using the identity[65]

$$e^{sA}e^{sB} = e^{sA+sB+s^2[A,B]/2+O(s^3)}, \tag{25}$$

where A, B belong to the Lie algebra of G, and s is a real number. This identity is easy to verify by expanding both sides in powers of s up to order s^2.

Use (25) to combine the first two factors in (20) and also to combine the last two factors in (20). This gives

$$\begin{aligned} u(\square) &= e^{(\epsilon_1 A_1 - \epsilon_2 A_2) - \epsilon_1\epsilon_2[A_2,A_1]/2 + O(\epsilon^3)} \\ &\quad\times e^{(\epsilon_2 A_2 - \epsilon_1 A_1) - \epsilon_1\epsilon_2[A_2,A_1]/2 + \epsilon_1\epsilon_2(\partial_1 A_2 - \partial_2 A_1) + O(\epsilon^3)}. \end{aligned} \tag{26}$$

Now use (25) again to combine the two factors in (26), which gives

$$u(\square) = e^r \qquad\qquad r \equiv \epsilon_1\epsilon_2 F_{12} + O(\epsilon^3) \tag{27}$$

with F_{12} defined by (22). The sum in (13) includes both orientations of each plaquette, and reversing the orientation replaces $u(\square)$ with its inverse, so the action depends only on the combinations $u(\square) + u^{-1}(\square)$. Use (27) in $u(\square) + u^{-1}(\square) = e^r + e^{-r}$ to get the desired result (21).

Article 76708 considers the holonomy around an infinitesimal closed loop in smooth spacetime. That leads to a smooth-spacetime version of the same result, where the quantities F_{ab} are the components of the field strength associated with a local potential A.

[65] A formula for all the terms in the exponent on the right-hand side and conditions under which it converges are reviewed in (Casas and Murua (2009), section 1 and theorem 3.2). Each term has the form $[X_1, [X_2, [X_3, \cdots]]]$ with each $X_k \in \{A, B\}$, which says that it belongs to the Lie algebra generated by A and B. This is the **Baker-Campbell-Hausdorff (BCH) theorem** (Hofstätter (2021); Hall (2015), section 5.3).

22 Time evolution for a single time step

Choose two consecutive times, t and $t' \equiv t + dt$. Write u and u' to denote link variables associated with times t and t', respectively. Write u_0 to denote link variables that have one endpoint at time t and the other at time t',[66] and write P_0 to denote the set of oriented plaquettes that use such timelike links. We can take the action for a single time step to be[67,68]

$$s[u', u_0, u] = \frac{\beta_{\text{t-s}}}{2} \sum_{\square \in P_0} \left(1 - \frac{W(\square)}{N}\right) + \frac{\beta_{\text{s-s}}}{2} \sum_{\square \in t} \left(1 - \frac{W(\square)}{N}\right), \tag{28}$$

where the sum over "$\square \in t$" is over oriented plaquettes that lie entirely within the spatial lattice at time t. The action (13) for multiple time steps is a sum of single-time-step actions (28). In terms of (28), the path integral for a single time step is

$$\Psi'[u'] \propto \int [du_0][du] \; e^{-s[u', u_0, u]} \, \Psi[u]. \tag{29}$$

Iterating this gives the path integral for an arbitrary number of time steps, which is written in equation (4) using slightly different notation.

Everything on the right-hand side of (29) – the Haar measure, the action, and the initial state – is invariant under interior gauge transformations, so the final state $\Psi'[u']$ is also invariant under interior gauge transformations, as it must be to represent an element of the Hilbert space.

[66] Mnemonic: 0 is the standard index-value for the timelike component of a vector, and here it's used to indicate a link in a timelike direction.

[67] Section 16 defined $\beta_{\text{t-s}}$ and $\beta_{\text{s-s}}$.

[68] Equation (28) is not symmetric with respect to the times t and t': it includes purely-spacelike plaquettes at time t but not at time t'. To prove that the transfer matrix is positive definite, article 43634 uses a symmetric version instead, and that's important for the argument in section 18. To simplify the notation, sections 23-24 will use the asymmetric version (28) to derive the hamiltonian. That derivation can easily be adapted to use the symmetric version, but that would complicate the notation without adding any further clarity to the derivation.

23 Temporal gauge

This section derives the identity

$$\int [du_0][du]\; e^{-s[u',u_0,u]}\,\Psi[u] = \int [du]\; e^{-s[u',1,u]}\,\Psi[u]. \tag{30}$$

Using (30) in (29) gives the path integral in the **temporal gauge**. Section 24 will explain how to derive the hamiltonian, starting with the path integral in the temporal gauge.

Start with the integral

$$\int [du]\; e^{-s[u',u_0,u]}\,\Psi[u]. \tag{31}$$

Integrating this over $[u_0]$ gives the left-hand side of (30). The definition of the lattice that was given in section 6 ensures that both endpoints of each timelike link are interior points, not boundary points, so every link in the set $[u_0]$ has an endpoint at the initial time t that is not a boundary point. Let h denote a gauge transformation function as in equation (2). For every interior point x at the initial time t, we can choose $h(x)$ so that $u^h(\ell) = 1$ for the timelike link ℓ that has x as one of its endpoints. This gauge transformation makes all the link variables in $[u_0]$ equal to 1. The Haar measure is invariant under gauge transformations,[69] the initial state $\Psi[u]$ is invariant under interior gauge transformations,[70] and the gauge transformation that we just constructed doesn't affect any of the links in the set $[u']$. This shows that the quantity (31) is independent of u_0, and the Haar measure is defined so that $\int [u_0] = 1$, so this establishes the identity (30).

Section 22 already established that the function of $[u']$ defined by the left-hand side of (30) is invariant under interior gauge transformations, so the function of $[u']$ defined by the right-hand side of (30) is, too.

[69] Section 9

[70] Section 11

24 The hamiltonian

Use equations (29) and (30) to write the path integral for a single time step as

$$\Psi'[u'] \propto \int [du]\; e^{-s[u',1,u]}\,\Psi[u]. \tag{32}$$

Equation (28) gives

$$s[u',1,u] = \beta_{\text{t-s}} \sum_{\ell} \left(1 - \frac{\text{trace}\Big(u'(\ell)u^{-1}(\ell)\Big) + \text{c.c.}}{2N}\right) + \frac{\beta_{\text{s-s}}}{2} \sum_{\square\in t} \left(1 - \frac{W(\square)}{N}\right),$$

where the sum over ℓ is over undirected links (more precisely, over only one of the two possible directions of each undirected link) in the spatial lattice.

The derivation of the hamiltonian works just like the derivation shown in article 51033 for principal chiral models,[71] so it won't be repeated here.[72] The coefficients are different, though,[73] so this section shows the result in an easy representative case, namely when G is a connected abelian group, such as a direct product of copies of $U(1)$ or $SO(2)$.

A matrix U representing an element of a connected Lie group G (abelian or not) may be written as the exponential of a matrix representing an element of the Lie algebra. The matrix U is unitary, so we can write

$$U = \exp\left(\sum_k \theta_k T_k\right)$$

[71] The field variables here are associated with links instead of with points, and the "potential" term (with no time derivatives) has a different structure, but the derivation of the hamiltonian still works the same way.

[72] Derivations are shown in Harlow and Ooguri (2021), appendix F (for discrete G, and a partial derivation for connected G); in Creutz (1983), chapter 15 (for connected G); in Fradkin and Susskind (1978) (for $G = \mathbb{Z}_2$); and in Kogut (1983), section V.A (for $G = SU(2)$)

[73] In this article, the continuum limit of the action has the form $\sim \frac{1}{g^2}\int d^dx\; \text{trace}(F^2)$, where the field strength F has the same units as $1/|x|^2$. For the principal chiral models considered in article 51033, the continuum limit of the action has the form $\sim \frac{1}{g^2}\int d^dx\; (\partial U)^2$, where U is dimensionless. (U is an element of G, whereas F is an element of the Lie algebra of G.) The action must be dimensionless, so g^2 has dimensions $|x|^{d-4}$ in this article but has dimensions $|x|^{d-2}$ in article 51033.

using a set of real variables θ_k, where $T_1, T_2, ...$ is a set of linearly independent antihermitian generators of the Lie algebra, normalized as in equation (11). Now suppose that G is abelian, so that all the generators T_j commute with each other. Then the hamiltonian may be derived using an easy generalization of the process that article 51033 uses for the $O(2)$ nonlinear sigma model. The result is[74,75]

$$H = \frac{-c_{\text{t-s}}}{2} \sum_{\ell} \sum_{k} \left(\frac{\partial}{\partial \theta_k(\ell)} \right)^2 + V \tag{33}$$

where $\theta_k(\ell)$ is defined by

$$u(\ell) = \exp\left(\sum_k \theta_k(\ell) T_k \right),$$

the operator V is defined by[76]

$$V\Psi[u] = \frac{c_{\text{s-s}}}{2} \sum_{\square} \left(1 - \frac{W(\square)}{N} \right) \Psi[u],$$

and the coefficients $c_{\text{t-s}}$ and $c_{\text{s-s}}$ are defined by

$$i\,dt\,c_{\text{t-s}} = \frac{N}{\nu \beta_{\text{t-s}}} \qquad\qquad i\,dt\,c_{\text{s-s}} = \beta_{\text{s-s}}.$$

Time evolution is unitary (the hamiltonian is hermitian) when the Wick rotation parameter ω is zero, and then the coefficients are

$$c_{\text{t-s}} = \frac{g^2}{\epsilon^{d-3}} \qquad\qquad c_{\text{s-s}} = \frac{N\,\epsilon^{d-5}}{\nu g^2}.$$

The generalization of (33) to not-necessarily-abelian gauged groups G is called the **Kogut-Susskind Hamiltonian**. The hamiltonian used for $U(1)$ electrodynamics in article 51376 is a special case of (33) with only one term in the sum over k.

[74] The sum over ℓ is over undirected links in the spatial lattice.

[75] For general G, the second-derivative term is the laplacian on the group manifold.

[76] The sum over $\square$ is over oriented plaquettes in the spatial lattice.

25 Gauge invariance and reversibility

This section shows that if time evolution is reversible (which is a prerequisite for being unitary), then states must be $\mathcal{G}$-invariant, as required in section 11.

Let let $\Psi^{(h)}[u] \equiv \Psi[u^{(h)}]$ be the state obtained from $\Psi[u]$ by an interior gauge transformation (2). After a single time step, the state $\Psi^{(h)}[u]$ gives this final state:[77]

$$\int [du_0][du]\; e^{-s[u',u_0,u]}\,\Psi[u^{(h)}].$$

The action s is $\mathcal{G}$-invariant, so this is the same as

$$\int [du_0][du]\; e^{-s[u',u_0^{(h)},u^{(h)}]}\,\Psi[u^{(h)}]. \tag{34}$$

The variables u' are not affected because we can choose the function h in equation (2) to satisfy $h(x) = I$ for all points x at the final time t', independently of its values at the initial time t.[78] The Haar measure is both left- and right-invariant, so this is the same as

$$\int [du_0^{(h)}][du^{(h)}]\; e^{-s[u',u_0^{(h)},u^{(h)}]}\,\Psi[u^{(h)}], \tag{35}$$

which can be written as

$$\int [du_0][du]\; e^{-s[u',u_0,u]}\,\Psi[u] \tag{36}$$

just by changing the way the integration variables are labelled. This shows that the final state produced by $\Psi^{(h)}[u]$ is the same as the one produced by $\Psi[u]$, so time evolution would not be reversible if states were not $\mathcal{G}$-invariant.[79]

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[77] Equation (29)

[78] The action is still invariant even if h were time-independent (the same at times t and t'), but then u' would be affected, which would change the final state. Requiring reversibility doesn't preclude global symmetries.

[79] This argument wouldn't apply if the action were only invariant under time-independent transformations in $\mathcal{G}$, but then the action wouldn't have a Lorentz symmetric continuum limit.

26 Gauge invariance and the boundary

Let $\mathcal{G}'$ be the group of gauge transformations that leave every plaquette variable invariant.[80] The group $\mathcal{G}$ of interior gauge transformations is a subgroup of $\mathcal{G}'$, but if the lattice has boundary points, then $\mathcal{G}'$ might be larger than $\mathcal{G}$, so some transformations in $\mathcal{G}'$ might have a nontrivial effect on states in the Hilbert space, which are only required to be $\mathcal{G}$-invariant.[81] This section shows that gauge transformations in $\mathcal{G}'$ commute with time evolution: if the initial state is $\mathcal{G}$-invariant, then applying a gauge transformation $h \in \mathcal{G}'$ first and then applying time evolution gives the same final state as applying time evolution first and then applying h.

Every plaquette that contributes to the action involves at least one link variable, which implies that it involves at least two adjacent interior points. Every nearest neighbor to an interior point is either another interior point or a boundary point, so every plaquette that contributes to the action involves at least two link variables. Plaquette variables that involve four link variables are invariant under arbitrary gauge transformations, so $\mathcal{G}'$ is determined by plaquettes whose corner-lists (x_1, x_2, x_3, x_4) involve either two or three link variables. The corresponding plaquette variables have one of these two forms:

- x_2 and x_3 are interior points, and x_1 and x_4 are boundary points,
- x_2 is an interior point, x_1 and x_3 are boundary points, and x_4 is neither.

In both cases, if we write the plaquette variable as trace(g) where g is the product of the link variables, then the effect of an arbitrary gauge transformation is

$$\text{trace}(g) \to \text{trace}\big(gh^{-1}(x_k)h(x_1)\big)$$

with $k = 4$ in the first case and $k = 3$ in the second case.[82] To be invariant

[80] Transformations in $\mathcal{G}'$ leave the action invariant. Proving that these are the *only* gauge transformations that leave the action invariant would require ruling out the possibility of cancellations between different terms in the sum over plaquettes.

[81] Section 11

[82] The point x_4 is not involved in the second case because in that case the links (x_3, x_4) and (x_4, x_1) don't have associated link variables, and gauge transformations only affect link variables.

for all g, this must at least be invariant when $g = I$, which gives the condition $\text{trace}(h^{-1}(x_k)h(x_1)) = \text{trace}(I)$, and then the relationships (8) imply

$$h(x_k) = h(x_1). \tag{37}$$

Such a pair of boundary points $\{x_1, x_k\}$ will be called a **constrained pair**. The spatial lattice is the same at every time,[83] so if x and y are two points on the boundary that differ only in the time coordinate, then they are connected to each other by a sequence of constrained pairs, which gives $h(x) = h(y)$. This shows that gauge transformations in $\mathcal{G}'$ are independent of time on the boundary. From here, the claim at the beginning of this section may be established by using an argument similar to the one in section 25, but with $(u')^{(h)}$ instead of u' in equations (34)-(36).

The reasoning in the preceding paragraph may also be used to determine the effect of transformations in $\mathcal{G}'$ at points on the spatial boundary at a single time, which determines its effect on states in the Hilbert space. The group $\mathcal{G}'$ includes transformations for which $h(x)$ is the same at all boundary points, but it may also include transformations with $h(x) \neq h(y)$ for some pairs of boundary points x and y. This occurs if x and y are not connected to each other by any sequence of constrained pairs, a possibility that is not excluded by the rules established in section 6. The analog of this phenomenon in smooth space would be having a spatial boundary with two or more components that cannot be connected to each other by paths in the boundary. In that case, $\mathcal{G}'$ includes gauge transformations for which $h(x)$ is constant on each connected component of the boundary but may differ from one connected component to another.

The group $\mathcal{G}'$ includes gauge transformations for which the function h in (2) is constant in space and time. These are called **global gauge transformations**. Global gauge transformations are examples of **internal symmetries**.[84,85]

[83] Section 6 imposed this condition on the spacetime lattice.

[84] A symmetry is called **internal** if it doesn't change where/when any observables are localized in spacetime.

[85] Some observables (like observables corresponding to *Wilson lines*, defined in sections 27-31) may be sensitive to global gauge transformations for which h is not in the center of the gauged group G (Harlow and Ooguri (2021), last sentence in the paragraph with equation (8.8)).

27 Wilson loops and Wilson lines

If r is any matrix representation of G, not necessarily faithful, then the function

$$\text{trace}\Big(r\big(u(x_1,x_2)\big)r\big(u(x_2,x_3)\big)\cdots r\big(u(x_{n-1},x_n)\big)\Big) \qquad \text{with } x_n = x_1 \tag{38}$$

is $\mathcal{G}$-invariant. This function will be called a **Wilson loop**. The traces ensures that the factor of $h^{-1}(x_n) = h^{-1}(x_1)$ cancels the factor of $h(x_1)$ when a gauge transformation is applied.[86] The representation r is used to define the trace. Even if G is already defined as a matrix group, we can use a different representation r in (38). Different choices of r define different $\mathcal{G}$-invariant functions. A plaquette variable is a special case in which the loop is the perimeter of a plaquette and the representation r is the one used to define the gauged group G.

Another example of a $\mathcal{G}$-invariant function is the **Wilson line**

$$r\big(u(x_1,x_2)\big)r\big(u(x_2,x_3)\big)\cdots r\big(u(x_{n-1},x_n)\big) \text{ if } x_1 \text{ and } x_n \text{ are boundary points.} \tag{39}$$

No trace is needed in this case,[87] because transformations in $\mathcal{G}$ have $h(x) = I$ at all boundary points x.

Wilson loops and Wilson lines will both be denoted $W(C)$, where C (for **curve**) is the sequence of directed links defined by the sequence of points $x_1, x_2, ..., x_n$. These functions correspond to a linear operators on the Hilbert space, using a correspondence that sections 28-30 will describe. Such an operator will be called a **Wilson (loop or line) operator** and denoted $\hat{W}(C)$. The operators are often just called *Wilson loops* and *Wilson lines*, without the word *operator*,[88,89] but distinguishing between the functions (38)-(39) and the corresponding operators will be important in sections 28-32.

[86] If G is abelian, then this cancellation occurs without the trace, but the trace is essential when G is nonabelian.

[87] Harlow and Ooguri (2021), text around equations (3.1) and (3.2)

[88] Peskin and Schroeder (1995), section 15.3

[89] Some authors use these names for the result of evaluating a path integral with this function in the integrand (Montvay and Münster (1997), section 3.2.4).

28 Observables

In quantum theory, observables are represented by linear operators on a Hilbert space. The Hilbert space defined in section 11 uses only $\mathcal{G}$-invariant functions, so observables in this model must preserve that condition: the result of applying an observable to a $\mathcal{G}$-invariant function must be another $\mathcal{G}$-invariant function. In the present model, every normal operator that satisfies this condition will be included in the set of observables.[90]

One example is the operator $\hat{f}$ defined by[91]

$$(\hat{f}\Psi)[u] \equiv f[u]\Psi[u] \tag{40}$$

for all $\Psi[u]$ in the Hilbert space, where $f[u]$ is any complex-valued $\mathcal{G}$–invariant function of the link variables in the spatial lattice.[92] If $\Psi[u]$ is $\mathcal{G}$-invariant, then so is $f[u]\Psi[u]$, so the operator that replaces Ψ with $f\Psi$ qualifies as an observable.

Quantum field theory is a refinement of quantum theory in which observables are associated with regions of spacetime. If the function $f[u]$ in (40) is made only of link variables in the spatial lattice at time t and that are all contained within a spatial region R, then $\hat{f}$ defines an observable localized in R at time t. One example is a Wilson operator $\hat{W}(C)$ that only involves points at time t. This observable is extended in space (along C) but not in time. Observables that are extended in time can also be described in the path integral formulation. Sections 29-31 will explain how that works.[93]

[90] A **normal operator** is an linear operator that commutes with its adjoint. At its core, an observable is represented by a collection of projection operators satisfying certain properties (article 03431). Article 74088 shows that every normal operator defines such a set of projection operators in a natural way, so observables can be represented as normal operators. Using a self-adjoint operator (a special type of normal operator) to represent an observable amounts to using real numbers to label the possible measurement outcomes. Using normal operators amounts to allowing complex numbers as labels. Using the raw set of projection operators amounts to leaving the labels unspecified.

[91] Most normal operators cannot be written this way.

[92] All such operators commute with each other because all complex-valued functions commute with each other. In particular, such an operator commutes with its adjoint (defined by replacing $\omega[u]$ with its complex conjugate), so it is a normal operator.

[93] Typical introductions don't mention some of the issues that sections 29-31 will emphasize. This article emphasizes them to clarify the connection to the general principles of quantum theory in article 03431.

29 Path integrals and the Heisenberg picture

From now on, the single-time-step version of equation (4) and the $\omega = 0$ version of equation (15) will be treated interchangeably. This is valid when approaching the limits $dt \to 0$ and $\omega \to 0$, in that order. The abbreviation $U(t) \equiv \exp(-iHt)$ will be used, and $X^\dagger$ will denote the adjoint of an operator X.

Consider the path integral

$$\Psi_f[u]_{t'} \propto \int_{<t'} [du]\; e^{-S_\omega[u]} f[u]\Psi[u]_t. \tag{41}$$

This is like (4) but with a factor of $f[u]$ inserted into the integrand, and the notation is slightly different: the final state is denoted Ψ_f (instead of Ψ') to indicate its dependence on the function f in the integrand. If the function $f[u]$ depends only on link variables at a single time t_f with $t \le t_f < t'$, then equation (41) becomes

$$|\Psi_f\rangle = U(t' - t_f)\hat{f}U(t_f - t)|\Psi\rangle \tag{42}$$

in the limits $dt \to 0$ and $\omega \to 0$, where $\hat{f}$ is defined as in section 28. In the Schrödinger picture,[94] we can describe (42) as the result of letting an initial state $|\Psi\rangle$ evolve forward in time from t to t_f, applying an operator $\hat{f}$, and then letting the resulting state continue evolving in time from t_f to t'.

Now let Φ be some other initial state, not necessarily the same as Ψ, and consider the inner product $\langle\Phi_1|\Psi_f\rangle$. The subscript 1 indicates that the function $\Phi_1[u]$ is defined using the trivial function $f[u] = 1$ in the integrand of the path integral. Equation (42) implies

$$\langle\Phi_1|\Psi_f\rangle = \langle\Phi|U^\dagger(t_f - t)\hat{f}U(t_f - t)|\Psi\rangle. \tag{43}$$

The combination $U^\dagger(t_f - t)\hat{f}U(t_f - t)$ on the right-hand side is the time-dependent version of $\hat{f}$ in the Heisenberg picture.[94] We could write left-hand side of (43)

[94]Article 22871

as a path integral by using equations (3) and (42), but a slight modification is appropriate when the Wick rotation parameter ω is nonzero.[95] To motivate the modification, define the operator M as in section 18, so that the time evolution equation for a single time-step is $\Psi \to M\Psi$. The operator M is defined for arbitrary ω, but the relationship $M^\dagger = M^{-1}$ holds only for $\omega = 0$. That matters because the the inner product defined in equation (3) gives $\langle M\Phi|M\Psi\rangle = \langle \Phi|M^\dagger M\Psi\rangle$, which is not necessarily equal to $\langle\Phi|\Psi\rangle$ unless $M^\dagger = M^{-1}$. If we want the inner product of $M\Psi$ with $M\Phi$ to be equal to the inner product of Ψ with Φ for arbitrary ω, then we need to generalize the inner product defined in equation (3) to account for time evolution. In the limit $dt \to 0$, equation (15) shows that replacing $M \to M^{-1}$ is the same as replacing $dt \to -dt$. In the same limit, replacing $dt \to -dt$ is also the same as replacing $S_\omega[u] \to -S_\omega[u]$, so the goal can be achieved by defining

$$\langle \Phi_1|\Psi_f\rangle \equiv \int [du]\; \big(\Phi_1^+[u]\big)^* \Psi_f^-[u] \tag{44}$$

with

$$\Psi_f^\pm[u]_{t'} \propto \int_{<t'} [du]\; e^{\pm S_\omega[u]} f[u]\Psi[u]_t. \tag{45}$$

Equation (45) is just like (41) but with the factor $\exp(-S_\omega[u])$ generalized to $\exp(\pm S_\omega[u])$. The identity

$$\exp\big(S_\omega[u]\big) = \Big(\exp\big(-S_\omega[u]\big)\Big)^* \qquad \text{when } \omega = 0$$

ensures that (44) reduces to (3) when $\omega = 0$.

Altogether, the relationship (43) illustrates how path integrals may be used to represent operators in the Heisenberg picture.

[95]Usually, instead taking $\omega = \pi/2$ as in this section, an infinitesimal value $0 < \omega \ll 1$ is used so that the path integral is almost lorentzian (the "$\pm i\epsilon$" prescription). This is essentially the "timefolding" formalism illustrated in section 1.2 in Grabovsky (2023), usually called the **closed time path** formalism (Cooper (1995)) or the **Schwinger–Keldysh** or **in-in** formalism (Nastase (2019), chapter 72; Mou *et al* (2019), section 1).

30 Operators extended in time

Now suppose that the function $f[u]$ in (41) is a Wilson loop or line $W(C)$ extended in the time direction. The range of times over which C extends must be contained between the initial and final times (t and t') in the path integral (41). This is implicit in the notation on the left-hand side, which says that link variables that lie in the spatial lattice at time t' are the only the only non-integrated link variables on the right-hand side.[96] If this condition is satisfied, then equation (41) defines a linear operator on the Hilbert space, but that operator is not localized in time. The time evolution of the state and the effect of the operator are intermingled with each other, so its effect cannot be properly described in the usual Schrödinger picture of a state evolving incrementally from one time-step to the next. That's okay, because we can use the Heisenberg picture instead.

If we use the symbol $\hat{f}$ to denote the operator defined by

$$\langle\Phi_1|\Psi_f\rangle = \langle\Phi|\hat{f}|\Psi\rangle \tag{46}$$

for a specific function $f[u]$ in equation (41) covering a specific range of times, then shifting the function $f[u]$ forward in time by an amount δt gives the operator $U^\dagger(\delta t)\hat{f}U(\delta t)$, just like in equation (43). The difference is that now $\hat{f}$ itself is not localized at any single time, but only in a finite time interval. That's not a problem: in the Heisenberg picture, states are timeless, so the applying an extended-in-time operator $\hat{f}$ to a state $|\Psi\rangle$ does not cause any conceptual trouble.

[96] Otherwise, the resulting function would depend on more than just the link variables in a single spatial slice, so it would not represent an element of the Hilbert space even though it would still be $\mathcal{G}$-invariant.

31 Observables extended in time

The operators considered in section 30 are relatively easy to describe in the path integral formulation, but they're not always observables. An observable is represented by a normal operator (an operator that commutes with its adjoint), and the operators considered in section 30 typically don't satisfy this condition. To see why, consider a simple special case: suppose that the function $f[u]$ in (41) is the product of two real-valued $\mathcal{G}$-invariant functions, $f[u] = f_A[u]f_B[u]$, where $f_A[u]$ depends only on link variables associated with time t_A, and $f_B[u]$ depends only on link variables associated with time $t_B \neq t_A$. We can define corresponding operators $\hat{f}$, $\hat{f}_A$, and $\hat{f}_B$ as in section 30. I don't know of any reason to expect $\hat{f}_A$ and $\hat{f}_B$ to commute with each other in general,[97] and if they don't, then $\hat{f}$ is not normal.

The combination $\hat{f} + \hat{f}^\dagger$ is normal, so it does qualify as an observable, but using a path integral to describe this observable would be awkward. To understand why, use sections 28-30 to deduce $\hat{f} = \hat{f}_A\hat{f}_B$ if $t_A > t_B$ and $\hat{f} = \hat{f}_B\hat{f}_A$ if $t_B > t_A$. This is called a **time-ordered product**, because the order in which $\hat{f}_A$ and $\hat{f}_B$ are multiplied is determined by their chronological order. The path integral formulation produces this chronological ordering naturally. In the same sense, its adjoint $\hat{f}^\dagger$ is reverse-time-ordered, so the combination $\hat{f} + \hat{f}^\dagger$ is not consistently time-ordered overall. This isn't a problem,[98] but it does suggest that using a path integral to describe extended-in-time *observables* will generally be awkward, even though many extended-in-time *operators* are relatively easy to describe.

A function $f[u]$ of the form (38) or (39) (Wilson loop or line) that is extended in time cannot be written as a product of $\mathcal{G}$-invariant functions that are each localized at a single time, but the operator $\hat{f}$ defined by (46) is still time-ordered in a natural sense, so the conclusion still applies: $\hat{f} + \hat{f}^\dagger$ is an observable, but $\hat{f}$ might not be.[99]

[97] In relativistic quantum field theory in smooth spacetime, $\hat{f}_A$ and $\hat{f}_B$ commute with each other if they're localized in regions of spacetime that cannot be connected to each other by any timelike worldline (article 21916), but they typically don't commute with each other otherwise.

[98] More carefully: I don't know any reason to insist that observables extended in time should also be time-ordered.

[99] I'm using the word **observable** as defined in article 03431. Some authors might use the word more liberally, such as referring to all Wilson-loop operators as *observables* even if they don't commute with their adjoints.

32 Temporal Wilson lines

The preceding sections considered functions $W(C)$ for which the time interval spanned by C fits inside the time interval spanned by the path integral, without touching the times that host the initial and final states. One virtue of the path integral formulation is that when $\omega > 0$, equation (15) implies that as the initial and final times approach $-\infty$ and $+\infty$ (without growing the time intervals spanned by any of the inserted operators), the path integral automatically projects the initial and final state onto the vacuum state.[100,101] Thanks to this property, we can use the path integral formulation to express vacuum expectation values without explicitly specifying the initial or final states. This is done by allowing time to wrap back on itself so that the number of integration variables (link variables) remains finite. Then we can consider a new kind of Wilson loop, one that wraps around the time dimension. This is called a **temporal Wilson line**.

Sections 28-30 explained how to define an operator on the Hilbert space from a function inserted in the path integral, but that correspondence assumes that the time interval spanned by the function doesn't intersect the initial or final time. A temporal Wilson line violates that condition, so a corresponding linear operator on the Hilbert space does not exist,[102] at least not if the states that comprise the Hilbert space are defined on spacelike slices of the path integral. A temporal Wilson line may be viewed as a modification of the action (and hamiltonian) instead of as an operator.[103]

[100] Article 63548

[101] When spontaneous symmetry breaking (SSB) is absent, the *vacuum state* is the unique state of lowest energy. In models with SSB, an extra ingredient is needed to select a vacuum state – a lowest-energy state satisfying the *cluster property* (article 21916).

[102] Some authors use the name *operator* for any insertion into the integrand of the path integral, without worrying about whether it corresponds to any linear operator on the Hilbert space. (Harlow and Ooguri (2021) acknowledge this in the text between equations (1.5) and (1.6).) That more liberal language is common when the path integral formulation of quantum field theory is studied as a purely mathematical subject – a welcome practice that has improved and will undoubtedly continue to improve our understanding of physics.

[103] Harlow and Ooguri (2021), text below equation (3.34)

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