The Force Mediated by a Scalar Quantum Field

Randy S

Abstract This article shows that a scalar quantum field can mediate a force between other objects with which it interacts, like a classical field does. The goal is to determine the properties of this force – whether it is attractive or repulsive, and how it varies with the distance between the objects.

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1 Introduction

Article 22792 showed that a free scalar quantum field can behave like a classical field. This article shows that a scalar quantum field can mediate a force between other objects with which it interacts, like a classical field does. The goal is to determine the properties of this force – whether it is attractive or repulsive, and how it varies with the distance between the objects.

To make the math easier, this article uses a model that has a suitable approximation built into it. Section 2 explains the idea behind the approximation. Section 5 defines the model, and section 7 uses it to calculate the force between two pointlike objects. This is analogous to the familiar case of two pointlike charges in electrodynamics, but here the mediating field is a scalar field instead of the electric field. The result is that the force between two objects with charges of equal sign is *attractive*. This is different than the situation in electrodynamics, where charges with the same sign repel each other.[1] It is more similar to the situation in general relativity, where masses with the same sign attract each other.[2]

[1]The table on page 126 in Peskin and Schroeder (1995) compares the direction of the forces mediated by a few different types of field.

[2]Mass can be regarded as the gravitational analog of charge, but analogies between gravity and other interactions should not be pushed too far. Empirically, all masses have the same sign, but electrostatic charge – and its scalar-field analog – may have both signs. This empirical observation about mass is related to various theoretical **energy conditions**, which are reviewed concisely in section 1.1 of Fewster (2012) and more extensively in Martin-Moruno and Visser (2017) and Curiel (2014). Witten (2019) explains some of their applications in detail.

2 Approach

A satisfying way to study the force between two objects would be to use a model with at least two quantum fields: one to mediate the force, and another to provide the material objects. In that model, influences goes both ways, as required by the **action principle**: the material objects influence the mediating field, and the mediating field influences the material objects. I'll call this a **two-way model**. Using a two-way model would be satisfying, but it would also be difficult, because the equations of motion for a system of interacting fields are necessarily nonlinear.

This article uses an easier approach. The model has only one quantum field, the one that mediates the force. The material objects are modeled as **(external) sources** – things that influence, but are not influenced by, the quantum field. I'll call this a **one-way model**,[3] because the influence goes only one way. We can use the one-way model to infer the direction and approximate magnitude of the force that would occur in a two-way model. Let $E(r)$ is the minimum possible energy of a state in the one-way model with two non-moving charges, as a function of the distance r between them. The rules are:

- If $E(r)$ increases with increasing r, then the force is attractive. If $E(r)$ decreases with increasing r, then it's repulsive.

- The magnitude of the force is approximately $|dE/dr|$.

These rules can be deduced just like they are in classical mechanics. Section 3 reviews the reasoning.

To use these rules, we need to know how the hamiltonian H in the one-way model depends on r, because H is the observable representing the system's total energy. The form of H is partially determined by the requirement that it generates translations in time, but that's not enough: if I is the identity operator and $f(r)$ is an arbitrary function of r, then the operators H and $H + f(r)I$ would generate the same time-evolution. Section 4 explains how to remove this ambiguity.

[3]This name is not standard. It is often called a **semiclassical model** (section 10).

3 Deducing the rules

Section 2 listed the rules that can be used in the one-way model to deduce what the direction and magnitude of the force would be in a two-way model. Those rules can be deduced just like they are in classical mechanics. This section reviews the reasoning.

To deduce the rule about the direction of the force, consider what would happen in a two-way model, starting with a configuration of two charges separated by a distance r with zero velocity and with no radiation in the mediating field. As a result of the two-way influence between the charges and the mediating field, the charges will start to move, either toward each other or away from each other, depending on whether the force is attractive or repulsive. The system's total energy is conserved, so some of the energy that was stored in the initial configuration is transferred to the charges' kinetic energy, and some is transferred to radiation carried away by the mediating field. After some time has passed, the moving charges will be in new locations. The energy of a static configuration (no radiation, no motion) with the charges in those new locations must be lower than the energy of the original static configuration, because some of the original energy was transfered to radiation and motion. Qualitatively, this justifies the rule described above regarding the direction of the force.

The rule regarding the magnitude of the force is a good approximation if the energy lost to radiation is negligible and the motion of the charges is slow. In that case, experience with nonrelativistic classical mechanics suggests that the equation of motion for each charged object should be something like $d\mathbf{p}/dt = -\nabla V$, where $\mathbf{p}$ is the charge's momentum and V would be the system's energy if the charges were not moving.[4] The force is $d\mathbf{p}/dt$ (by definition), and the magnitude of ∇V is what section 2 denoted $|dE/dr|$.

[4] Article 33629

4 Removing the ambiguity

To remove the ambiguity that was mentioned in section 2, the hamiltonian H for the one-way model should be based on the hamiltonian $H_{\text{two-way}}$ for a two-way model, which in turn should not have any r-dependence in its constant term – because in the two-way model, the locations of the material objects are encoded in a state,[5] not baked into the hamiltonian.

This article assumes that hamiltonian $H_{\text{two-way}}$ for the two-way model has the form

$$H_{\text{two-way}} = H_{\text{matter}} + H_\phi - \int d^D x\ \phi(t, \mathbf{x}) J(t, \mathbf{x}).$$

On the right-hand side, the first term depends only on observables associated with the material objects or their constituents, the second term depends only on the mediating field $\phi(t, \mathbf{x})$, and the third term is responsible for their mutual interaction.[6] The charge-density operator $J(t, \mathbf{x})$ is an observable associated with the material objects.[7]

If the material objects are massive enough that their velocities are not affected much by the mediating field during short periods of time (insofar as a quantum object can have a well-defined velocity at all), then we can consider states in which H_{matter} and $J(t, \mathbf{x})$ are effectively independent of what the mediating field is doing.

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[5] Article 03431

[6] In quantum field theory, interactions can re-arrange the correspondence between the original fields and the observables that detect particles, so the correspondence is not always as straightforward as it is for non-interacting fields. This section ignores that complication. At least heuristically, this can be justified by assuming that the interaction is weak enough that the correspondence is only slightly re-arranged, and then defining new field operators in terms of the particles. When expressed in terms of those new field operators, the dominant terms in the hamiltonian are expected to have the same form as the original hamiltonian, and the additional terms are expected to have only small effects (because of the weak-interaction assumption). This is an application of the Effective Field Theory idea, which is used to construct approximate models whose predictions should be good approximations to those of the original model under certain conditions, like low energy. This is the standard way of explaining why the model(s) called Non-Relativistic Quantum Electrodynamics (NRQED) should be good approximations to relativistic QED when the massive particles have low enough energy. The application of this idea to NRQED is reviewed in Paz (2015) and in section 3.4 of Lepage (1989), and an application to condensed matter is reviewed in Polchinski (1992).

[7] In the two-way model, the material objects and the mediating field are all quantum entities, so the charge density $J(\mathbf{x})$ is an operator on the Hilbert space, just like any other observable.

In the context of such states, the operators H_{matter} and $J(t,\mathbf{x})$ might as well be replaced with their expectation values, at least if the state is such that the material objects are not significantly entangled with the mediating field. If the velocities of the material objects are negligible, then H_{matter} is effectively constant and the charge density J is effectively independent of time. This reasoning leads to a one-way model with hamiltonian

$$H = H_\phi - \int d^D x\ \phi(t,\mathbf{x})J(\mathbf{x}) + \text{constant},$$

where now the charge density $J(\mathbf{x})$ is an ordinary function instead of an operator. This is equation (3) in the next section.

5 The one-way model

Let $\mathbf{x} = (x_1, ..., x_D)$ denote a point in D-dimensional space. The only quantum entity in the model is the mediating field $\phi(\mathbf{x})$, which will be just like a free scalar field except for one new term in the hamiltonian – and consequently in the equation of motion – to represent its interaction with the external source (section 4). As in article 22792,[8] the equal-time commutation relations are

$$\begin{aligned}\big[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})\big] &= 0\\ \big[\dot\phi(t,\mathbf{x}),\dot\phi(t,\mathbf{y})\big] &= 0\\ \big[\phi(t,\mathbf{x}),\dot\phi(t,\mathbf{y})\big] &= i\delta(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{1}$$

The equation of motion is

$$\ddot\phi(t,\mathbf{x}) - \nabla^2\phi(t,\mathbf{x}) + m^2\phi(t,\mathbf{x}) = J(\mathbf{x}), \tag{2}$$

and the hamiltonian is

$$H = \int d^Dx \ \left(\frac{\dot\phi^2(t,\mathbf{x}) + \big(\nabla\phi(t,\mathbf{x})\big)^2 + m^2\phi^2(t,\mathbf{x})}{2} - J(\mathbf{x})\phi(t,\mathbf{x})\right) + \kappa \tag{3}$$

with constant term κ. As explained in section 4, κ is independent of the configuration of the material objects, and $J(\mathbf{x})$ is an ordinary function (not an operator) representing the prescribed charge density of the configuration of material objects. The function $J(\mathbf{x})$ is taken to be independent of time in this article, which implies that the operator H is independent of time.[9]

The external source $J(\mathbf{x})$ is prescribed: it influences the quantum field but is not influenced by the quantum field. The fact that the hamiltonian depends on a prescribed function of $\mathbf{x}$ ruins translation symmetry, so momentum is not conserved. In a two-way model with no external sources, momentum would be conserved.

[8] This article uses the same notation as that one.

[9] The proof works just like the one in article 52890.

6 Lowest energy with a given external source

This section computes the minimum possible energy in the one-way model, for a given external source $J(\mathbf{x})$. In section 7, the external source will be specialized to represent a pair of pointlike "charges," and the dependence of the lowest energy on the distance between these charges will be used to obtain the force.

When $J = 0$, the commutation relations (1) and equation of motion (2) are both satisfied by $\phi = \phi_0$ with[10]

$$\phi_0(t,\mathbf{x}) = \int \frac{d^D p}{(2\pi)^D}\,\frac{a(\mathbf{p})e^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega}} + \text{adjoint} \tag{4}$$

if the operators $a(\mathbf{p})$ satisfy

$$[a(\mathbf{p}), a(\mathbf{p}')] = 0 \qquad\qquad [a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^D\,\delta(\mathbf{p}' - \mathbf{p}). \tag{5}$$

When $J \neq 0$, the conditions (1) and (2) are instead satisfied by

$$\phi(t,\mathbf{x}) = \phi_0(t,\mathbf{x}) + \phi_J(\mathbf{x}) \tag{6}$$

where $\phi_J(\mathbf{x})$ is any ordinary time-independent function satisfying

$$-\nabla^2\phi_J(\mathbf{x}) + m^2\phi_J(\mathbf{x}) = J(\mathbf{x}). \tag{7}$$

One solution is

$$\phi_J(\mathbf{x}) = \int d^D y\; G(\mathbf{x}-\mathbf{y})J(\mathbf{y}) \tag{8}$$

with[11]

$$G(\mathbf{x}-\mathbf{y}) \equiv \int \frac{d^D p}{(2\pi)^D}\,\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{p}^2 + m^2}. \tag{9}$$

[10] Article 37301 addresses technicalities in the massless case ($m = 0$). Those technicalities won't be important here.

[11] Article 58590

If the support of J is contained in a finite region of space, then this solution approaches zero as $|\mathbf{x}| \to \infty$, rapidly enough so that states with finite energy exist even if space is infinite.[12]

To calculate the energy of the lowest-energy state, substitute (6) into (3) to get

$$H = H_0 + H_1 + H_2$$

where H_n is the terms involving n factors of ϕ_0. The term H_0 is[13]

$$\begin{aligned}
H_0 &= \int d^Dx \left(\frac{(\nabla\phi_J)^2 + m^2\phi_J^2}{2} - J\phi_J \right) + \kappa \\
&= \int d^Dx \left(\frac{\nabla\cdot(\phi_J\nabla\phi_J)}{2} - \frac{1}{2}J\phi_J \right) + \kappa && \text{(use equation (7))} \\
&= -\frac{1}{2}\int d^Dx\; J\phi_J + \kappa && \text{(use equation (14), below)} \\
&= -\frac{1}{2}\int d^Dx\, d^Dy\; J(\mathbf{x})G(\mathbf{x}-\mathbf{y})J(\mathbf{y}) + \kappa. && \text{(use equation (8))}
\end{aligned}$$

Using equation (7), the term H_1 may be written

$$H_1 = \int d^Dx\; \nabla\cdot\big(\phi_0(t,\mathbf{x})\nabla\phi_J(\mathbf{x})\big).$$

This involves the operators $\phi_0(t,\mathbf{x})$ only at the boundary of space, so it's zero if we define the model in a finite volume with periodic boundary conditions and then take the infinite-volume limit. That leaves H_2 as the only part of the hamiltonian that involves ϕ_0, so we can construct a Hilbert space just like in article 00980, and then the energy of the minimum-energy state is

$$E[J] \equiv -\frac{1}{2}\int d^Dx\, d^Dy\; J(\mathbf{x})G(\mathbf{x}-\mathbf{y})J(\mathbf{y}) + \text{constant} \tag{10}$$

where the constant term is independent of J.

[12] This is clear from equation (14), below.

[13] The first three lines use the abbreviations $J \equiv J(\mathbf{x})$ and $\phi_J \equiv \phi_J(\mathbf{x})$.

7 The force between two point charges

Now let the external source be[14]

$$J(\mathbf{x}) = q_1\delta(\mathbf{x}_1 - \mathbf{x}) + q_2\delta(\mathbf{x}_2 - \mathbf{x}). \tag{11}$$

This is supposed to represent a pair of particles, one fixed at $\mathbf{x}_1$ and the other at $\mathbf{x}_2$, with charges q_1 and q_2. Then the lowest energy is[15]

$$E[J] = -q_1 q_2 G(\mathbf{x}_1 - \mathbf{x}_2) - q_1 q_2 G(\mathbf{0}) + \text{ constant}. \tag{12}$$

Equation (9) implies that this depends only on the distance $|\mathbf{x}_1 - \mathbf{x}_2|$. The force between the two particles is just the derivative of this with respect to the distance between them (section 2). The force is attractive if the sign of the derivative is positive, or repulsive if the sign of the derivative is negative. To determine the sign of the derivative, write $r = |\mathbf{x}_1 - \mathbf{x}_2|$, and suppose that the only nonzero component of $\mathbf{x}_1 - \mathbf{x}_2$ is the first component. Then

$$G(\mathbf{x}_1 - \mathbf{x}_2) = \int \frac{dp_2 \cdots dp_D}{(2\pi)^D} \int_{-\infty}^{\infty} dp_1 \; \frac{e^{ip_1 r}}{p_1^2 + b^2}$$

with

$$b \equiv \left(m^2 + \sum_{n\geq 2} p_n^2\right)^{1/2}. \tag{13}$$

After writing the denominator as

$$p_1^2 + b^2 = (p_1 + ib)(p_1 - ib),$$

[14] Article 22792 uses an external source of the form $J(\mathbf{x}) \propto \delta(\mathbf{x})$ to calculate the associated "force field" – the expectation value of the field operator in the space around an isolated pointlike charge.

[15] The term $G(\mathbf{0})$ in (12) can be absorbed by the J-independent constant term in (10) by defining the model on a lattice, choosing the constant term to absorb the $G(\mathbf{0})$ term, and then taking the continuum limit with that new constant term held fixed. This doesn't affect the calculation in this section, because $G(\mathbf{0})$ is independent of the distance between the particles.

the integral over p_1 can be evaluated easily as a contour integral:[16] the integrand is an analytic function of p_1 everywhere in the finite complex plane except $p_1 = \pm ib$, and it goes to zero for $|p_1| \to \infty$, so the value of the integral is unchanged if we deform the integration contour to be the closed curve

$$p_1(\theta) = ib + \epsilon e^{i\theta}$$

with $0 \le \theta < 2\pi$ and $\epsilon \ll 1$. Use

$$dp_1 = i\epsilon e^{i\theta} d\theta$$

and take the limit $\epsilon \to 0$ to get

$$G(\mathbf{x}_1 - \mathbf{x}_2) = \int \frac{dp_2 \cdots dp_D}{(2\pi)^D} \int_0^{2\pi} d\theta \; \frac{e^{-br}}{2b} = \int \frac{dp_2 \cdots dp_D}{(2\pi)^{D-1}} \; \frac{e^{-br}}{2b}. \tag{14}$$

This implies

$$\frac{d}{dr} G(\mathbf{x}_1 - \mathbf{x}_2) = \int \frac{dp_2 \cdots dp_D}{(2\pi)^{D-1}} \; \frac{-e^{-br}}{2} < 0 \tag{15}$$

because the integrand is negative for all $p_2, ..., p_D$. According to (12), this implies that dE/dr has the same sign as $q_1 q_2$, so the force is attractive if the charges have the same sign.

For any given value of b, increasing the value of r decreases the magnitude of the integrand in (15), so equations (12) and (15) also imply that the magnitude of the force decreases with increasing distance, approaching zero as $r \to \infty$. Section 8 evaluates the integral completely for $D = 3$, which gives

$$G(\mathbf{x}) = \frac{e^{-m\,|\mathbf{x}|}}{4\pi\,|\mathbf{x}|} \qquad\qquad \text{(if } D = 3\text{)}. \tag{16}$$

[16] Article 22050

8 Derivation of (16)

The derivation of (16) from (9) is a standard exercise, which is summarized here[17] using the abbreviation $r \equiv |\mathbf{x}|$:

$$
\begin{aligned}
G(\mathbf{x}) &\equiv \int \frac{d^3p}{(2\pi)^3}\,\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2+m^2} \\
&= (2\pi)^{-3}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^\infty p^2\,dp\,\frac{e^{ipr\cos\theta}}{p^2+m^2} \\
&= (2\pi)^{-2}\int_{-1}^{1} d\cos\theta \int_0^\infty p^2\,dp\,\frac{e^{ipr\cos\theta}}{p^2+m^2} \\
&= (2\pi)^{-2}\int_0^\infty dp\,p\,\frac{e^{ipr}-e^{-ipr}}{(p^2+m^2)ir} \\
&= (2\pi)^{-2}\int_{-\infty}^\infty dp\,\frac{p\,e^{ipr}}{(p^2+m^2)ir} \\
&= (2\pi)^{-2}\int_{-\infty}^\infty dp\,\frac{p\,e^{ipr}}{(p+im)(p-im)ir}.
\end{aligned}
$$

The remaining integral may be evaluated as a contour integral by deforming the contour to be a small circle enclosing the point $p = im$, like in section 7. The result is (16).

[17] I chose to summarize it here because this took less time than finding a good online reference.

9 An application

Electrically charged particles exert a force on each other mediated by the (quantum) electromagnetic field, not by a scalar field, but the result derived in this article does have at least a rough physical application: the longest-range part of the so-called *strong force* between two nucleons (protons or neutrons) is mediated by a scalar field.[18] The field's associated particle is called a **pion**,[19,20] so I'll call this scalar field the **pion field**.[21,22] The force it mediates between nucleons is attractive, and the fact that it is stronger than electromagnetism allows it to hold protons together in a nucleus even though protons repel each other electrically. Under appropriate circumstances (like in a high-energy scattering experiment), individual pions may be produced, just like individual photons may be produced. However, unlike photons, pions are massive: the mass parameter m in equation (13) is positive, not zero. Thanks to the factor e^{-br} in equation (15), with b defined by (13), this means that the force mediated by the pion field is short-ranged. Quantitatively, the range is

$$\sim \frac{\hbar}{m_{\text{pion}}\, c} \approx 1.4 \times 10^{-15} \text{ meter}$$

in standard units. This is comparable to the size of a typical nucleus.

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[18]Actually, it's a *pseudo-scalar* field, a distinction that relates to how it transforms under space-time reflections. This distinction does not affect the result about the direction of the force.

[19]The fact that pions are associated with the longest-range part of the strong force between nucleons is related to the fact that pions are the lightest mesons.

[20]Pions come in three varieties, differing from each other in their electric charge, but if we ignore the electric interaction and consider only the so-called strong interaction, then we can treat all pions as identical (and we can treat protons and neutrons as identical).

[21]The interaction mediated by the pion field is a residual effect of the full gluon-field-mediated interaction that confines quarks into hadrons.

[22]This is only a convenient abbreviation. People often abbreviate even further by using expressions like "the force mediated by pions" or, in the electromagnetic case, "the force mediated by photons." A more careful way to say it would be "the force mediated by the same quantum field that triggers pion-detectores (respectively, photon-detectors)." Thinking about the thing I'm calling the *pion field* as being made of pions is no more helpful than thinking of the atmosphere as being made of tornadoes. Article 22792 helps put this in context.

10 Semiclassical models

Section 2 introduced the name *one-way model* for a model in which one thing (the material objects in this case) influences another thing (the mediating field in this case) but not conversely. That type of model is used routinely in both classical and quantum physics, but the name *one-way model* is not the standard name for it. In the context of quantum physics, a standard name is **semiclassical model**, because the model includes both quantum things (the mediating field in this case) and classical things (the material objects in this case). Here, the words *quantum* and *classical* mean something specific: observables associated with the quantum thing don't all commute with each other, but observables associated with the classical thing commute with all observables. Mathematically, that only makes sense if the influence is one-way: classical things can influence quantum things, but not conversely. A semiclassical model is necessarily a one-way model.[23,24]

The word *classical* is sometimes (especially in older literature) used to mean something entirely different: it is sometimes used to refer to a *quantum* system that is macroscopic and microscopically complicated, so that its most important observables approximately commute with each other for most practical purposes. That's what it means in a sentence like "measurement involves interaction with a classical system." If the words *classical system* in that sentence were misinterpreted as "a system whose observables all strictly commute with each other," then the sentence would be mathematically nonsensical. Mathematically, a system whose observables all commute with each other cannot be influenced by – and therefore cannot implement any measurement of[25] – a system whose observables don't all commute with each other.

[23] A one-way model is not necessarily a semiclassical model. One-way models are used routinely in classical physics, too, where nothing is quantum.

[24] This sentence is valid if *semiclassical model* is defined the way I've defined it here, but beware that the name *semiclassical model* is sometimes used differently.

[25] Article 03431

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(Open-access items include links.)

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