Matrix Representations of Clifford Algebras

Randy S

Abstract Article 03910 introduced **Clifford algebra**, which can be used to construct the **spin group** (article 08264), a special double cover of the part of the group of Lorentz transformations that is generated by pairs of reflections. Clifford algebras admit matrix representations in which each basis vector is represented by a **Dirac matrix**. This article shows how to construct matrix representations for Clifford algebras of any signature in any number of dimensions.

Contents

1 Clifford algebra: review and notation

Let V be a d-dimensional vector space over $\mathbb{R}$, the field of real numbers. Let g be a symmetric bilinear form[1] on V. In this article, g is always nondegenerate, which means that V has a basis $\mathbf{e}_1, ..., \mathbf{e}_d$ for which

$$g(\mathbf{e}_j, \mathbf{e}_k) = \begin{cases} \pm 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are called **orthogonal** if $g(\mathbf{a}, \mathbf{b}) = 0$. A nonzero vector that is orthogonal to itself, $g(\mathbf{v}, \mathbf{v}) = 0$, will be called **self-orthogonal**.

The **signature** of g is (p, m), where p and m are the numbers of plus-signs and minus-signs, respectively,[2] among the quantities $g(\mathbf{e}_k, \mathbf{e}_k)$. Clearly $p + m = d$. The signature is called **euclidean** if p or m is zero. If both are nonzero and either p or m is equal to 1, then it's called **lorentzian**.

The **Clifford algebra**[1] $\text{Cliff}(V, g)$ is the largest[3] associative algebra generated by V that has a unit 1 (an identity element for multiplication) and that satisfies[4]

$$\mathbf{v}^2 = g(\mathbf{v}, \mathbf{v}) \tag{2}$$

for all $\mathbf{v} \in V$. The product in this algebra is called the **Clifford product** and will be denoted by juxtaposition, so the Clifford product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted $\mathbf{ab}$. In equation (2), $\mathbf{v}^2$ is an abbreviation for the Clifford product of $\mathbf{v}$ with itself. The Clifford algebra $\text{Cliff}(V, g)$ is also commonly denoted $\text{Cliff}(p, m)$, where (p, m) is the signature of g. Each notation has its own advantages, and this article uses both of them. The notation $\text{Cliff}(V, g)$ will be used when emphasizing that the algebra is generated by the vector space V, and the notation $\text{Cliff}(p, m)$ will be used when emphasizing that the signature is (p, m).

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[2] Sometimes the word *signature* is used for the difference $p - m$, but in this article it always means the pair (p, m).

[3] Here, *largest* is an allusion to a *universal property*. Roughly, it means that the only relationships in the algebra are those that can be derived from $\mathbf{v}^2 = g(\mathbf{v}, \mathbf{v})$ together with the general rules of associative algebra.

[4] Sometimes the opposite sign convention is used ($\mathbf{v}^2 = -g(\mathbf{v}, \mathbf{v})$). Beware of this when comparing signature-dependent results from different sources.

2 The concept of a matrix representation

Let W be a vector space. If Ω is an associative algebra, such as a Clifford algebra, then a **representation** of Ω on W is a homomorphism from the algebra Ω into the algebra of linear transformations of W. In this article, W will always be a finite-dimensional vector space over either $\mathbb{R}$ or $\mathbb{C}$, and the representation will be called a "representation over $\mathbb{R}$" or "representation over $\mathbb{C}$," respectively. The dimension N of the vector space W is called the **dimension** of the representation.

After choosing a basis for W, every linear transformation of W may be represented as a square matrix with components in either $\mathbb{R}$ or $\mathbb{C}$, respectively, so a representation may also be defined as something that assigns each element $A \in \Omega$ to a matrix $\gamma(A)$ satisfying these conditions:[5]

$$\begin{gathered}\gamma(AB) = \gamma(A)\gamma(B) \\ \gamma(A+B) = \gamma(A) + \gamma(B) \\ \gamma(rA) = r\gamma(A)\end{gathered}$$

for all $A, B \in \Omega$ and for all $r \in \mathbb{R}$. To emphasize this perspective, a representation may be called a **matrix representation**. Each matrix in the representation has order N (size $N \times N$), where N is the dimension of W.

Two matrix representations are **equivalent** to each other if one of them can be converted to the other one by choosing a different basis for W.

[5] This is what *homomorphism* means.

3 Representations over $\mathbb{R}$ and over $\mathbb{C}$

This article is about matrix representations of Clifford algebras $\text{Cliff}(V, g)$, where V is a vector space over $\mathbb{R}$. If A is an element of the Clifford algebra, then so is rA whenever r is a real number, but rA is undefined (or at least not in the Clifford algebra) when r is a non-real complex number.

Two different vector spaces will usually be in play at the same time:

- One of them is V, a vector space over $\mathbb{R}$ that generates the Clifford algebra.
- The other is W, a vector space W whose linear transformations host a representation of $\text{Cliff}(V, g)$. The vector space W may be over $\mathbb{R}$ or over $\mathbb{C}$. In either case, the dimension of W may be different than the dimension of V.

When the vector space W is over $\mathbb{C}$, the matrix $\gamma(A)$ representing $A \in \text{Cliff}(V, g)$ may have complex components, but the matrix $r\gamma(A)$ does not necessarily represent any element of the algebra unless r is a real number. If r is a real number, then $r\gamma(A) = \gamma(rA)$.

The vector space W will be over $\mathbb{C}$ except where specified otherwise, but sections 18-22 will focus on **real representations** in which W is a vector space over $\mathbb{R}$, so that the components of each matrix must be real numbers.[6]

This article does not consider representations in which the components of a matrix may be quaternions.[7] In this article, W is always a vector space over $\mathbb{R}$ or $\mathbb{C}$, so the components of each matrix are real numbers or complex numbers, respectively.

[6] In physics, the name **complex representation** is often reserved for a representation that is not equivalent (by a change of basis) to any real representation of the same dimension, so that *real representation* and *complex representation* are mutually exclusive.

[7] Allowing quaternions as components can be useful because of **Wedderburn's theorem**, which says that every *simple* algebra is isomorphic to a full matrix algebra over a division algebra (Martin (2016), theorem 2.2.6). An algebra is **simple** if it doesn't have any nontrivial ideals (Martin (2016), page 62). The quaternion algebra is an example of a division algebra.

4 Examples

The Clifford algebra Cliff$(2,0)$ is generated by two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfying

$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \qquad\qquad \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1.$$

Every element of this algebra has the form

$$r_0 + r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_1\mathbf{e}_2$$

with real coefficients r_0, r_1, r_2, r_3. This algebra has a matrix representation γ with

$$\gamma(\mathbf{e}_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad \gamma(\mathbf{e}_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this representation, each matrix has only real components.

The Clifford algebra Cliff$(0,2)$ is is generated by two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfying

$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \qquad\qquad \mathbf{e}_1^2 = \mathbf{e}_2^2 = -1.$$

As before, every element of this algebra has the form

$$r_0 + r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_1\mathbf{e}_2$$

with real coefficients r_0, r_1, r_2, r_3. This algebra has a matrix representation γ with

$$\gamma(\mathbf{e}_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad\qquad \gamma(\mathbf{e}_2) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

This representation uses some matrices with complex components, but it doesn't use all of them. The matrix

$$-i\gamma(\mathbf{e}_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

does not represent any element of this Clifford algebra.

5 Dirac matrices

Equation (2) implies[8]

$$\mathbf{ab} + \mathbf{ba} = 2g(\mathbf{a}, \mathbf{b}), \tag{3}$$

so if γ is a matrix representation of a Clifford algebra Cliff(V, g), then the matrices representing vectors $\mathbf{a}, \mathbf{b} \in V$ must satisfy

$$\boxed{\gamma(\mathbf{a})\gamma(\mathbf{b}) + \gamma(\mathbf{b})\gamma(\mathbf{a}) = 2g(\mathbf{a}, \mathbf{b})\, I,} \tag{4}$$

where I is the identity matrix. Let $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d$ be a set of vectors that are mutually orthogonal and normalized so that $g(\mathbf{e}_k, \mathbf{e}_k) = \pm 1$. The matrix $\gamma(\mathbf{e}_k)$ that represents $\mathbf{e}_k$ will be called a **Dirac matrix**, and the abbreviation

$$\gamma_k \equiv \gamma(\mathbf{e}_k)$$

will be used. Equation (4) implies

$$\boxed{\gamma_j\gamma_k + \gamma_k\gamma_j = 2g(\mathbf{e}_j, \mathbf{e}_k)I.} \tag{5}$$

The Clifford algebra is generated by its vectors, so if we have a set of matrices that satisfy equation (5), then using them as Dirac matrices gives a representation of the whole Clifford algebra.

[8]To deduce this, set $\mathbf{v} = \mathbf{a} + \mathbf{b}$ in (2) and use $g(\mathbf{b}, \mathbf{a}) = g(\mathbf{a}, \mathbf{b})$.

6 Faithful representations, irreducible representations

Let Ω be an algebra, and let γ be a representation of Ω by linear transformations of a vector space W.

- The representation is called **faithful** if distinct elements $A \neq B$ of the algebra are always represented by distinct matrices, $\gamma(A) \neq \gamma(B)$.

- The representation is called **reducible** if W has a proper subspace[9] that is self-contained under all of the linear transformations that represent elements of Ω. If W doesn't have any such subspace, then the representation is called **irreducible**. Intuitively: a reducible representation is one that contains a smaller representation inside of itself, and an irreducible representation is one that does not.

- The representation is called **completely reducible** if it is a direct sum of irreducible representations. This means that we can choose a basis for W in which every matrix $\gamma(A)$ is block-diagonal, and each block constitutes an irreducible representation. According to this definition, an irreducible representation is completely reducible (in a trivial way). Every finite-dimensional representation of $\text{Cliff}(p, m)$ is completely reducible.[10]

This article focuses on representations that have the minimum possible nonzero dimension, which implies that they are irreducible.

[9] A *proper subspace* is any subspace that is not all of W but that includes more than just the zero vector.

[10] This can be deduced from the fact that $\text{Cliff}(p, m)$ is **semisimple** (Ablamowicz (2016), section 3, with the understanding that *simple* is a special case of *semisimple*), combined with the fact that every finite-dimensional representation of a semisimple algebra is completely reducible (Etingof *et al* (2011), proposition 2.16).

7 The minimum dimension

For any positive integer d, define

$$N_{\mathbb{C}}(d) \equiv \begin{cases} 2^{d/2} & \text{if } d \text{ is even,} \\ 2^{(d-1)/2} & \text{if } d \text{ is odd.} \end{cases} \tag{6}$$

For every signature (p, q), section 11 will construct a representation of $\text{Cliff}(p, m)$ on a complex vector space with dimension $N_{\mathbb{C}}(p + m)$. This section shows that a nontrivial[11] representation over $\mathbb{C}$ cannot be smaller than this.

Consider the complexified Clifford algebra[12]

$$\text{Cliff}_{\mathbb{C}}(p + m) = \text{Cliff}(p, m) \otimes \mathbb{C}, \tag{7}$$

which is obtained from $\text{Cliff}(p, m)$ by allowing complex coefficients instead of only real coefficients. This doesn't change the minimum dimension among representations over $\mathbb{C}$, because any representation of $\text{Cliff}(p, m)$ on a complex vector space gives a representation of $\text{Cliff}_{\mathbb{C}}(p+m)$ of the same dimension just by allowing complex coefficients in the algebra. The complexified algebra depends only on $p + m$.

The structure of these algebras is described by the isomorphisms[13]

$$\begin{aligned} \text{Cliff}(2n) &\simeq M_{\mathbb{C}}(2^n) \\ \text{Cliff}(2n + 1) &\simeq M_{\mathbb{C}}(2^n) \oplus M_{\mathbb{C}}(2^n), \end{aligned}$$

where $M_{\mathbb{C}}(k)$ is the algebra of all matrices of size $k \times k$ with components in $\mathbb{C}$. The minimum dimension of a nontrivial representation of $M_{\mathbb{C}}(k)$ on a complex vector space is k.[14] Altogether, this implies that the minimum dimension of a nontrivial representation of $\text{Cliff}(p, m)$ on a complex vector space is $N_{\mathbb{C}}(p + m)$.

[11] The representation that maps every element of the algebra to zero is called the **trivial representation**. Its dimension is zero.

[12] A more careful notation is $\text{Cliff}(p, m) \otimes_{\mathbb{R}} \mathbb{C}$. The subscript $\mathbb{R}$ means that $(rA) \otimes_{\mathbb{R}} B = A \otimes_R (rB)$ for all $r \in \mathbb{R}$.

[13] Figueroa-O'Farrill (2015), section 4, table 3. Equations (2.7.4a)-(2.7.4b) in Benn and Tucker (1989) are meant to say this, too, but equation (2.7.4b) has a significant typographical error.

[14] Etingof *et al* (2011), theorem 2.6

8 Canonical basis: definition

Section 9 will construct the *regular representation* of $\text{Cliff}(V, g)$. The construction refers to a *canonical basis* for the algebra $\text{Cliff}(V, g)$. This section explains what that means.

First, let $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d$ be a list of vectors in V that are orthogonal to each other and that satisfy $\mathbf{e}_k^2 = \pm 1$. This will be called a **canonical basis for $\boldsymbol{V}$**. The quantities

$$\begin{aligned} &1 \\ &\mathbf{e}_j \\ &\mathbf{e}_j\mathbf{e}_k \quad \text{with } j < k \\ &\mathbf{e}_j\mathbf{e}_k\mathbf{e}_\ell \quad \text{with } j < k < \ell \\ &\text{...and so on...} \end{aligned} \tag{8}$$

are all linearly independent, and every element of the Clifford algebra is a linear combination of these.[15] The list of quantities (8) will be called a **canonical basis for the Clifford algebra**.

The vector factors in each product (8) may be written in any order, with an overall minus sign if the permutation is odd, because equation (3) implies that orthogonal vectors anticommute with each other:

$$\mathbf{ab} = -\mathbf{ba} \qquad \text{if } g(\mathbf{a}, \mathbf{b}) = 0. \tag{9}$$

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9 The regular representation

This section constructs a faithful matrix representation of Cliff(p, m). The dimension of this representation is greater than $N_{\mathbb{C}}(p+m)$, but this section constructs it anyway to introduce some of the ingredients that will also be used in section 11 to construct representations that have the minimum dimension $N_{\mathbb{C}}(p+m)$.

Define $d \equiv p+m$, and let $X_1, X_2, X_3, ...$ be the elements of the canonical basis[16] for the Clifford algebra. For each X_j, let $\Phi(X_j)$ be the column matrix with 2^d components, all of which are zero except the jth component, which is 1:

$$\Phi(X_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad \Phi(X_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \qquad \Phi(X_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \qquad \text{and so on.}$$

Then $\Phi(A)$ may be defined for all elements A of the Clifford algebra using

$$\Phi\left(\sum_k r_k X_k\right) \equiv \sum_k r_k \Phi(X_k).$$

For each A in the Clifford algebra, let $\gamma(A)$ be the $N \times N$ matrix defined by

$$\gamma(A)\Phi(X_k) = \Phi(AX_k).$$

These matrices satisfy

$$\gamma(A)\gamma(B) = \gamma(AB)$$

for all elements A, B of the Clifford algebra, so this gives a real representation of dimension 2^d called the **regular representation**.

[16] This was defined in section 8.

10 Is the regular representation reducible?

The algebra Cliff$(2,0)$ is generated by two vectors $\mathbf{e}_1, \mathbf{e}_2$ satisfying

$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \qquad\qquad \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1.$$

In this case, the regular representation has dimension 4. The regular representation is reducible: the subspace of dimension 2 spanned by

$$\Phi\big((1 \pm \mathbf{e}_1)(1 + \mathbf{e}_2)\big) = \Phi(1) + \Phi(\mathbf{e}_2) \pm (\Phi(\mathbf{e}_1) + \Phi(\mathbf{e}_1\mathbf{e}_2))$$

is self-contained under the action of $\gamma(A)$ for all $A \in$ Cliff$(2,0)$.[17] This gives a representation of dimension $N_{\mathbb{C}}(2) = 2$ that is still faithful and still real.

The algebra Cliff$(0,2)$ is generated by two vectors $\mathbf{e}_1, \mathbf{e}_2$ satisfying

$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \qquad\qquad \mathbf{e}_1^2 = \mathbf{e}_2^2 = -1.$$

Again, the regular representation has dimension 4. This representation does not contain any smaller representation over $\mathbb{R}$, but it can be re-packaged as a smaller representation over $\mathbb{C}$, because the vector space of dimension 2 over $\mathbb{C}$ spanned by

$$\Phi(1) + i\Phi(\mathbf{e}_2) \pm (\Phi(\mathbf{e}_1) + i\Phi(\mathbf{e}_1\mathbf{e}_2))$$

is self-contained under the action of $\gamma(A)$ for all $A \in$ Cliff$(0,2)$. This representation is still faithful, but it's not real: it is not equivalent to any real representation of dimension 2.

[17]Checking this for $\gamma(\mathbf{e}_1)$ and $\gamma(\mathbf{e}_2)$ is sufficient, because these generate the whole algebra.

11 Irreducible representations over $\mathbb{C}$

Section 9 constructed the regular representation of $\text{Cliff}(V, g)$. The regular representation is a faithful real representation with dimension 2^d, where d is the dimension of V. This section explains how to construct a representation of dimension $N_{\mathbb{C}}(d)$ over $\mathbb{C}$. A nontrivial representation over $\mathbb{C}$ cannot be smaller than this.[18]

As in section 7, let $\text{Cliff}_{\mathbb{C}}(V)$ be the algebra obtained from $\text{Cliff}(V, g)$ by allowing complex coefficients. The construction described here will give a representation of $\text{Cliff}_{\mathbb{C}}(V)$ on a complex vector space W. Restricting that representation to the subset $\text{Cliff}(V, g) \subset \text{Cliff}_{\mathbb{C}}(V)$ gives a representation of $\text{Cliff}(V, g)$.

Let $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_d \in V$ be a canonical basis for V.[19] Let n be the largest integer that is $\leq d/2$, and for each $\ell \in \{1, 2, ..., n\}$, use the abbreviation Γ_ℓ for either $\mathbf{e}_{2\ell-1}\mathbf{e}_{2\ell}$ or $i\mathbf{e}_{2\ell-1}\mathbf{e}_{2\ell}$, whichever makes $\Gamma_\ell^2 = 1$. If d is odd, then also define Γ_{n+1} to be either $\mathbf{e}_d$ or $i\mathbf{e}_d$, whichever makes $\Gamma_{n+1}^2 = 1$. All of these quantities Γ_ℓ commute with each other. Now define $P \equiv \prod_\ell (1 + \Gamma_\ell)$, where the product is over all n or $n + 1$ values of ℓ when d is even or odd, respectively. The fact that each Γ_ℓ squares to 1 implies

$$\Gamma_\ell P = P. \tag{10}$$

Let X_1, X_2, X_3, ... be the elements of a canonical basis for the Clifford algebra[19] $\text{Cliff}(V, g)$. Using only the n vectors $\mathbf{e}_{2\ell}$, we can construct 2^n of the elements in the canonical basis. Every other element of the canonical basis is proportional to one of these times one or more of the Γ_ℓs.[20] Equation (10) then implies that of the 2^d quantities $X_k P$, only 2^n of them are linearly independent (over $\mathbb{C}$).

Now define Φ as in section 9. The previous paragraph deduced that only 2^n of the vectors $\Phi(X_k P)$ are linearly independent (over $\mathbb{C}$), so this gives a representation of $\text{Cliff}_{\mathbb{C}}(V)$ on a complex vector space W of dimension $2^n = N_{\mathbb{C}}(d)$. Restricting to the subset $\text{Cliff}(V, g) \subset \text{Cliff}_{\mathbb{C}}(V)$ then gives a representation of $\text{Cliff}(V, g)$ on W.

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[18] Section 7

[19] Section 8

[20] To confirm this, use $\mathbf{e}_{2\ell-1} \propto \mathbf{e}_{2\ell}\Gamma_\ell$.

12 Example: Cliff$(2,0)$

The algebra Cliff$(2,0)$ is generated by vectors $\mathbf{e}_1, \mathbf{e}_2$ satisfying $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$ and $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$. The recipe described in section 11 gives

$$P = 1 + i\mathbf{e}_1\mathbf{e}_2.$$

The quantities that were denoted $X_k P$ in section 11 are P, $\mathbf{e}_1 P$, $\mathbf{e}_2 P$, and $\mathbf{e}_1\mathbf{e}_2 P$. These can all be written as QP or as $\mathbf{e}_2 QP$ for some Q that belongs to the algebra generated by $\Gamma_1 \equiv i\mathbf{e}_1\mathbf{e}_2$, so the identity $\Gamma_1 P = P$ implies that only two of $X_k P$s are linearly independent over $\mathbb{C}$. The two quantities P and $\mathbf{e}_2 P$ are linearly independent, so the two vectors $\Phi(P)$ and $\Phi(\mathbf{e}_2 P)$ span a representation of Cliff$(2,0)$. To express this as a matrix representation, write

$$\Phi(P) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Phi(\mathbf{e}_2 P) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Use the general relationship $\gamma(A)\Phi(B) = \Phi(AB)$ to confirm that this gives

$$\gamma(\mathbf{e}_1) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \gamma(\mathbf{e}_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This representation uses complex components, but we can get a real representation (a representation in which each matrix has only $\mathbb{R}$-valued components) by using

$$\Phi(P + i\mathbf{e}_2 P) = \Phi(P) + i\Phi(\mathbf{e}_2 P)$$
$$\Phi(iP + \mathbf{e}_2 P) = i\Phi(P) + \Phi(\mathbf{e}_2 P)$$

as the basis vectors instead of $\Phi(P)$ and $\Phi(\mathbf{e}_2 P)$. If we write

$$\Phi(P + i\mathbf{e}_2 P) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Phi(iP + \mathbf{e}_2 P) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then the matrices representing $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$\gamma(\mathbf{e}_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \gamma(\mathbf{e}_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

13 Example: Cliff$(3,0)$

The algebra Cliff$(3,0)$ is generated by vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ that satisfy $\mathbf{e}_k^2 = 1$ and that anticommute with each other. The recipe described in section 11 gives

$$P = (1 + i\mathbf{e}_1\mathbf{e}_2)(1 + \mathbf{e}_3).$$

The quantities that were denoted $X_k P$ in section 11 are

$$\begin{matrix} P & \mathbf{e}_1 P & \mathbf{e}_2 P & \mathbf{e}_3 P \\ \mathbf{e}_1\mathbf{e}_2 P & \mathbf{e}_2\mathbf{e}_3 P & \mathbf{e}_3\mathbf{e}_1 P & \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 P. \end{matrix}$$

These can all be written as[21] QP or as $\mathbf{e}_2 QP$ for some Q that belongs to the algebra generated by the quantities $\Gamma_1 \equiv i\mathbf{e}_1\mathbf{e}_2$ and $\Gamma_2 \equiv \mathbf{e}_3$, so the identities

$$\Gamma_1 P = P \qquad\qquad \Gamma_2 P = P$$

imply that only two of $X_k P$s are linearly independent over $\mathbb{C}$. The two quantities P and $\mathbf{e}_2 P$ are linearly independent, so the two vectors $\Phi(P)$ and $\Phi(\mathbf{e}_2 P)$ span a representation of Cliff$(3,0)$. To express this as a matrix representation, write

$$\Phi(P) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad \Phi(\mathbf{e}_2 P) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Use the general relationship $\gamma(A)\Phi(B) = \Phi(AB)$ to confirm that this gives

$$\gamma(\mathbf{e}_1) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad\qquad \gamma(\mathbf{e}_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad\qquad \gamma(\mathbf{e}_3) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this representation, every 2×2 matrix with complex components represents an element of Cliff$(3,0)$, so this is not equivalent to any real representation of the same dimension, in contrast to the example in section 12.

[21] Checking this for $\mathbf{e}_k$ is sufficient, because the three basis vectors $\mathbf{e}_k$ generate all of Cliff$(3,0)$. The relationships $\mathbf{e}_1 = i\mathbf{e}_2\Gamma_1$ and $\mathbf{e}_3 = \Gamma_2$ show that each of the three basis vectors $\mathbf{e}_k$ may be written as Q or $\mathbf{e}_2 Q$ with Q in the algebra generated by the Γ_ℓs.

14 Example: Cliff$(1, 3)$

The algebra Cliff$(1, 3)$ is generated by vectors $\mathbf{e}_k$ with $k \in \{1, 2, 3, 4\}$ that all anticommute with each other, normalized so that $\mathbf{e}_1^2 = -1$ and $\mathbf{e}_k^2 = 1$ if $k \in \{2, 3, 4\}$. The recipe described in section 11 gives

$$P = (1 + \mathbf{e}_1\mathbf{e}_2)(1 + i\mathbf{e}_3\mathbf{e}_4). \tag{11}$$

The 16 quantities that were denoted $X_k P$ in section 11 may all be written in one of the forms[22]

$$QP \qquad \mathbf{e}_2 QP \qquad \mathbf{e}_4 QP \qquad \mathbf{e}_2\mathbf{e}_4 QP$$

with Q in the algebra generated by the quantities $\Gamma_1 \equiv \mathbf{e}_1\mathbf{e}_2$ and $\Gamma_2 \equiv i\mathbf{e}_3\mathbf{e}_4$. The identities $\Gamma_1 P = P$ and $\Gamma_2 P = P$ then imply that only four of the 16 quantities $X_k P$ are linearly independent over $\mathbb{C}$, so the vectors

$$\Phi(P) \qquad \Phi(\mathbf{e}_2 P) \qquad \Phi(\mathbf{e}_4 P) \qquad \Phi(\mathbf{e}_2\mathbf{e}_4 P)$$

span a representation of dimension 4. The components of each matrix $\gamma(\mathbf{e}_k)$ may be worked out by writing

$$\Phi(P) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Phi(\mathbf{e}_2 P) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \Phi(\mathbf{e}_4 P) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \Phi(\mathbf{e}_2\mathbf{e}_4 P) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and using the general relationship $\gamma(A)\Phi(B) = \Phi(AB)$.

[22] To confirm this, use $\mathbf{e}_1 = \mathbf{e}_2\Gamma_1$ and $\mathbf{e}_3 = -i\mathbf{e}_4\Gamma_2$.

15 Another way to construct representations

This section describes another way[23] to construct a representation of Cliff(p, m) on a complex vector space of dimension $N_{\mathbb{C}}(p+m)$.[24] In this construction, each Dirac matrix is manifestly either hermitian ($\gamma_k^\dagger = \gamma_k$) or antihermitian ($\gamma_k^\dagger = -\gamma_k$), according to whether $\gamma_k^2 = I$ or $\gamma_k^2 = -I$, respectively.[25]

Start with the 2×2 matrices

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{12}$$

These satisfy

$$\begin{aligned} XY &= -YX = -Z \\ YZ &= -ZY = -X \\ ZX &= -XZ = Y \end{aligned} \tag{13}$$

and

$$X^2 = I \qquad Y^2 = -I \qquad Z^2 = I \tag{14}$$

where I is the identity matrix. We can use these matrices to construct other (larger) matrices using the **tensor product**. The tensor product of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with an $N \times N$ matrix B is the $2N \times 2N$ matrix $A \otimes B$ defined by

$$A \otimes B \equiv \begin{bmatrix} aB & bB \\ cB & dB \end{bmatrix},$$

[23]This method is also reviewed in Appendix B of Polchinski (1998).

[24]Recall (section 7) that $N_{\mathbb{C}}(p+m)$ is the smallest possible dimension of a complex vector space that can host a nontrivial representation of Cliff(p, m).

[25]$M^\dagger$ denotes the conjugate transpose of a matrix M.

where each entry on the right-hand side is an $N \times N$ block. Using this, we may iteratively define the tensor product of any number of 2×2 matrices, like this:

$$A \otimes B \otimes C \equiv A \otimes (B \otimes C).$$

The generalization to matrices of arbitrary order should be clear. The tensor product satisfies

$$(A \otimes B \otimes C)(A' \otimes B' \otimes C') = (AA') \otimes (BB') \otimes (CC'), \tag{15}$$

with an obvious generalization to any number of factors.

To construct a representation of Cliff(p, m) when $d \equiv p + m$ is even, define n by $d = 2n$, and define the matrices

$$\begin{aligned}
\gamma_1 &= \epsilon_1 X \otimes I \otimes I \otimes I \otimes \cdots \\
\gamma_2 &= \epsilon_2 Y \otimes I \otimes I \otimes I \otimes \cdots \\
\gamma_3 &= \epsilon_3 Z \otimes X \otimes I \otimes I \otimes \cdots \\
\gamma_4 &= \epsilon_4 Z \otimes Y \otimes I \otimes I \otimes \cdots \\
\gamma_5 &= \epsilon_5 Z \otimes Z \otimes X \otimes I \otimes \cdots \\
\gamma_6 &= \epsilon_6 Z \otimes Z \otimes Y \otimes I \otimes \cdots
\end{aligned} \tag{16}$$

and so on up to γ_{2n}. The number of factors on each line is n. Each ϵ_k is either 1 or i, chosen to control the sign in $\gamma_k^2 = \pm$(identity matrix). Equations (13)-(14) may be used to confirm that the matrices (16) satisfy (5), so this gives a representation of Cliff(p, m) of dimension $2^n = N_{\mathbb{C}}(2n)$, using matrices with complex components.

When d is odd, define n by $d = 2n + 1$, construct the first $2n$ Dirac matrices as shown above, and define the remaining one by

$$\gamma_{2n+1} \equiv \epsilon_{2n+1} \gamma_1 \gamma_2 \cdots \gamma_{2n} \tag{17}$$

with ϵ_{2n+1} chosen to control the sign in $\gamma_{2n+1}^2 = \pm$(identity matrix). The fact that the γs in (16) all anticommute with each other implies that they also anticommute with (17), so this gives a representation of Cliff(p, m) of the same dimension $2^n = N_{\mathbb{C}}(2n + 1)$.

16 Representations with a custom form

This section shows that if the dimension of V is even, then Cliff(V, g) has a minimum-size representation in which all of the diagonal elements of every Dirac matrix are zero.

Start with Dirac matrices that generate a representation of Cliff(p, m):

$$\bar{\gamma}_1,\ \bar{\gamma}_2, ...,\ \bar{\gamma}_d \tag{18}$$

with $d \equiv p + m$. The $d + 1$ matrices[26]

$$\gamma_k = X \otimes \bar{\gamma}_k = \left[\begin{array}{c|c} 0 & \bar{\gamma}_k \\ \hline \bar{\gamma}_k & 0 \end{array}\right] \qquad \text{for } k \in \{1, 2, ..., d\} \tag{19}$$

$$\gamma_{d+1} \propto Y \otimes I = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right] \tag{20}$$

all anticommute with each other. The proportionality factor in (20) can be chosen so that γ_{d+1} squares to $\pm$(identity matrix) with the desired sign, so this gives a complete set of Dirac matrices for Cliff$(p + 1, m)$ or Cliff$(p, m + 1)$, depending on the proportionality factor.

If the matrices (18) each have order 2^n, then the matrices (19)-(20) each have order 2^{n+1}. If d is odd and $2^n = N_{\mathbb{C}}(d)$, then $2^{n+1} = N_{\mathbb{C}}(d + 1)$, so in this case the $d + 1$ Dirac matrices (19)-(20) have the minimum possible size.

In this representation, the quantity that will be denoted Γ in section 24 is a diagonal matrix. This is convenient for separating an irreducible representation of the full Clifford algebra into two irreducible representations of the even subalgebra, as described in section 24.

[26] In these equations, 0 and I stand for the zero matrix and the identity matrix, respectively, of the same size as the $\bar{\gamma}$'s.

17 Representations with another custom form

This section shows that every Cliff(p,m) has a minimum-size representation in which only one Dirac matrix has nonzero elements on the diagonal.

Start with Dirac matrices that generate a representation of Cliff(p,m):

$$\bar{\gamma}_1,\ \bar{\gamma}_2, ...,\ \bar{\gamma}_d \tag{21}$$

with $d \equiv p+m$. The $d+1$ matrices

$$\gamma_k = X \otimes \bar{\gamma}_k = \left[\begin{array}{c|c} 0 & \bar{\gamma}_k \\ \hline \bar{\gamma}_k & 0 \end{array}\right] \qquad \text{for } k \in \{1,2,...,d\} \tag{22}$$

$$\gamma_{d+1} \propto Z \otimes I = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right] \tag{23}$$

all anticommute with each other. The proportionality factor in (23) can be chosen so that γ_{d+1} squares to $\pm$(identity matrix) with the desired sign, so this gives a complete set of Dirac matrices for Cliff$(p+1,m)$ or Cliff$(p,m+1)$, depending on the proportionality factor.

If the matrices (21) each have order 2^n, then the matrices (22)-(23) each have order 2^{n+1}. If d is odd and $2^n = N_{\mathbb{C}}(d)$, then $2^{n+1} = N_{\mathbb{C}}(d+1)$, so in this case the $d+1$ Dirac matrices (22)-(23) have the minimum possible size.

When d is odd, the new Dirac matrix defined by $\gamma_{d+2} \propto \gamma_1\gamma_2\cdots\gamma_{d+1}$ anticommutes with all of the others, and it can be normalized so that it squares to $\pm$(identity matrix) with the desired sign. This gives a complete set of Dirac matrices for Cliff$(p+2,m)$, Cliff$(p+1,m+1)$, or Cliff$(p,m+2)$, still using matrices with the minimum possible size, and γ_{d+1} is still the only one with nonzero elements on the diagonal.

Representations of this form can be convenient when the signature is lorentzian, because then we can take the diagonal Dirac matrix to be the one corresponding to the timelike dimension.

18 Real representations

The constructions described in sections 11 and 15 give representations of Cliff(p,m) of dimension $N_{\mathbb{C}}(p+m)$, which is the smallest possible dimension for a nontrivial irreducible representation.[27] In those constructions, the matrices were allowed to have components in $\mathbb{C}$, even though the algebra being represented – namely Cliff(p,m) – requires all coefficients to be in $\mathbb{R}$. Those representations may turn out to be real (with all matrix components in $\mathbb{R}$) for some signatures but not for others. Real representations of larger dimension exist for any signature,[28] but here we are interested in real representations of the smaller dimension $N_{\mathbb{C}}(p+m)$.

Let $N_{\mathbb{R}}(p,m)$ denote the minimum dimension among real representations of Cliff(p,m). Clearly $N_{\mathbb{R}}(p,m) \leq 2N_{\mathbb{C}}(p+m)$, because any representation of dimension N over $\mathbb{C}$ may be regarded as[29] a representation of dimension $2N$ over $\mathbb{R}$. The interesting question is, when is $N_{\mathbb{R}}(p,m)$ less than $N_{\mathbb{C}}(p,m)$? This table summarizes the answer:[30]

$(p-m)$ mod 8	$N_{\mathbb{R}}(p,m)$
0,1,2	$N_{\mathbb{C}}(p+m)$
3,4,5,6,7	$2N_{\mathbb{C}}(p+m)$

Sections 19-21 will describe a few methods that may be used to construct real representations of various signatures iteratively, starting with $(p,m) = (1,0)$ and incrementing $p+m$ with each step. Using those methods, real representations of dimension $N_{\mathbb{C}}(p+m)$ may be constructed for every signature (p,m) in which such representations exist. This will be illustrated in section 22.

[27]Section 7

[28]Section 9 constructed a real representation of dimension 2^{p+m} for any Cliff(p,m).

[29]If $\mathbf{v}$ is a nonzero vector, then $\mathbf{v}$ and $i\mathbf{v}$ are linearly independent over $\mathbb{R}$: the only pair of real numbers a,b for which $a\mathbf{v} + bi\mathbf{v} = 0$ is $a = b = 0$.

[30]Benn and Tucker (1989), table 2.10 (The "dimension" column in that table is the dimension over $\mathbb{R}$, and their n is my d.)

19 Real representations, method 1

This section shows that if we start with a real representation of Cliff(p,m) with $p+m$ even, then we can get real representations of the algebras tabulated here (with n defined by $p+m=2n$) without increasing the dimension of the representation:

algebra	when it works
Cliff$(p+1,m)$	when $p+m$ is even and $n+m$ is even
Cliff$(p,m+1)$	when $p+m$ is even and $n+m$ is odd

To do this, suppose that we have Dirac matrices for a representation of Cliff(p,m) with $p+m=2n$:

$$\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{2n}. \tag{24}$$

Now define an additional Dirac matrix γ_{2n+1} by

$$\gamma_{2n+1}=\gamma_1\gamma_2\cdots\gamma_{2n}. \tag{25}$$

This matrix is clearly real if the matrices (24) are real. The matrices $\gamma_1,\gamma_2,\ldots,\gamma_{2n+1}$ generate a representation of another Clifford algebra. To determine its signature, use the fact that (25) squares to $(-1)^{n+m}$. To see this, note that it is a product of n mutually commuting bivectors $\gamma_j\gamma_{j+1}$. If each of the original $2n$ Dirac matrices squared to $+1$, then the square of each bivector would be -1. This gives a factor of $(-1)^n$. The other factor $(-1)^m$ accounts for the fact that m of the Dirac matrices actually square to -1 instead. Altogether, this gives the results tabulated above.

20 Real representations, method 2

This section shows that if we start with a real representation of Cliff(p, m) with $p + m$ odd, then we can get real representations of the algebras tabulated here by doubling the dimension of the representation:

algebra	when it works
Cliff$(p + 1, m)$	when $p + m$ is odd
Cliff$(p, m + 1)$	when $p + m$ is odd
Cliff$(m + 1, p)$	when $p + m$ is odd

To do this, define n by $p + m = 2n + 1$, and suppose that we have a real representation of dimension $2^n = N_{\mathbb{C}}(p, m)$ for the Clifford algebra Cliff(p, m). For any matrix M, let $M \otimes (p, m)$ denote the set of matrices $M \otimes \gamma_a$, where γ_a are the matrices in the original representation of Cliff(p, m).[31] Let I denote the identity matrix of order 2^n, and define X, Y, Z as in equation (12). The matrices

$$Z \otimes (p, m) \qquad\qquad X \otimes I$$

give a real representation of Cliff$(p + 1, m)$. The matrices

$$Z \otimes (p, m) \qquad\qquad Y \otimes I$$

give a real representation of Cliff$(p, m + 1)$. The matrices

$$Y \otimes (p, m) \qquad\qquad Z \otimes I,$$

give a real representation of Cliff$(m + 1, p)$. All of these real representations have dimension $2^{n+1} = N_{\mathbb{C}}(p + m + 1)$.

[31] The tensor product $\otimes$ is defined in section 15.

21 Real representations, method 3

This section shows that if we start with a real representation of Cliff(p, m), then we can get real representations of the algebras tabulated here without changing the dimension of the representation:

algebra	when it works
Cliff$(p+4, m-4)$	when $m \geq 4$
Cliff$(p-4, m+4)$	when $p \geq 4$

To do this, suppose we have a real representation in which the first four Dirac matrices

$$\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4$$

all square to $+1$. Then the matrix

$$M \equiv \gamma_1\gamma_2\gamma_3\gamma_4 \tag{26}$$

squares to $+1$, anti-commutes with each of the first four Dirac matrices, and commutes with each of the others. This implies that the new matrices

$$\bar{\gamma}_k \equiv \begin{cases} M\gamma_k & \text{if } k = 1, 2, 3, 4, \\ \gamma_k & \text{if } k \geq 5 \end{cases} \tag{27}$$

still anti-commute with each other, and the first four of them square to -1. So the new set of Dirac matrices $\bar{\gamma}$ has signature $(p-4, m+4)$.

Similarly, if we have a real representation in which the first four Dirac matrices square to -1, then equations (26)-(27) give a new set of Dirac matrices with signature $(p+4, m-4)$.

22 Real representations for $d = 2$ through $d = 11$

Define X, Y, Z as in equation (12). Start with the real representation of Cliff$(1,1)$ defined by

$$\gamma_1 = X \qquad \qquad \gamma_2 = Y.$$

and the real representation of Cliff$(2,0)$ defined by

$$\gamma_1 = X \qquad \qquad \gamma_2 = Z.$$

From those starting points, we can proceed like this:

- Applying method 1 to these representations of Cliff$(1,1)$ and Cliff$(2,0)$ gives (in either case) a real representation of Cliff$(2,1)$ using matrices of order 2.

- Applying method 2 to this representation of Cliff$(2,1)$ gives real representations of Cliff$(2,2)$ and Cliff$(3,1)$ using matrices of order 2^2.

- Applying method 1 to these representations of Cliff$(2,2)$ and Cliff$(3,1)$ gives (in either case) a real representation of Cliff$(3,2)$ using matrices of order 2^2.

- Applying method 2 to this representation of Cliff$(3,2)$ gives real representations of Cliff$(3,3)$ and Cliff$(4,2)$ using matrices of order 2^3.

- Applying method 3 to this representation of Cliff$(4,2)$ gives a real representation of Cliff$(0,6)$ using matrices of order 2^3.

- Applying method 1 to these representations of Cliff$(3,3)$ and Cliff$(4,2)$ gives (in either case) a real representation of Cliff$(4,3)$ using matrices of order 2^3,

- Applying method 3 to this representation of Cliff$(4,3)$ gives a real representation of Cliff$(0,7)$ using matrices of order 2^3.

Figure 1 shows this pattern extended to $d = 11$.

$$
\begin{tikzcd}
1 & 1\times 1 & & (1,0) \arrow[d, "2"'] \arrow[dr, "2"] \\
2 & 2\times 2 & & (1,1) \arrow[d, "1"'] & (2,0) \\
3 & 2\times 2 & & (2,1) \arrow[d, "2"'] \arrow[dr, "2"] \\
4 & 4\times 4 & & (2,2) \arrow[d, "1"'] & (3,1) \\
5 & 4\times 4 & & (3,2) \arrow[d, "2"'] \arrow[dr, "2"] \\
6 & 8\times 8 & & (3,3) \arrow[d, "1"'] & (4,2) \arrow[r, "3"] & (0,6) \\
7 & 8\times 8 & (0,7) & (4,3) \arrow[l, "3"'] \arrow[d, "2"'] \arrow[drr, "2"] \\
8 & 16\times 16 & (0,8) & (4,4) \arrow[l, "3"'] \arrow[r, "3"] \arrow[d, "1"'] & (8,0) & (5,3) \arrow[r, "3"] & (1,7) \\
9 & 16\times 16 & (1,8) & (5,4) \arrow[l, "3"'] \arrow[r, "3"] \arrow[d, "2"'] \arrow[drr, "2"] & (9,0) \\
10 & 32\times 32 & (1,9) & (5,5) \arrow[l, "3"'] \arrow[r, "3"] \arrow[d, "1"'] & (9,1) & (6,4) \arrow[r, "3"] & (2,8) \\
11 & 32\times 32 & (2,9) & (6,5) \arrow[l, "3"'] \arrow[r, "3"] & (10,1)
\end{tikzcd}
$$

Figure 1 – In this figure, (p, m) denotes a real representation with that signature. The labels 1,2,3 on the arrows refer to methods 1,2,3 for obtaining representations in other signatures, as described in sections 19-21. (Some arrows are not shown because they don't lead to anything new.) The first column is the dimension $d = p + m$ of the original vector space, and the second column is the size of each Dirac matrix ($N \times N$).

23 Traces

When using a matrix representation γ of $\text{Cliff}(p, m)$, we often need to compute the trace of a product of Dirac matrices. This section derives one general result about such traces.

Let $\mathbf{x}, \mathbf{y}, ...$ be any list of $N \geq 1$ mutually orthogonal vectors that are not self-orthogonal. If N is even, or if N is odd with $N < p + m$, then

$$\text{Trace}(\gamma(\mathbf{x})\gamma(\mathbf{y})\cdots) = 0. \tag{28}$$

This holds for all representations of any dimension, not just for irreducible representations of the minimum possible dimension.

To deduce this, use the fact that the trace of a product is invariant under a cyclic permutation of the factors: $\text{Trace}(AB) = \text{Trace}(BA)$. In particular,

$$\text{Trace}\Big(\gamma(\mathbf{x})\gamma(Q)\Big) = \text{Trace}\Big(\gamma(Q)\gamma(\mathbf{x})\Big),$$

which implies

$$\text{Trace}\Big(\gamma(\mathbf{x}Q)\Big) = \text{Trace}\Big(\gamma(Q\mathbf{x})\Big). \tag{29}$$

for any vector $\mathbf{x}$ and any other element Q of the Clifford algebra. If Q is a product of an odd number of vectors, each of which is orthogonal to $\mathbf{x}$, then $\mathbf{x}Q = -Q\mathbf{x}$, so equation (29) says that the trace must be zero. This proves that equation (28) holds when N is even, as long as $N > 0$

Now suppose that N is odd. Let Q be a product of an odd number of vectors, and let $\mathbf{v}$ denote some other vector that is orthogonal to all of the vector factors in Q but not self-orthogonal. Such a vector $\mathbf{v}$ exists if $N < p+m$. Then we have $\mathbf{v}Q\mathbf{v} = -\mathbf{v}^2 Q$ where $\mathbf{v}^2$ is a non-zero real number. The trace is invariant under cyclic permutations of the factors, so this implies that the quantity $\text{Trace}(\gamma(\mathbf{v}^2 Q)) = \mathbf{v}^2\text{Trace}(\gamma(Q))$ is zero. This proves that equation (28) holds when N is odd if $N < p + m$.

24 Representations of the even subalgebra

The **even subalgebra** of a Clifford algebra consists of those elements which are linear combinations of products of even numbers of vectors. The even subalgebra of $\text{Cliff}(p,m)$ will be denoted $\text{Cliff}_{\text{even}}(p,m)$.

Section 7 determined the minimum possible dimension $N_{\mathbb{C}}(p+m)$ of a nontrivial representation of $\text{Cliff}(p,m)$ on a complex vector space. That result, combined with the isomorphism[32]

$$\text{Cliff}_{\text{even}}(p,m) \simeq \text{Cliff}(m,p-1), \tag{30}$$

may be used to infer the minimum possible dimension $N_{\text{even},\mathbb{C}}(p+m)$ of a nontrivial representation of $\text{Cliff}_{\text{even}}(p,m)$ on a complex vector space. The conclusion is:

- When $p+m$ is odd, $N_{\text{even},\mathbb{C}}(p+m) = N_{\mathbb{C}}(p+m)$.
- When $p+m$ is even, $N_{\text{even},\mathbb{C}}(p+m) = N_{\mathbb{C}}(p+m)/2$.

When $p+m$ is even, a representation of $\text{Cliff}(p,m)$ of dimension $N_{\mathbb{C}}(p+m)$ becomes reducible when restricted to $\text{Cliff}_{\text{even}}(p,m)$, separating into two irreducible representations that each have dimension $N_{\mathbb{C}}(p+m)/2$. To construct these two irreducible representations, choose vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{p+m} \in \text{Cliff}(p,m)$ that satisfy the conditions in section 8. Define $\Gamma = \epsilon\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_{p+m}$ with $\epsilon \in \{1, i\}$ chosen so that $\Gamma^2 = 1$. Then the quantities

$$P_\pm \equiv \frac{1 \pm \Gamma}{2}$$

are both projections ($P_+^2 = P_+$ and $P_-^2 = P_-$), and they annihilate each other ($P_+P_- = 0$). The projections $P_\pm$ belong to $\text{Cliff}_{\text{even}}(p,m)$, and they commute with everything in that algebra. Now, start with a representation γ of $\text{Cliff}(p,m)$ of dimension N, and restrict it to $\text{Cliff}_{\text{even}}(p,m)$. Then we can get two smaller representations of $\text{Cliff}_{\text{even}}(p,m)$ by multiplying everything by $\gamma(P_+)$ or by multiplying everything by $\gamma(P_-)$. Both of these representations have dimension $N/2$.

[32] Benn and Tucker (1989), page 39, equation (2.3.1). Article 03910 also derives this isomorphism.

25 Example: Cliff$(1,3)$

Section 14 explained how to construct a representation of Cliff$(1,3)$ on a four-dimensional vector space over $\mathbb{C}$. That representation is irreducible. To illustrate the recipe that was described in section 24, this section shows how that representation of Cliff$(1,3)$ decomposes into two irreducible representations of $\text{Cliff}_{\text{even}}(1,3)$. The notation in this section is the same as in section 14.

The algebra $\text{Cliff}_{\text{even}}(1,3)$ is generated by the six bivectors $\mathbf{e}_j\mathbf{e}_k$ with $j<k$. The quantity that was denoted Γ in section 24 is

$$\Gamma = i\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4.$$

This satisfies $\Gamma^2=1$, so the quantities

$$P_\pm \equiv \frac{1\pm\Gamma}{2}$$

are both projections. The quantity Γ commutes with everything in $\text{Cliff}_{\text{even}}(1,3)$, so the projections $P_\pm$ do, too. However, Γ anticommutes with $\mathbf{e}_k$, so

$$P_+\mathbf{e}_k = \mathbf{e}_kP_- \qquad\qquad P_-\mathbf{e}_k = \mathbf{e}_kP_+.$$

When P is defined by equation (11) as before, these projections satisfy

$$\begin{aligned}
P_+P &= P & P_-P &= 0\\
P_+\mathbf{e}_2\mathbf{e}_4P &= \mathbf{e}_2\mathbf{e}_4P & P_-\mathbf{e}_2\mathbf{e}_4P &= 0\\
P_+\mathbf{e}_2P &= 0 & P_-\mathbf{e}_2P &= \mathbf{e}_2P\\
P_+\mathbf{e}_4P &= 0 & P_-\mathbf{e}_4P &= \mathbf{e}_4P.
\end{aligned}$$

The first column of relationships says that applying $\gamma(P_+)$ to Φ leaves a representation of $\text{Cliff}_{\text{even}}(1,3)$ spanned by the two vectors $\Phi(P)$ and $\Phi(\mathbf{e}_2\mathbf{e}_4P)$. The second column of relationships says that applying $\gamma(P_-)$ to Φ leaves a representation of $\text{Cliff}_{\text{even}}(1,3)$ spanned by the two vectors $\Phi(\mathbf{e}_2P)$ and $\Phi(\mathbf{e}_4P)$. These are irreducible representations of $\text{Cliff}_{\text{even}}(1,3)$.

26 Real representations of the even subalgebra

Section 24 explained how to produce representations of $\text{Cliff}_{\text{even}}(p, m)$ on a complex vector space of the smallest possible dimension $N_{\text{even},\mathbb{C}}(p + m)$. Depending on the signature (p, m), real representations may exist with that same dimension $N_{\text{even},\mathbb{C}}(p+m)$. One way to determine when such representations exist is to use the isomorphism (30) together with the table in section 18. The result is summarized in this table, using $N_{\text{even},\mathbb{R}}(p, m)$ to denote the minimum dimension among real representations of $\text{Cliff}_{\text{even}}(p, q)$:

$(p - m)$ mod 8	$N_{\text{even},\mathbb{R}}(p, m)$
7,0,1	$N_{\text{even},\mathbb{C}}(p, m)$
2,3,4,5,6	$2N_{\text{even},\mathbb{C}}(p, m)$

To construct real representations of dimension $N_{\text{even},\mathbb{C}}(p, m)$ when they exist, start with a real representation of $\text{Cliff}(p, m)$ of dimension $N_{\mathbb{C}}(p+m)$. Such representations exist whenever $p-q$ modulo 8 is either 0, 1, or 2.[33] A real representation of $\text{Cliff}(p, m)$ is still real when restricted to $\text{Cliff}_{\text{even}}(p, m)$. Multiplying every Dirac matrix by i reverses the signature from (p, m) to (m, p), giving a representation of $\text{Cliff}(m, p)$ that becomes real when restricted to $\text{Cliff}_{\text{even}}(m, p)$. Altogether, this gives real representations of $\text{Cliff}_{\text{even}}(p, m)$ of dimension $N_{\mathbb{C}}(p + m)$ whenever $p - q$ modulo 8 belongs to $\{6, 7, 0, 1, 2\}$.[34]

That already accounts for the cases with odd $p-m$ in the first row of the table, because $N_{\text{even},\mathbb{C}}(p + m) = N_{\mathbb{C}}(p + m)$ when $p - q$ is odd.[35] When $p - q$ is even, $N_{\text{even},\mathbb{C}}(p + m)$ is equal to $N_{\mathbb{C}}(p + m)/2$, so the remaining question is for which signatures (p, m) the real representations described in the preceding paragraph remain real when the dimension-halving projections described in section 24 are applied. The answer can be deduced like this:

[33] Section 18

[34] This is consistent with the symmetry $\text{Cliff}_{\text{even}}(m, p) \simeq \text{Cliff}_{\text{even}}(p, m)$ (article 08264).

[35] Section 24

- When $p-q$ modulo 8 is 0 or 4, the product $\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_{p+m}$ squares to 1, so in this case the quantity that was denoted Γ that section doesn't include a factor of i, so we get real representations of $\text{Cliff}_{\text{even}}(p,m)$ of dimension $N_{\text{even},\mathbb{C}}(p,m)$.

- When $p-q$ modulo 8 is 2 or 6, the product $\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_{p+m}$ squares to -1, so in this case the quantity that was denoted Γ that section does include a factor of i, so we don't get real representations of $\text{Cliff}_{\text{even}}(p,m)$ of dimension $N_{\text{even},\mathbb{C}}(p,m)$.

This accounts for the cases with even $p-m$ in the table shown above.

27 Summary of minimum dimensions

This section gathers the minimum-dimension results that were introduced in the preceding sections. Notation:

- $N_{\mathbb{C}}(p+m)$ is the smallest dimension of a complex vector space that can host a nontrivial representation of Cliff(p,m).
- $N_{\text{even},\mathbb{C}}(p+m)$ is the smallest dimension of a complex vector space that can host a nontrivial representation of Cliff$_{\text{even}}(p,m)$.
- $N_{\mathbb{R}}(p+m)$ is the smallest dimension of a real vector space that can host a nontrivial representation of Cliff(p,m).
- $N_{\text{even},\mathbb{R}}(p+m)$ is the smallest dimension of a real vector space that can host a nontrivial representation of Cliff$_{\text{even}}(p,m)$.

For representations hosted on complex vector spaces, the smallest dimensions are[36]

$$N_{\mathbb{C}}(d) = \begin{cases} 2^{d/2} & \text{if } d \text{ is even,} \\ 2^{(d-1)/2} & \text{if } d \text{ is odd} \end{cases} \qquad N_{\text{even},\mathbb{C}}(d) = \begin{cases} 2^{(d/2)-1} & \text{if } d \text{ is even,} \\ 2^{(d-1)/2} & \text{if } d \text{ is odd.} \end{cases}$$

For representations hosted on real vector spaces, the smallest dimensions are[37]

$p-m$ mod 8	$N_{\mathbb{R}}(p,m)$	$N_{\text{even},\mathbb{R}}(p,m)$
0	$2^{d/2}$	$2^{(d/2)-1}$
1	$2^{(d-1)/2}$	$2^{(d-1)/2}$
2	$2^{d/2}$	$2^{d/2}$
3	$2^{(d+1)/2}$	$2^{(d+1)/2}$

$p-m$ mod 8	$N_{\mathbb{R}}(p,m)$	$N_{\text{even},\mathbb{R}}(p,m)$
4	$2^{(d/2)+1}$	$2^{d/2}$
5	$2^{(d+1)/2}$	$2^{(d+1)/2}$
6	$2^{(d/2)+1}$	$2^{d/2}$
7	$2^{(d+1)/2}$	$2^{(d-1)/2}$

with $d \equiv p+m$. This agrees with table 2.10 in Benn and Tucker (1989).

[36]Sections 7 and 24
[37]Sections 18 and 26

28 Some terminology

A recurring theme in physics is that even when the math is clear, the language can be confusing because words aren't always used consistently. This section summarizes some of the different dialects that are used in the context of representations of Clifford algebras. This section uses the abbreviation $d \equiv p + m$.

- This terminology is common in the math literature:[38] a representation of Cliff(p, m) of dimension $N_{\mathbb{C}}(p, m)$ is called a **pinor representation**, and a representation of Cliff$_{\text{even}}(p, m)$ of dimension $N_{\text{even},\mathbb{C}}(p, m)$ is called a **spinor representation**. Elements of the vector space W on which the representation acts are called **pinors** or **spinors**, respectively.[39] When d is odd, pinors and spinors are the same. When d is even, one pinor consists of two spinors.[40]

- This terminology is also common in the math literature:[41] when Cliff(p, m) or Cliff$_{\text{even}}(p, m)$ is a **simple** algebra,[42] a minimum-dimension representation of that algebra is called a **spinor representation**. When Cliff(p, m) or Cliff$_{\text{even}}(p, m)$ is not simple, a minimum-dimension representation is called a **semi-spinor representation**.

- This terminology is common in the physics literature: In a representation of Cliff(p, m) on a vector space W of dimension $N_{\mathbb{C}}(p, m)$, an element of W is called a **Dirac spinor**. When d is even, Cliff$_{\text{even}}(p, m)$ has representations on a vector space W of dimension $N_{\mathbb{C}}(p, m)/2$, and an element of that W

[38] Example: Figueroa-O'Farrill (2015), section 3, page 7

[39] Warning: for all d, Harvey (1990) defines a *pinor representation* to be an irreducible representation of the Clifford algebra, but then for some odd values of $p - m$, the *space of pinors* is defined to be the direct sum of the two vector spaces on which two inequivalent pinor representations act. That language in definition 11.10, which is almost one and a half pages long. In that book, for those values of $p - q$, a *pinor* is an element of the space of pinors (example: the text between equations 12.97 and 12.98) but is not an element of the space on which a pinor representation acts. The opportunities for confusion are abundant.

[40] Section 24

[41] Example: Benn and Tucker (1989), section 2.5, page 55

[42] A simple algebra is isomorphic to a full matrix algebra in which the components of each matrix are elements of an associative division algebra (Bresar (2009), theorem 3.4).

is called a **chiral spinor**[43] or **Weyl spinor**.[44] When d is even, one Dirac spinor consists of two chiral spinors (Weyl spinors).[45]

- A real representation of Cliff(p, m) of dimension $N_{\mathbb{R}}(p, m)$ is called a **Majorana representation**.[46]

- Starting with a real representation of Cliff(p, m) of dimension $N_{\mathbb{R}}(p, m)$, multiplying all Dirac matrices by i (which flips the signature from (p, m) to (m, p)) gives what is sometimes called a **pesudo-Majorana representation**[47] or sometimes just **Majorana representation**.[48]

[43]Example: Wipf (2016), section 4.4

[44]Peskin and Schroeder (1995), section 3.2, page 44

[45]The two chiral spinors come from using the projections $(1 \pm \Gamma)/2$ that were defined in section 24. These two representations are exchanged with each other by a reflection along any individual vector, because such a reflection flips the sign of Γ. The name *chiral* alludes to this reflection.

[46]Example: Figueroa-O'Farrill (2015), section 3, pages 8-9. When comparing that source to this article, beware that we use opposite conventions for writing the signature: the quantity denoted $s - t$ in that source corresponds to $-(p - m)$ in this article. (Recall footnote 4 in section 1.)

[47]Page 15 in Figueroa-O'Farrill (2015) criticizes this as "a nebulous concept best kept undisturbed."

[48]Example: Wipf (2016), section 4.3, below equation (4.32)

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