(In)equivalence of Irreducible Representations of Clifford Algebras

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Abstract Given an irreducible matrix representation of a Clifford algebra, we can get another one by replacing every Dirac matrix with its negative, with its complex conjugate, with its transpose, or any composition of these. These representations may or may not be equivalent to each other (related to each other by a linear transformation), depending on the number d of dimensions of the vector space that generates the Clifford algebra, and depending on the signature. This article determines which ones are equivalent to each other for each number of dimensions and each signature. When d is even, the answer is simple: they are all equivalent to each other. When d is odd, the pattern is more complicated.

Contents

1 Clifford algebra: notation and conventions

Let V be a d-dimensional vector space over $\mathbb{R}$, the field of real numbers. Let g be a nondegenerate symmetric bilinear form[1] on V. Then V has a basis $\mathbf{e}_1, ..., \mathbf{e}_d$ for which

$$g(\mathbf{e}_j, \mathbf{e}_k) = \begin{cases} \pm 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The **signature** of g is (p, m), where p and m are the numbers of plus-signs and minus-signs, respectively,[2] among the quantities $g(\mathbf{e}_k, \mathbf{e}_k)$. Clearly $p + m = d$.

The **Clifford algebra** $\text{Cliff}(p, m)$ is the largest[3] associative algebra generated by V that has a unit (an identity element for multiplication) and that satisfies

$$\mathbf{v}^2 = g(\mathbf{v}, \mathbf{v}) \tag{2}$$

for all vectors $\mathbf{v} \in V$. In equation (2), $\mathbf{v}^2$ is an abbreviation for the Clifford product of $\mathbf{v}$ with itself. Sometimes the opposite sign convention is used ($\mathbf{v}^2 = -g(\mathbf{v}, \mathbf{v})$), which exchanges the roles of p and m in the Clifford algebra. Beware of this when comparing signature-dependent results from different sources.

[1] Article 03910

[2] Sometimes the word *signature* is used for the difference $p - m$, but in this article it means the pair (p, m).

[3] Here, *largest* is an allusion to a *universal property*. Roughly, it means that the only relationships in the algebra are those that can be derived from $\mathbf{v}^2 = g(\mathbf{v}, \mathbf{v})$ together with the general rules of associative algebra.

2 Representations: notation and conventions

Let W be a finite-dimensional vector space over the field $\mathbb{C}$ of complex numbers.[4] A **representation** of $\text{Cliff}(p,m)$ on W is a homomorphism from the algebra $\text{Cliff}(p,m)$ into the algebra of linear transformations of W.[5] In a given basis for W, every linear transformation of W may be represented as a square matrix with components in $\mathbb{C}$, so a representation may also be defined as something that assigns a matrix $\gamma(A)$ to each element $A \in \text{Cliff}(p,m)$ in a way that respects these conditions:[6]

$$\begin{gathered}\gamma(AB) = \gamma(A)\gamma(B)\\ \gamma(A+B) = \gamma(A) + \gamma(B)\\ \gamma(rA) = r\gamma(A)\end{gathered}$$

for all $A, B \in \text{Cliff}(p,m)$ and for all $r \in \mathbb{R}$. To emphasize this perspective, a representation may be called a **matrix representation**.

A representation is called **irreducible** if it if W doesn't have any proper subspace that is self-contained under all of the linear transformations that represent elements of $\text{Cliff}(p,m)$. Intuitively: an irreducible representation is one that doesn't contain any smaller representation inside of itself. The name *irreducible representation* is often abbreviated **irrep**.

In the context of a given basis $\mathbf{e}_1, ..., \mathbf{e}_d$ for the vector space V that generates $\text{Cliff}(p,m)$, the matrix $\gamma_a \equiv \gamma(\mathbf{e}_a)$ representing a basis vector $\mathbf{e}_a$ is called a **Dirac matrix**. This definition is basis-dependent: each Dirac matrix in one basis is typically a linear combination of the Dirac matrices in a different basis. Equation (2) implies

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2g(\mathbf{e}_a, \mathbf{e}_b)I \tag{3}$$

where I denotes the identity matrix.

[4] Representations on vector spaces over other fields may also be considered, but this article doesn't.

[5] Article 86175

[6] This is what **homomorphism** means.

3 The questions

Two representations γ and γ' are **equivalent** to each other if the condition

$$\gamma(\mathbf{v})M = M\gamma'(\mathbf{v})$$

is satisfied for all vectors $\mathbf{v}$ by some invertible matrix M. The matrix M is said to **intertwine** the two representations. This condition may also be written

$$M^{-1}\gamma(\mathbf{v})M = \gamma'(\mathbf{v}),$$

which says that one representation can be obtained from the other by a change of basis of the vector space W on which the representation acts.[7]

Given one irrep of the Clifford algebra, we can get another one by replacing every Dirac matrix with its negative, with its complex conjugate, with its transpose, or any composition of these. The resulting representation may or may not be equivalent to the original one. This article focuses on two questions:

- Which of these irreps are equivalent to each other?
- When two of these irreps are equivalent, what specific matrix M intertwines them?

The first question will be answered generally (section 8). Sections 10-12 will show that every irrep is equivalent to one in which each Dirac matrix is either real or imaginary[8] and each Dirac matrix is also either symmetric or antisymmetric. The second question will be answered only for irreps having this special property (section 14).

[7]This is a vector space over $\mathbb{C}$. It should not be confused with the vector space to which the spacetime vectors $\mathbf{v}$ belong, which is a vector space over $\mathbb{R}$. These two vector spaces typically have different numbers of dimensions.

[8]A Dirac matrix is called **real** or **imaginary** if its nonzero components are all real numbers or all imaginary numbers, respectively.

4 A clarification

Consider these statements:

1. The algebra Cliff$(3,0)$ is isomorphic to the algebra of 2×2 matrices over $\mathbb{C}$.

2. For each N, all irreducible representations of $M_{\mathbb{C}}(N)$, the algebra of $N\times N$ matrices over $\mathbb{C}$, are equivalent to each other.[9]

3. The algebra Cliff$(3,0)$ has two irreducible representations that are not equivalent to each other.

Each of these statements is true when interpreted properly, but they might appear to contradict each other. The apparent contradiction is resolved by being more explicit about which field ($\mathbb{R}$ or $\mathbb{C}$) plays which role in which context.

Each of the Clifford algebras Cliff(p,m) is an algebra over $\mathbb{R}$. The matrix algebra $M_{\mathbb{C}}(N)$ is an algebra over $\mathbb{C}$. It can also be interpreted as an algebra over $\mathbb{R}$, which is why statement 1 above makes sense, but statement 2 above tacitly refers to representations that are linear over $\mathbb{C}$, which is a more restrictive condition than merely being linear over $\mathbb{R}$. The algebra Cliff$(3,0)$ (over $\mathbb{R}$) has two irreps that are each other's complex conjugates. These two irreps cannot be equivalent to each other as defined in section 3, because a matrix M (with components in $\mathbb{C}$) cannot satisfy $AM = MA^*$ for every matrix A.[10] This confirms statement 3. Each one of those two irreps of Cliff$(3,0)$ is isomorphic to $M_{\mathbb{C}}(2)$ as an algebra over $\mathbb{R}$, but this algebra over $\mathbb{R}$ may be repackaged as an algebra over $\mathbb{C}$ in two different ways, and those two ways are not related to each other by any $\mathbb{C}$-linear transformation.

[9] Each full matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the quaternions) is a **simple** algebra (Martin (2016), page 63, exercise 2.1.17), and all irreps of a simple algebra are equivalent to each other (Benn and Tucker (1989), page 329, statement (A21)).

[10] Example: consider $A = zI$ for some complex number z with $z^* \neq z$.

5 Three lemmas

In this article, these lemmas will be used but not proved:

- **Lemma 1:** When d is even, all irreps are equivalent to each other.[11]
- **Lemma 2:** When d is odd, two inequivalent irreps exist, and every other irrep is equivalent to one of these.[12]
- **Lemma 3:** When d is odd, every irrep has the property that each Dirac matrix is proportional to the product of all of the others.[13]

Lemma 3 implies part of lemma 2, but not all of it. When d is odd, lemma 3 implies that the product of all Dirac matrices is proportional to the identity matrix. This implies the existence of at least two inequivalent irreps, because every matrix M commutes with the identity matrix, so two irreps whose all-Dirac-matrix products have different signs cannot be equivalent.

[11] When d is even, each Clifford algebra over $\mathbb{R}$ is isomorphic to a full matrix algebra over the field of real numbers $\mathbb{R}$ or the quaternions $\mathbb{H}$ (article 03910). All irreps of such a matrix algebra are equivalent to each other (Figueroa-O'Farrill (2015), page 8).

[12] Each odd-dimensional Clifford algebra over $\mathbb{R}$ is isomorphic either to a full matrix algebra over the field of complex numbers $\mathbb{C}$ or to the direct sum of two full matrix algebras over $\mathbb{R}$ or $\mathbb{H}$ (article 03910). In both cases, the algebra has two inequivalent irreps. For a full matrix algebra over $\mathbb{C}$, this is acknowledged on page 8 in Figueroa-O'Farrill (2015) and illustrated in section 4. For the other case, this follows from footnote 11.

[13] Article 86175

6 The all-Dirac-matrix product

Define

$$\omega \equiv \gamma_1\gamma_2\cdots\gamma_d. \tag{4}$$

When d is even, the matrix ω anticommutes with every Dirac matrix. When d is odd, ω commutes with every Dirac matrix. In both cases, $\omega^2 = \pm I$. This sign will be determined below.

When d is even,

$$\omega^2 = \begin{cases} I & \text{if } d = 4n \text{ and } p \text{ is even,} \\ -I & \text{if } d = 4n \text{ and } p \text{ is odd,} \\ -I & \text{if } d = 4n+2 \text{ and } p \text{ is even,} \\ I & \text{if } d = 4n+2 \text{ and } p \text{ is odd,} \end{cases} \tag{5}$$

To derive this, using an indexing convention in which $\gamma_a^2 = I$ for $a \leq p$ and $\gamma_a^2 = -I$ for $a > p$. Think of the product (4) as a product of pairs $\gamma_{2k+1}\gamma_{2k+2}$ with $k \in \{0, 1, 2, ...\}$. All of these pairs commute with each other. This implies:

- If p and m are both even, then the square of every pair $\gamma_{2k+1}\gamma_{2k+2}$ is $-I$, so the square of ω is I or $-I$ if the number of pairs is even ($d = 4n$) or odd ($d = 4n + 2$), respectively.

- If p and m are both odd, then the pair $\gamma_p\gamma_{p+1}$ squares to I, and the other pairs all square to $-I$, so the square of ω is I or $-I$ if the number of *other* pairs is even ($d = 4n + 2$) or odd ($d = 4n$), respectively.

Altogether, this gives (5).

When d is odd, lemma 3 implies

$$\omega = \epsilon I \tag{6}$$

for some complex number ϵ, and then the relationship $\omega^2 = \pm I$ implies

$$\epsilon \equiv \begin{cases} \pm 1 & \text{if } \omega^2 = I \\ \pm i & \text{if } \omega^2 = -I. \end{cases} \tag{7}$$

The two cases depend on the value of d and on which of p or m is odd:

$$\omega^2 = \begin{cases} I & \text{if } d = 4n+1 \text{ and } p \text{ is odd,} \\ -I & \text{if } d = 4n+1 \text{ and } m \text{ is odd,} \\ -I & \text{if } d = 4n+3 \text{ and } p \text{ is odd,} \\ I & \text{if } d = 4n+3 \text{ and } m \text{ is odd,} \end{cases} \tag{8}$$

This can be derived using reasoning like what was used above for d even, but now one Dirac matrix in the product (4) is un-paired, so the sign of ω^2 is affected by whether square of the un-paired Dirac matrix is I or $-I$.

Given a product of Dirac matrices, define its **reverse** to be the product of the same Dirac matrices in the opposite order.[14] In particular, the reverse of ω is

$$\omega^{\text{rev}} = \gamma_d \cdots \gamma_2 \gamma_1. \tag{9}$$

This is equal to either ω or $-\omega$, depending on the value of d:

$$\omega^{\text{rev}} = \begin{cases} \omega & \text{if } d \in \{4n,\, 4n+1\} \\ -\omega & \text{if } d \in \{4n+2,\, 4n+3\}. \end{cases} \tag{10}$$

[14] An **anti-automorphism** is like an automorphism except that it reverses the order of the factors in a product. Reversion may be defined as the unique anti-automorphism of the Clifford algebra that leaves $\gamma(\mathbf{v})$ invariant for every vector $\mathbf{v}$. According to the first page of chapter 2 in Deligne (1999), Bourbaki calls reversion the **principal anti-automorphism** of the Clifford algebra.

7 Relationship to the signature

This section shows how the results in section 6 relate to the quantity

$$\sigma \equiv (p-m) \text{ modulo } 8. \tag{11}$$

Here's a summary:

- If d is even and $\omega^2 = I$, then $\sigma \in \{0,4\}$.
- If d is even and $\omega^2 = -I$, then $\sigma \in \{2,6\}$.
- If d is odd and $\omega^2 = I$, then $\sigma \in \{1,5\}$.
- If d is odd and $\omega^2 = -I$, then $\sigma \in \{3,7\}$.

To derive these relationships when d is even, let j denote either p or m, whichever is smaller. By re-ordering the factors in ω (which doesn't affect the value of ω^2), we can write it so that the first j Dirac-matrix pairs square to I and the remaining pairs square to $-I$. The number of pairs that square to $-I$ is $|p-m|/2$, so ω^2 is I or $-I$ according to whether $|p-m|/2$ is even or odd.

To derive these relationships when d is odd, use with the results for even d together with the fact that one Dirac matrix remains un-paired:

- If the un-paired Dirac matrix squares to I, then the results are the same as when d was even but with the possible values for σ all incremented by 1, so $\sigma \in \{0,4\}$ and $\sigma \in \{2,6\}$ become $\sigma \in \{1,5\}$ and $\sigma \in \{3,7\}$, respectively.
- If the un-paired Dirac matrix squares to $-I$, then the results are the same as when d was even but with the opposite sign for ω^2 and with the possible values for σ all decremented by 1, so $\sigma \in \{0,4\}$ and $\sigma \in \{2,6\}$ become $\sigma \in \{7,3\}$ and $\sigma \in \{1,5\}$, respectively, both with the opposite sign for ω^2.

8 (In)equivalence relations when d is odd

Given any irrep defined by $\mathbf{v} \to \gamma(\mathbf{v})$ for all vectors $\mathbf{v}$, these abbreviations will be used:

- $-\gamma$ denotes the representation defined by $\mathbf{v} \to -\gamma(\mathbf{v})$ for all vectors $\mathbf{v}$.
- γ^* denotes the representation defined by $\mathbf{v} \to \gamma^*(\mathbf{v})$ for all vectors $\mathbf{v}$.
- γ^T denotes the representation defined by $\mathbf{v} \to \gamma^T(\mathbf{v})$ for all vectors $\mathbf{v}$.
- $\gamma^\dagger$ denotes the representation defined by $\mathbf{v} \to \gamma^\dagger(\mathbf{v})$ for all vectors $\mathbf{v}$.

When d is even, all of these representations are equivalent to each other.[15] Section 9 will determine which ones are (in)equivalent to each other when d is odd. Here's a summary of the results:

Representations	Equivalent when d is odd?
γ and $-\gamma$	no
γ and γ^*	yes if $\omega^2 = I$, no if $\omega^2 = -I$
γ and γ^T	yes if $\omega^{\text{rev}} = \omega$, no if $\omega^{\text{rev}} = -\omega$
γ and $\gamma^\dagger$	yes if m is even, no if m is odd

The (in)equivalence of γ with $-\gamma^*$, $-\gamma^T$, and $-\gamma^\dagger$ may be deduced by combining these results with lemma 2 in section 5.

[15] Lemma 1 in section 5

9 Derivation of relations when d is odd

To deduce the results that were tabulated in section 8, let $\omega[\gamma \to \tilde{\gamma}]$ denote the result of replacing each Dirac matrix γ_a in the product (4) with $\tilde{\gamma}_a$, where $\tilde{\gamma}$ may be any of the other representations listed at the beginning of section 8. If the representations γ and $\tilde{\gamma}$ are equivalent, then[16] a nonzero matrix M exists for which $\omega M = M\omega[\gamma \to \tilde{\gamma}]$. Lemma 3 says that ω and $\omega[\gamma \to \tilde{\gamma}]$ are both proportional to I, and I commutes with everything, so if $\omega[\gamma \to \tilde{\gamma}] \neq \omega$, then the representations cannot be equivalent. Combine this with lemma 2 to conclude that the representations are equivalent to each other if and only if $\omega[\gamma \to \tilde{\gamma}] = \omega$. Using that insight, the results tabulated in section 8 can be deduced like this:

- $\omega[\gamma \to -\gamma] = -\omega$ when d is odd. This gives the first row in the table.
- $\omega[\gamma \to \gamma^*] = \omega^*$, and equations (6)-(7) imply
$$\omega^* = \begin{cases} \omega & \text{if } \omega^2 = I, \\ -\omega & \text{if } \omega^2 = -I \end{cases} \qquad \text{when } d \text{ is odd.}$$
This gives the second row in the table.
- $\omega[\gamma \to \gamma^T] = (\omega^T)^{\text{rev}}$, and equations (6)-(7) imply $\omega^T = \omega$ when d is odd. This gives the third row in the table.
- In a standard representation, each Dirac matrix is either hermitian or antihermitian, so if the signature is (p, m), then the number of antihermitian factors in ω is even if m is even, and it's odd if m is odd. This gives
$$\omega[\gamma \to \gamma^\dagger] = \begin{cases} \omega & \text{if } m \text{ is even,} \\ -\omega & \text{if } m \text{ is odd.} \end{cases}$$
This gives the fourth row in the table.

[16] Section 3

10 Existence of standard representations

Most (maybe all) of the explicit matrix representations used in the physics literature have the property that each Dirac matrix is either real or imaginary and each Dirac matrix is either symmetric or antisymmetric. In this article, an irrep with that property will be called a **standard representation**.[17] This definition is basis-dependent, because the definition of *Dirac matrix* is basis dependent.[18] Section 14 will explain how intertwiners[19] between the representations listed in section 8 may be constructed explicitly in a standard representation.

Sections 11-12 will demonstrate that an irrep with this property always exists, for every d and every signature.

Section 12 will also explore whether the number of real Dirac matrices is even or odd, and section 13 will explore whether the number of symmetric Dirac matrices is even or odd. These properties will be important when constructing intertwiners in section 14. The answers are summarized in tables 1 and 2 on the next page. In cases where the answer is *no*, the non-existence of such a standard representation can be inferred from the results shown in section 8: if such a representation did exist, then it could be used[20] to construct an intertwiner between two representations that sections 8-9 already determined cannot be equivalent.

[17]This name is not standard. (This isn't meant to be a joke, but it was fun to write.)
[18]Section 2
[19]Section 3
[20]Section 14 will explain how this could be done.

	Can the # of real γ_as be even?	Can the # of real γ_as be odd?
$p - m = 4k$	yes	undetermined
$p - m = 4k + 1$	**no**	yes
$p - m = 4k + 2$	undetermined	yes
$p - m = 4k + 3$	yes	**no**

Table 1 – Options for the number of real Dirac matrices in a standard representation of Cliff(p, m). The integer k is arbitrary (positive or negative). The *yes* results will be derived in section 12. The entries marked *undetermined* will not be determined (or needed) in this article.

	Can the # of symmetric γ_as be even?	Can the # of symmetric γ_as be odd?
$d = 4n$	yes	undetermined
$d = 4n + 1$	**no**	yes
$d = 4n + 2$	undetermined	yes
$d = 4n + 3$	yes	**no**

Table 2 – Options for the number of symmetric Dirac matrices in a standard representation of Cliff(p, m), as a function of $d \equiv p + m$. The integer n is nonnegative but otherwise arbitrary. The *yes* results will be derived in section 13. The entries marked *undetermined* will not be determined (or needed) in this article.

11 Standard representations for split signatures

This section shows how to construct a standard representation when the signature (p, m) is **split**, which means $p = m$. The representation will be assembled using tensor products of the 2×2 matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

as explained in article 86175. When the signature is split, we can use

$$\begin{aligned}
\gamma_1 &= X \otimes 1 \otimes 1 \otimes \cdots \\
\gamma_2 &= Y \otimes 1 \otimes 1 \otimes \cdots \\
\gamma_3 &= Z \otimes X \otimes 1 \otimes \cdots \\
\gamma_4 &= Z \otimes Y \otimes 1 \otimes \cdots \\
\gamma_5 &= Z \otimes Z \otimes X \otimes \cdots \\
\gamma_6 &= Z \otimes Z \otimes Y \otimes \cdots ,
\end{aligned}$$

and so on, as the Dirac matrices in a standard representation.

12 Standard representations for other signatures

Starting with a standard representation γ with $p = m$, a standard representation $\tilde{\gamma}$ with any even value of $p - m$ may be constructed by setting

$$\tilde{\gamma}_a = \begin{cases} i\gamma_a & \text{for some } a, \\ \gamma_a & \text{for other } a. \end{cases}$$

Each factor of i changes the value of $p - m$ by either $+2$ or -2. If we start with a $p = m$ irrep γ in which every Dirac matrix is real, then the number of real Dirac matrices in the resulting representation $\tilde{\gamma}$ is even if it has $p - m = 4k$ and odd if it has $p - m = 4k + 2$, where k denotes an arbitrary integer (positive or negative). This accounts for the *yes* results shown in table 1 for even values of $p - m$.

To construct a standard representation when d is odd, start with the standard representation for $d - 1$ with split signature that was constructed in section 11, and set $\gamma_d \equiv \gamma_1\gamma_2 \cdots \gamma_{d-1}$. If $d = 4n + 1$, then $\gamma_d^2 = I$, so this gives a representation with $p - m = 1$ in which every Dirac matrix is real. If $d = 4n + 3$, then $\gamma_d^2 = -I$, so this gives a representation with $p - m = -1$ in which every Dirac matrix is real. We can change the signature by using overall factors of i as before. In the resulting representation, the number of real Dirac matrices is odd if $p - m = 4k + 1$, and the number is even if $p - m = 4k + 3$. This accounts for the *yes* results shown in table 1 with odd values of $p - m$.

13 (Anti)symmetry of standard representations

When determining whether the number of symmetric Dirac matrices is even or odd (table 2), the factors of i don't matter. We constructed a standard representation for Cliff(p,p) in which the number of symmetric matrices is even if p is even ($d = 4n$) and is odd if p is odd ($d = 4n+2$). This accounts for two of the *yes* entries in table 2.

To construct a standard representation when d is odd, start with a standard representation for $d-1$ with split signature in which half of the Dirac matrices are symmetric, and set $\gamma_d \equiv \gamma_1\gamma_2\cdots\gamma_{d-1}$. The (anti)symmetry of the resulting matrix depends on two items:[21]

- whether the number of antisymmetric factors in the product is even or odd,
- whether reversing the order of the factors in the product changes its sign or not.

This implies that γ_d is symmetric, both for $d = 4n+1$ and for $d = 4n+3$. This accounts for the other two *yes* entries in table 2.

[21] The second item matters because the Dirac matrices anticommute with each other, and the transpose reverses the order of the factors.

14 Constructing intertwiners

Given one irrep γ of Cliff(p, m), this section explains construct intertwiners M between γ and each of the related representations listed in section 8 whenever they are equivalent to the original irrep γ.

The idea is simple. Let X be any set of Dirac matrices in the representation γ (not necessarily all of them). Let $|X|$ denote the number of Dirac matrices in X, and let ω_X denote the product of all of the Dirac matrices in X. The fact that every Dirac matrix anticommutes with all of the others but commutes with itself implies:

- If $|X|$ is even, then ω_X anticommutes with every Dirac matrix in X, and ω_X commutes with every Dirac matrix that is not in X.

- If $|X|$ is odd, then ω_X commutes with every Dirac matrix in X, and ω_X anticommutes with every Dirac matrix that is not in X.

Examples:[22]

- Let X be the set of all Dirac matrices in the representation γ. If $|X|$ is even, then ω_X satisfies $\gamma_a X = -X\gamma_a$ for every Dirac matrix γ_a, so ω_X intertwines the representations γ and $-\gamma$.

- Let X be the set of all real Dirac matrices in the representation γ. If $|X|$ is odd, then ω_X satisfies $\gamma_a X = X\gamma_a^*$ for every Dirac matrix γ_a, so ω_X intertwines the representations γ and γ^*. If $|X|$ is even, then ω_X satisfies $\gamma_a X = -X\gamma_a^*$ for every Dirac matrix γ_a, so ω_X intertwines the representations γ and $-\gamma^*$.

- Let X be the set of all symmetric Dirac matrices in the representation γ. If $|X|$ is odd, then ω_X satisfies $\gamma_a X = X\gamma_a^T$ for every Dirac matrix γ_a, so ω_X intertwines the representations γ and γ^T. If $|X|$ is even, then ω_X satisfies $\gamma_a X = -X\gamma_a^T$ for every Dirac matrix γ_a, so ω_X intertwines the representations γ and $-\gamma^T$.

[22] These examples use the notation that was defined in section 8.

15 Tables of intertwiners

The next page gives explicit examples of intertwiners for irreps of Cliff(p, m) in the context of the standard representations that were constructed in sections 11-12. Abbreviations:

- ω is the product of all Dirac matrices.
- ω_{real} is the product of all real Dirac matrices.
- ω_{imag} is the product of all imaginary Dirac matrices.
- ω_{sym} is the product of all symmetric Dirac matrices.
- ω_{antisym} is the product of all antisymmetric Dirac matrices.
- ω_{herm} is the product of all hermitian Dirac matrices.
- ω_{antiherm} is the product of all antihermitian Dirac matrices.

In these tables, k denotes an arbitrary integer (positive or negative):

Example of matrix M that satisfies $\gamma_a M = M\gamma_a^*$ for all a		
$p-m$	M	references
$4k$	ω_{imag}	table 1, §14
$4k+1$	ω_{imag}	table 1, §14
$4k+2$	ω_{real}	table 1, §14
$4k+3$	**none**	eqn (8), §7-8

Example of matrix M that satisfies $\gamma_a M = -M\gamma_a^*$ for all a		
$p-m$	M	references
$4k$	ω_{real}	table 1, §14
$4k+1$	**none**	eqn (8), §7-8
$4k+2$	ω_{imag}	table 1, §14
$4k+3$	ω_{real}	table 1, §14

In these tables, n denotes an arbitrary nonnegative integer:

Example of matrix M that satisfies $\gamma_a M = M\gamma_a^T$ for all a		
d	M	references
$4n$	ω_{antisym}	table 2, §14
$4n+1$	ω_{antisym}	table 2, §14
$4n+2$	ω_{sym}	table 2, §14
$4n+3$	**none**	eqn (5), §8

Example of matrix M that satisfies $\gamma_a M = -M\gamma_a^T$ for all a		
d	M	references
$4n$	ω_{sym}	table 2, §14
$4n+1$	**none**	eqn (5), §8
$4n+2$	ω_{antisym}	table 2, §14
$4n+3$	ω_{sym}	table 2, §14

In these tables, m denotes the number of Dirac matrices whose square is $-I$:

Example of matrix M that satisfies $\gamma_a M = M\gamma_a^\dagger$ for all a		
d, m	M	references
even, even	ω_{antiherm}	§14
even, odd	ω_{herm}	§14
odd, even	ω_{antiherm}	§14
odd, odd	**none**	§8

Example of matrix M that satisfies $\gamma_a M = -M\gamma_a^\dagger$ for all a		
d, m	M	references
even, even	ω_{herm}	§14
even, odd	ω_{antiherm}	§14
odd, even	**none**	§8
odd, odd	ω_{herm}	§14

16 Specialization to lorentzian signature

For every Clifford algebra Cliff(p, m), section 8 summarized when various intertwiners exist. This section specializes those results to lorentzian signatures, which are signatures (p, m) in which p and m are both nonzero and one of them is equal to 1.

When the signature is lorentzian, the relationships (5) and (8) reduce to these:

	mostly-minus convention	mostly-plus convention
$d = 4n$	$\omega^2 = -I$	$\omega^2 = -I$
$d = 4n + 1$	$\omega^2 = I$	$\omega^2 = -I$
$d = 4n + 2$	$\omega^2 = I$	$\omega^2 = I$
$d = 4n + 3$	$\omega^2 = -I$	$\omega^2 = I$

Using these relationships, and using the notation that was introduced at the beginning of section 8, the results that were shown in that section reduce to these when the signature is lorentzian:

signature convention	d	γ is equivalent to	γ is not equivalent to
either	odd		$-\gamma$
mostly minus	$4n + 1$	γ^*	$-\gamma^*$
mostly minus	$4n + 3$	$-\gamma^*$	γ^*
mostly plus	$4n + 1$	$-\gamma^*$	γ^*
mostly plus	$4n + 3$	γ^*	$-\gamma^*$
either	$4n + 1$	γ^T	$-\gamma^T$
either	$4n + 3$	$-\gamma^T$	γ^T
mostly minus	odd	$\gamma^\dagger$	$-\gamma^\dagger$
mostly plus	odd	$-\gamma^\dagger$	$\gamma^\dagger$

Explicit intertwiners may be inferred from the tables in section 15.

17 References

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Article **03910** (https://cphysics.org/article/03910):
"Clifford Algebra, also called Geometric Algebra" (version 2023-05-08)

Article **86175** (https://cphysics.org/article/86175):
"Matrix Representations of Clifford Algebras" (version 2023-06-02)