Topological ’t Hooft Operators in Path Integrals

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Abstract Article 09181 introduces the concept of a **topological operator** in quantum field theory. This article uses the path integral formulation to construct a family of topological operators called **’t Hooft operators** in models whose only field is a gauge field. The gauged group can be any compact Lie group. A topological ’t Hooft operator is nominally localized on a $(d-2)$-dimensional submanifold Σ of d-dimensional spacetime. This article treats spacetime as discrete so the math is straightforward. Article 53519 uses the canonical (continuous time) formulation to construct topological ’t Hooft operators for which the submanifold Σ is restricted to a single time. This article explains how to relate that description of the operators to the one introduced here.

Contents

1 Introduction

Wilson operators and *'t Hooft operators* are are among the simplest examples of gauge invariant operators in quantum models with gauge fields. Article 22721 gives an overview of Wilson and 't Hooft operators, distinguishing two different types of each based on the nature of the submanifold X of d-dimensional spacetime on which the operator is nominally localized:[1,2,3]

operator	notation	number of dimensions of X	type of submanifold	this article
Wilson, type 1	W°	1	proper and neat	
Wilson, type 2	$W^\bullet$	2	proper	
't Hooft, type 1	T°	$d-3$	proper and neat	
't Hooft, type 2	$T^\bullet$	$d-2$	proper	✓

Other articles in this series describe the first three operators in the table.[4] This article defines 't Hooft operators of type 2 for any compact gauged group G in a model whose only field is a gauge field. Such an operator will be denoted $T(\Sigma)$, where Σ is the submanifold on which the operator is nominally localized.[5] The operators $T(\Sigma)$ are *topological operators*, and they generate a *one-form symmetry*.[6] Section 11 will show that some Wilson operators of type 1 are *charged* with respect to this symmetry, which simply means they are not invariant under the symmetry transformation.

This article uses the word **operator** for any modification of the integrand of the path integral, like article 09181 does. Article 02242 explains how this relates to the concept of a linear operator on the Hilbert space.

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[1] Here, "nominally localized on X" means "localized in an arbitrarily small neighborhood of X."

[2] The names **type 1** and **type 2** and the superscripts ∘ and • are not standard.

[3] Article 22721 explains the last column.

[4] Article 22721 cites the articles that introduce these operators.

[5] In article 22721, this type of 't Hooft operator is denoted $T^\bullet$.

[6] Article 09181 explains what this means.

2 Two formulations

Article 53519 uses the canonical (hamiltonian) formulation to construct $T(\Sigma)$ for any compact gauged group G when Σ is a submanifold of the spatial manifold at a single time. A state is represented as a function of the link variables at that time, and $T(\Sigma)$ is defined by shifting the link variables whose links are intersected by Σ.[7] *Shifting* a link variable means multiplying it by an element of the center of G.

This article uses the path integral formulation to construct $T(\Sigma)$ without the single-time restriction on Σ. Spacetime is treated as a discrete lattice. In the path integral formulation, an 't Hooft operator $T(\Sigma)$ is defined by shifting the plaquette variables whose plaquettes are intersected by Σ.[8] *Shifting* a plaquette variable means multiplying it by an element of the center of G.

Section 14 will explain how to relate these two formulations to each other when Σ is localized at a single time.

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[7] In the canonical formulation, when space is treated as a discrete lattice, Σ does not intersect any points of the lattice and does not intersect any link more than once.

[8] In the path integral formulation, when spacetime is treated as a discrete lattice, Σ does not intersect any points or links of the lattice and does not intersect any plaquette more than once.

3 Notation and conventions

- d is the number of dimensions of the spacetime manifold, except in section 10, where d is the exterior derivative.
- Spacetime will be treated as a lattice. To simplify the description of 't Hooft operators, this article views the discrete structure as being embedded in an underlying smooth spacetime manifold. The underlying smooth spacetime manifold may be curved but is assumed to be static.
- The construction in this article works for a generic lattice of the type described in article 46333,[9] with one restriction that will be imposed in section 14.
- The lattice is made of k-cells for each $k \in \{0, 1, 2, ..., d\}$. A 0-cell is a **point**. A 1-cell (a directed **link**) is denoted ℓ, and a 2-cell (oriented **plaquette**) is denoted $\square$.[10] Despite the notation, plaquettes aren't necessarily square.
- $\square^{-1}$ is the orientation-reversed counterpart of $\square$.
- G is the gauged group.
- $u(\ell)$ is the G-valued **link variable** associated with the directed link ℓ.
- $W(\square)$ is a complex-valued **traced plaquette variable**, defined to be the trace of the product of link variables around the perimeter of the plaquette.
- The Wilson operator associated with a closed curve[11] C and a representation r of G is denoted $W_r(C)$.[12]
- The 't Hooft operator associated with a $(d-2)$-dimensional submanifold Σ of spacetime and an element z of the center of G is denoted $T_z(\Sigma)$.[13]

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[9] The hypercubic lattice used in article 89053 for flat spacetime is a special case. The pictures in this article depict that case because it's relatively easy to draw.

[10] Article 91116 defines **oriented**, and article 46333 applies it to k-cells. For links, **directed** means **oriented**.

[11] If spacetime has a boundary, then *closed curve* can be generalized to *one-dimensional properly embedded neat submanifold* (article 22721).

[12] In article 22721, this type of Wilson operator is denoted $W_r^\circ(C)$.

[13] In article 22721, this type of 't Hooft operator is denoted $T_z^\bullet(\Sigma)$.

4 The path integral

The path integral has the form[14]

$$\Psi'[u]_{t'} \propto \int_{<t'} [du]\; e^{iS[u]}\Psi[u]_t \tag{1}$$

where

- Ψ and Ψ' are the initial and final states,
- $[u]_t$ denotes the set of link variables whose endpoints are both at time t,
- the integral is over of the link variables that have at least one endpoint in the range $\geq t$ and $< t'$ (with no more than one endpoint at time t'),
- each link variable is integrated over the gauged group G.

The properties of the action $S[u]$ that will be important in this article include:[15]

- It depends on link variables $u(\ell)$ only through the traced plaquette variables $W(\square)$.[16]
- It is a sum of terms that each depends on only one traced plaquette variable:

  $$S[u] = \sum_{\square} c(\square)W(\square) \tag{2}$$

  where $c(\square)$ are fixed complex-valued coefficients[17] whose values are not important in this article.

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[14] Article 89053

[15] The Wilson action introduced in article 89053 has these properties.

[16] This implies that $S[u]$ is invariant under gauge transformations.

[17] This allows for Wick rotation to a euclidean action (article 89053).

5 Shift operators

This section introduces two families of shift operators: one in the path integral formulation, and in the canonical formulation.

Let z be an element the center of G, and choose plaquette $\square$ within the time interval covered by the path integral. Define an operator $T_z(\square)$ by

$$T_z(\square)\int [du]\ e^{iS[u]}\Psi[u] \equiv \int [du]\ e^{iS'[u]}\Psi[u] \tag{3}$$

where S' is obtained from the action (2) by replacing $W(\square)\to zW(\square)$ and $W(\square^{-1})\to z^{-1}W(\square^{-1})$, like this:[18,19]

$$S'\big(...,\, W(\square),\, W(\square^{-1}),\, ...\big) = S\big(...,\, zW(\square),\, z^{-1}W(\square^{-1}),\, ...\big). \tag{4}$$

All other traced plaquette variables (represented by "...") are unaffected. The initial state $\Psi[u]$ is not affected. The operator $T_z(\square)$ will be called a **shift operator**. It's an *operator* in the generalized sense used in article 09181.

The canonical formulation uses another **shift operator** $T_z(\ell)$ that acts directly on the initial state $\Psi[u]$, where ℓ is a link whose endpoints are both at the initial time. It is defined by[20]

$$T_z(\ell)\Psi[u] = \Psi'[u] \tag{5}$$

with

$$\Psi'\big(...,\, u(\ell),\, u(\ell^{-1}),\, ...\big) = \Psi\big(...,\, zu(\ell),\, z^{-1}u(\ell^{-1}),\, ...\big).$$

All other link variables (represented by "...") are unaffected. This is an *operator* in the conventional sense of a linear operator on the Hilbert space.

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[18] This respects the fact that traced plaquette variables (shifted or not) whose plaquettes have opposite orientations must be each other's inverses (article 89053).

[19] The effect of shifting a single traced plaquette variable (and its conjugate) typically cannot be achieved by any transformation of the link variables, but the expression (4) is unambiguous because the action (2) depends on the link variables only through traced plaquette variables.

[20] Article 53519

6 The orientation of an intersection

If M is an oriented manifold and X and Y are oriented submanifolds of M with $\dim X + \dim Y = \dim M$, then X and Y generically intersect each other only at isolated points. This section explains how to define the orientation of such an intersection, specialized to the case where one of the two submanifolds is either a plaquette or a link. This section assumes familiarity with differential forms.[21]

At any given point of an n-dimensional manifold, all n-forms are proportional to each other.[21] The manifold is called **orientable** if it admits an n-form that is not zero anywhere, called an **orientation form**, and any choice of such an n-form defines an **orientation** for the manifold. Two n-forms define the same orientation if one can be obtained from the other by multiplying by a positive function, so any orientable manifold has two possible orientations corresponding to the two possible signs of an n-form.

Let M_d be the d-dimensional orientable spacetime manifold, and suppose an orientation for M_d has been specified. Choose a point $p \in M_d$ and a coordinate system $x_0, x_1, ..., x_{d-1}$ in a neighborhood of p so that M_d's orientation form may be written[22]

$$\omega_d \equiv \partial_0 \wedge \partial_1 \wedge \cdots \wedge \partial_{d-1} \tag{6}$$

in that neighborhood. Let M_{d-1} be the $(d-1)$-dimensional orientable spatial manifold given by fixing the value of the time coordinate x_0. Suppose $p \in M_{d-1}$ and use the coordinate system $x_1, ..., x_{d-1}$ for M_{d-1} in a neighborhood of p so that M_{d-1}'s orientation form may be written

$$\omega_{d-1} \equiv \partial_1 \wedge \cdots \wedge \partial_{d-1} \tag{7}$$

in that neighborhood. These two orientation forms are related to each other by $\omega_d = \partial_0 \wedge \omega_{d-1}$.

Now consider the orientation forms associated with links and plaquettes in a neighborhood of the point p. If a directed link ℓ happens to be parallel to the

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[21] Article 91116

[22] This expression treats each partial derivative $\partial_k \equiv \partial/\partial x_k$ as a vector field (article 09894).

j-direction, then its orientation form is either ∂_j or $-\partial_j$. Choose the sign so that the orientation form associated with the link variable $u(\ell)$ is ∂_j. More generally, if a link ℓ is not necessarily parallel to a coordinate axis, then its orientation form $\omega(\ell)$ is a linear combination of the coordinate vector fields. The orientation form associated with the oppositely-directed link is $\omega(\ell^{-1}) = -\omega(\ell)$. Next consider the traced plaquette variable

$$W(\square) \equiv \text{trace}\big(u(\ell_1)u(\ell_2)\cdots\big)$$

where $\ell_1, \ell_2, ...$ is the sequence of links around the perimeter of the plaquette. Choose the associated orientation form to be

$$\omega(\square) = \omega(\ell_1) \wedge \omega(\ell_2).$$

Cyclically permuting the factors in the trace may change the overall factor by selecting a different pair of consecutive links, but it doesn't change the sign. The orientation form associated with the oppositely-oriented plaquette is

$$\omega(\square^{-1}) = \omega(\ell_2^{-1}) \wedge \omega(\ell_1^{-1})$$

up to an overall positive factor, which implies $\omega(\square^{-1}) = -\omega(\square)$ because the wedge product is anticommutative.

Finally, consider intersections. First consider an intersection with a plaquette, thinking of the plaquette as a tiny oriented element of area. Let Σ be a $(d-2)$-dimensional orientable manifold of the d-dimensional orientable spacetime M_d, and suppose that Σ intersects a plaquette $\square$ in M_d transversely,[23] so that the intersection is a single point p. Specify an orientation for Σ by choosing a $(d-2)$-form $\omega(\Sigma)$. The orientation of the intersection will be called **positive** or **negative** if

$$\omega(\square) \wedge \omega(\Sigma) = f\omega_d \tag{8}$$

with $f > 0$ or $f < 0$, respectively, and with ω_d defined by (6). A nowhere-zero function f satisfying equation (8) always exists, so this definition is sufficient.

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[23] Chapter 6 in Lee (2013) defines **transverse** intersection.

Next consider an intersection with a link, thinking of the link as a tiny line segment. Suppose that the $(d-2)$-dimensional submanifold Σ is contained in the $(d-1)$-dimensional spatial manifold M_{d-1}. Suppose that Σ intersects a link ℓ in M_{d-1} transversely, so that the intersection is a single point p. The orientation of the intersection will be called **positive** or **negative** if

$$\omega(\ell)\wedge\omega(\Sigma) = f\omega_{d-1} \tag{9}$$

with $f > 0$ or $f < 0$, respectively.[24]

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[24] Article 53519 described the positive case as the link intersecting Σ "back-to-front," because the submanifold Σ has codimension 1 relative to the spatial manifold M_{d-1}, so choosing an orientation for Σ in that context amounts to choosing which side of Σ to call the "front." This doesn't work in the context of the spacetime manifold M_d, where Σ has codimension 2. (Analogy: choosing which side of a curve to call the "front" makes sense in 2-dimensional euclidean space, but it doesn't make sense in 3-dimensional euclidean space.)

7 Participating plaquettes

In smooth d-dimensional spacetime, an 't Hooft operator would be localized on a connected manifold Σ with $d-2$ dimensions. This article treats spacetime as a lattice, in the sense that the gauge field variables will be associated with links in the lattice, but defining 't Hooft operators is easier if we think of the lattice as a special set of points in smooth d-dimensional spacetime. With that picture in mind, let Σ be a connected $(d-2)$-dimensional manifold with these properties:

- Σ does not intersect any links of the spacetime lattice.
- All intersections of Σ with plaquettes are all transverse.
- Σ does not intersect any plaquette more than once, and the plaquettes pierced by Σ do not extend outside the time interval covered by the path integral.
- The boundary $\partial\Sigma$ does not intersect any plaquettes.

Figure 1 shows an example.

Define Ω to be the set of plaquettes intersected by Σ with positively-oriented intersection.[25] Section 8 will define a topological 't Hooft operator localized on the set of plaquettes. This is the lattice version of being localized on Σ itself. To simplify the language, this article will say that the 't Hooft operator is **localized on $\boldsymbol{\Sigma}$**, with the understanding that this really means localized on the plaquettes that Σ intersects.

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[25] Section 6 defined the orientation of an intersection.

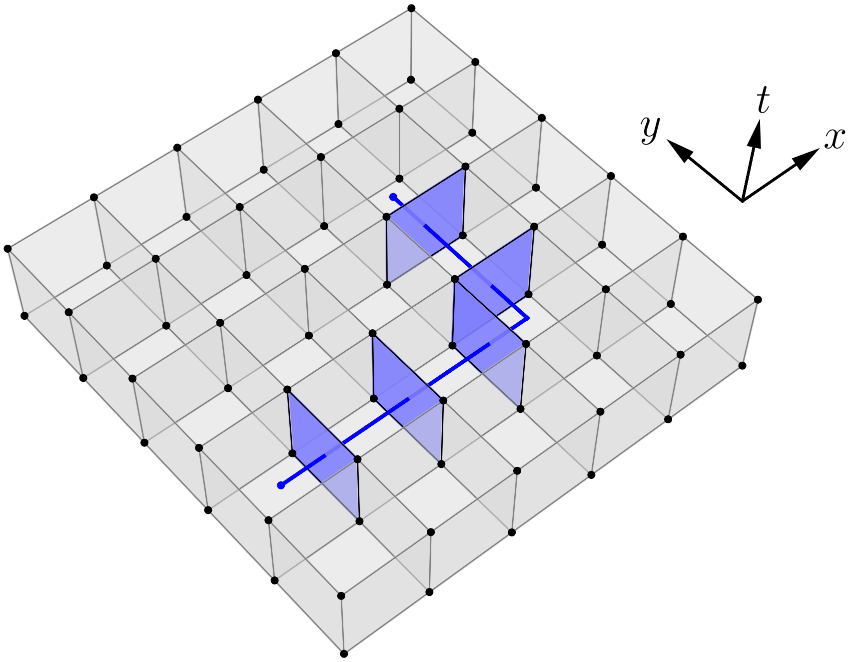

Figure 1 – When $d = 3$, the submanifold Σ is a one-dimensional curve (colored blue in this picture) that doesn't intersect any links of the 3d spacetime lattice, but it does pierce some plaquettes (also shaded blue). Its boundary $\partial\Sigma$ is a pair of points (blue). To reduce clutter, only one layer of cubettes is shown. The layer is sandwiched between two consecutive times.

8 Topological ’t Hooft operators

Let Σ an oriented submanifold of spacetime with codimension 2 satisfying the conditions in section 7. Section 7 defined a set Ω of oriented plaquettes that intersect Σ with a given orientation of the intersection. For each z in the center of G, the corresponding **(topological) ’t Hooft operator** $T_z(\Sigma)$ is defined by[26,27]

$$\boxed{T_z(\Sigma) \equiv \prod_{\square\in\Omega} T_z(\square)} \tag{10}$$

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[26] Samuel (1983), equation 2.4

[27] Some sources that describe ’t Hooft operators in smooth spacetime describe the operator’s effect as a “singular gauge transformation.” Section 3.3 in Greensite (2003) uses this language even when defining the operator in discrete space.

9 A shift-invariance property

Consider the path integral (1). Let ℓ be any link, either spacelike or timelike, with at least one endpoint after the initial time and at least one endpoint before the final time. In other words, ℓ is any link whose corresponding link variable is not one of the arguments of either the initial state or the final state. Let Ω_ℓ be the set of oriented plaquettes that have the oriented link ℓ in their perimeters. This will be called the **bouquet** of plaquettes around the link ℓ. Figure 2 shows an example. The conditions on ℓ ensure that none of the plaquettes in the bouquet extends outside the time interval covered by the path integral.

For any link ℓ that satisfies those conditions, consider the operator

$$\prod_{\square\in\Omega_\ell} T_z(\square). \tag{11}$$

For each plaquette $\square$ in the bouquet around ℓ, this operator replaces the traced plaquette variable $W(\square)$ by $zW(\square)$ wherever that traced plaquette variable appears in the action. This affects the action, but it doesn't change the path integral. To prove this, use the fact that the action depends on link variables only through traced plaquette variables,[28] so the shift described above is equivalent to replacing $u(\ell)$ by $zu(\ell)$ for the chosen link ℓ. Then use the identity

$$\int du\; f(zu) = \int du\; f(u), \tag{12}$$

which is a property of the Haar measure.[29] This shows that if the action is the only thing in the path integral that depends on the chosen link variable $u(\ell)$, then the operator (11) leaves the path integral invariant.

This result holds if the action S is the only thing in the integrand that depends on the integration variable $u(\ell)$. If some other operator (like a Wilson loop) that depends on $u(\ell)$ is inserted into the integrand, then this shift-invariance result might not hold for that link ℓ. This will be important in section 12.

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[28] Section 4

[29] Article 89053

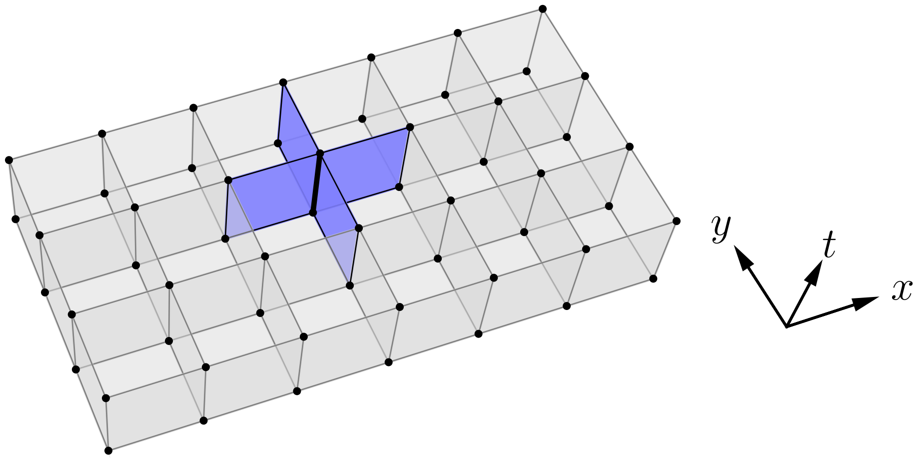

Figure 2 – Example of a bouquet of plaquettes in 3-dimensional spacetime. In this example, the bouquet is around a timelike link. Only one layer of cubettes is shown, like in figure 1. In d-dimensional spacetime, a bouquet consists of $2(d-1)$ plaquettes around one link.

10 Shift invariance and Maxwell's equations

When $G = U(1)$, the shift-invariance property derived in section 9 is related to Maxwell's equations. To deduce this, write the link variables as $u(\ell) = e^{i\,\theta(\ell)}$. Then the shift described in section 9 is equivalent to replacing $\theta(\ell)$ by $\theta(\ell) + \delta\theta$ for the chosen link ℓ. Expressed in terms of θ, the shift-invariance property (12) becomes

$$\int_{\text{one period}} d\theta\ f(\theta + \delta\theta) = \int_{\text{one period}} d\theta\ f(\theta) \tag{13}$$

for every real number $\delta\theta$. The infinitesimal version of (13) is

$$\int_{\text{one period}} d\theta\ \frac{\partial}{\partial\theta} f(\theta) = 0.$$

When applied to an integration variable $\theta(\ell)$ on which nothing but the action depends, this gives[30]

$$\int [d\theta]\ \frac{\delta S}{\delta\theta(\ell)} e^{-S}\Psi = 0, \tag{14}$$

which is the lattice version of one of Maxwell's equations,[31] namely the part of Maxwell's equations that would be written $d(\star F) = 0$ in smooth spacetime, where F is the field-strength two-form,[32] $\star F$ is its Hodge dual,[33] and d is the exterior derivative.

In a model that includes matter, the shift-invariance property highlighted in section 9 would involve the matter fields in addition to a bouquet of traced plaquette variables. The equation of motion (14) would still hold, but when written out in terms of field operators, it would include a term involving the matter fields.[34]

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[30] If we want to fix the gauge, we can do that after taking the derivative in (14).

[31] Compare this to how Maxwell's equations are expressed using the action principle in classical electrodynamics (article 98002).

[32] The rest of Maxwell's equations ($dF = 0$) are implied by defining the field-strength tensor in terms of the link variables ($F = dA$).

[33] Article 91116

[34] In smooth spacetime, it would involve both $d(\star F)$ and an operator representing the charge/current density of the matter field(s).

11 A topological property

This section shows that the 't Hooft operators defined in section 8 are topological operators as defined in article 09181.

Starting with one manifold Σ, choose one of the links ℓ on the perimeter of one of the plaquettes intersected by Σ. By shifting the traced plaquette variables in the bouquet around that link as described in section 9, we can convert $T_z(\Sigma)$ to $T_z(\Sigma')$, where Σ' is defined by rerouting Σ so that it intersects all the other plaquettes in the bouquet instead of the original one. This is illustrated in figures 3 and 4. If the integrand of the path integral doesn't contain any other insertions that depend on the link variable $u(\ell)$, then this shift leaves the path integral invariant,[35] so $T_z(\Sigma) = T_z(\Sigma')$ in that case. In words: the 't Hooft operator is invariant under all deformations of Σ that can be reached using any number of these single-bouquet deformations, as long as nothing else in the integrand of the path integral (other than the action) depends on the link variable at the core of any of those bouquets.

If Σ doesn't have a boundary ($\partial\Sigma = \varnothing$) and is such that we can use incremental deformations (like in the previous paragraph) to shrink Σ to nothing, then $T_z(\Sigma)$ is the identity operator when viewed as an ordinary operator on the Hilbert space.[36,37]

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[35] Section 9

[36] The qualifier "when viewed as an ordinary operator on the Hilbert space" is important. Article 02242 explains why in general terms, and section 12 demonstrates that $T_z(\Sigma)$ can have a nontrivial effect on other operators in the integrand of the path integral even when Σ does not have a boundary.

[37] For $G = U(1)$, the smooth-spacetime version of this topological property is $\int_{\partial V} \star F = \int_V d(\star F) = 0$ where F is the field-strength two-form satisfying the equation of motion $d(\star F) = 0$ (section 10), V is a manifold with codimension 1 in spacetime and with boundary $\partial V = \Sigma^{-1} \cup \Sigma'$, and Σ^{-1} is obtained from Σ by reversing its orientation. This is the one-form symmetry analog of a calculation that article 09181 shows for a zero-form symmetry.

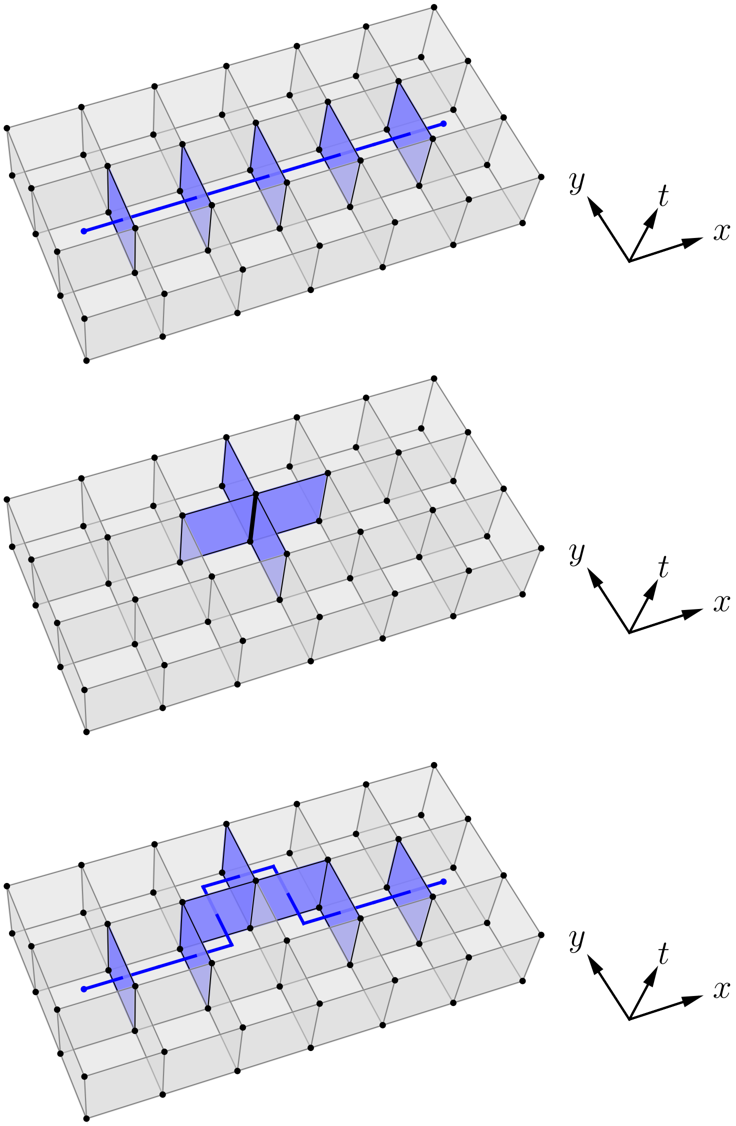

Figure 3 – The top picture shows a spacelike ’t Hooft line. Only one layer of cubettes is shown, like in figure 1. The middle picture shows the plaquettes whose bounding links are affected by a shift applied to the link highlighted by the thick black line (section 9). The bottom picture shows the result of applying that shift to the configuration in the top picture, tuned so that one of the previously-pierced plaquettes cancels. This shows that the ’t Hooft operator associated with the manifold Σ represented by the thick blue line in the bottom picture is the same as the ’t Hooft operator whose Σ is shown in the top picture.

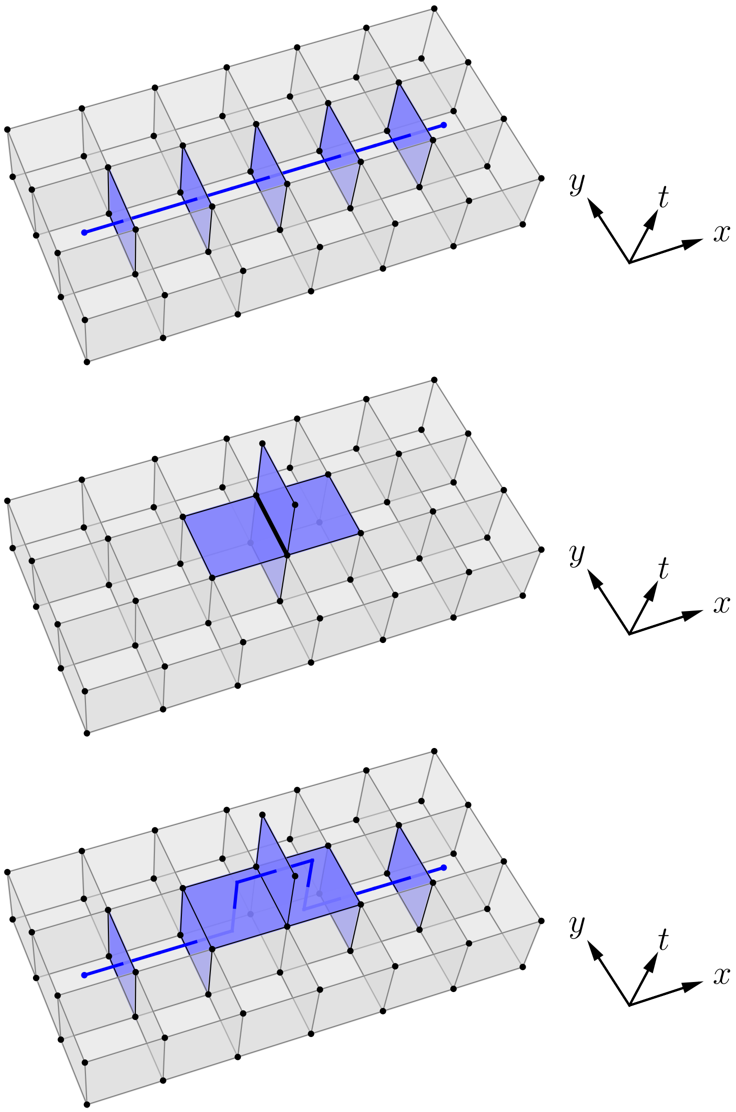

Figure 4 – Like figure 3, but for a spacelike link (which introduces a timelike bend into Σ) instead of a timelike link (which would introduce a spacelike bend into Σ, as shown in figure 3).

12 One-form symmetry

The derivation in section 11 assumes that nothing else in the integrand of the path integral (other than the action) depends on any of the link variables through which Σ must pass during this shrinking process. In particular, it assumes that the integrand does not include any Wilson operators. This section mentions what can happen when a Wilson operator is also present.

Let C be a closed curve made of links in the spacetime lattice. Suppose that a Wilson operator $W(C)$ is present in the integrand of the path integral.[38] Let Σ_1 and Σ_2 be two $(d-2)$-dimensional submanifolds that can both be obtained from Σ by incremental deformations as described in section 11. Using Σ_1^{-1} to denote the orientation-reversed counterpart of Σ_1, the closed manifold $\Sigma_1^{-1}\cup\Sigma_2$ is the boundary ∂V of a $(d-1)$-dimensional submanifold V. If Σ_1, and Σ_2 both satisfy the conditions outlined in section 7, then ∂V does not intersect C (because it doesn't intersect any links at all), but the volume V may intersect C at a finite number of points. Each of these intersections has a well-defined orientation. Assign the value ±1 to each of these intersections with the sign depending on the orientation. The sum of these values is the **intersection number** $\eta(C,V)$.[39]

The same bouquet-shifting argument that was used in section 11 may also be used here, but now a factor of $z^{\pm 1}$ is acquired whenever an incremental deformation of Σ passes through a link in C.[40] The net effect of morphing Σ_1 to Σ_2 is

$$T_z(\Sigma_1)\int[du]\;e^{iS[u]}W_r(C)\Psi[u]=\big(r(z)\big)^{\eta(C,V)}T_z(\Sigma_2)\int[du]\;e^{iS[u]}W_r(C)\Psi[u]\qquad(15)$$

where r is the representation of the gauged group G used to define the trace of the product of link variables in the Wilson operator $W_r(C)$. This is a **one-form symmetry**.[41] It's the path integral version of the **center symmetry** derived in article 53519.

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[38] Article 89053

[39] Article 44113

[40] Section 9

[41] Article 09181

13 A shift operator identity

Section 5 defined two families of shift operators, one associated with plaquettes in the path integral formulation, and one associated with links in the canonical formulation. This section highlights a relationship between these two families of shift operators. Section 14 will use this to relate the path integral and canonical formulations of 't Hooft operators to each other.

Section 9 showed that if ℓ is any link whose corresponding link variable is not one of the arguments of either the initial state or the final state, then the operator (11) leaves the path integral invariant. Now suppose instead that ℓ is a link whose endpoints are both at the initial time, so the initial state $\Psi[u]$ in the path integral (1) is a function of the corresponding link variable $u(\ell)$. Define Ω_ℓ as in section 9. Then

$$\begin{aligned}\prod_{\square\in\Omega_\ell} T_z(\square)\int[du]\; e^{iS[u]}\Psi[u] &= \int[du]\;\Psi[u]T_z(\ell)e^{iS[u]} &&\text{(equations (3) and (5))}\\ &= \int[du]\; e^{iS[u]}\,T_{z^{-1}}(\ell)\Psi[u] &&\text{(equation (12))}\\ &= \int[du]\; e^{iS[u]}\,T_z(\ell^{-1})\Psi[u] &&\big(u(\ell)^{-1}=u(\ell^{-1})\big).\end{aligned}$$

On the right side of the first line, $T_z(\ell)$ acts on the factor $e^{iS[u]}$ just like it acts on any function of the link variables (equation (5)). The resulting identity

$$\prod_{\square\in\Omega_\ell} T_z(\square)\int[du]\; e^{iS[u]}\Psi[u] = \int[du]\; e^{iS[u]}\,T_z(\ell^{-1})\Psi[u] \tag{16}$$

will be used in section 14.

14 Relationship to the canonical formulation

Article 53519 describes topological ’t Hooft operators at a single time as operators that act directly on the Hilbert space, without using path integrals.[42] This section explains how to relate that formulation to the path integral formulation.[43]

To give *single time* a clear meaning, two conditions will be imposed:

- Spacetime is globally hyperbolic, so its topology has the form $\mathbb{R}\times M_{d-1}$. The factor $\mathbb{R}$ is parameterized by a time coordinate, and M_{d-1} is the spatial manifold at any given value of that time coordinate. The spacetime manifold has d dimensions, and the spatial manifold has $d-1$ dimensions.
- The discrete version of spacetime consists of time-shifted copies of a discrete version of M_{d-1}.[44,45]

Such a lattice has two types of plaquettes:

- The perimeter of an **electric** plaquette has both timelike and spacelike edges.
- The perimeter of a **magnetic** plaquette has only spacelike edges.

Start with the single-time formulation used in article 53519. Let Σ' be a $(d-2)$-dimensional oriented submanifold of the $(d-1)$-dimensional spatial manifold, and suppose that Σ' satisfies the conditions imposed in article 53519. That article defines the single-time ’t Hooft operator $T_z(\Sigma')$ by

$$T_z(\Sigma')\Psi[u]\equiv\prod_{\ell\in\Omega'}T_z(\ell)\Psi[u] \tag{17}$$

where Ω' is the set of directed links that pierce Σ' with positive orientation.[46]

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[42] This is the context for the hamiltonian formulation, where time evolution is generated by the hamiltonian.

[43] This section doesn’t derive the hamiltonian, but article 89053 does when spacetime is flat and $G=U(1)$.

[44] The usual hypercubic lattice has this form. Article 46333 describes a generalization that still has this form.

[45] This is certainly possible if the components g_{ab} of the spacetime metric are all independent of the time coordinate and the components with one time index and one space index are zero, like in article 26542.

[46] Section 6

The goal is to relate equation (17) to the path integral version of the 't Hooft operator, which was described in section 8. To do that, we need to replace Σ' with another $(d-2)$-dimensional submanifold Σ that doesn't intersect any links but whose projection into the spatial manifold at time $t=0$ is Σ'. We could do this by simply shifting Σ' into the future by a fraction of one time step, but – for a technical reason that will be explained soon – we will do something slightly different. Let t denote the time coordinate, and take the manifold Σ' defined above to be situated at $t=0$. Define Σ to be the union of these two parts:[47]

- a **plateau** consisting of a copy of Σ' translated to time $t=dt/2$, where dt is the time interval between consecutive copies of the spatial lattice,
- a **skirt** $I\times\partial\Sigma'$, where I is the time interval from $-dt/2$ to $dt/2$.

This is illustrated in figure 5. The whole manifold Σ is within one discrete step-size of the time $t=0$. Plaquettes intersected by the plateau part of Σ are electric plaquettes with an edge at time $t=0$ that was intersected by Σ'. Plaquettes intersected by the skirt part of Σ are magnetic plaquettes at time $t=0$ that were intersected by $\partial\Sigma'$.

Now let Ω be the set of oriented plaquettes defined in section 7 for this Σ. Every plaquette whose perimeter includes a link in the set Ω' defined above is either in Ω or has two links that were intersected by Σ'. In the latter case, one of those two links is in Ω', and the other is the oppositely oriented counterpart of a link in Ω'. (In figure 5, the parallel links highlighted in the bottom picture belong to Ω' and all have the same orientation.) This gives

$$\left(\prod_{\ell\in\Omega'}\prod_{\square\in\Omega_\ell}T_z(\square^{-1})\right)\int[du]\;\Psi[u]e^{iS[u]}=\left(\prod_{\ell\in\Omega}T_z(\square)\right)\int[du]\;\Psi[u]e^{iS[u]}\tag{18}$$

with Ω_ℓ defined as in section 13. Equation (18) is deduced by paying close attention to orientations (section 6) and using the fact that variables $W(\square)$ with $\square\notin\Omega$ are

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[47] The manifold Σ is just an auxiliary device used to define Ω, so allowing Σ to straddle the initial time is legal as long as the plaquettes in Ω don't involve any points before the initial time.

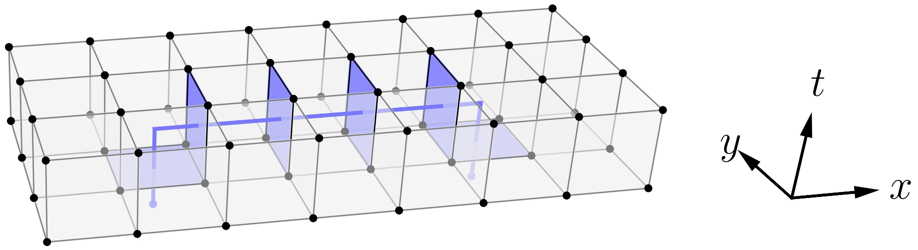
not affected because the link-variable shifts on the left side of (18) cancel each other whenever $\square \notin \Omega$. Altogether,

$$\begin{aligned}
\int [du]\; e^{iS[u]} T_z(\Sigma')\Psi[u] &= \int [du]\; e^{iS[u]} \left(\prod_{\ell\in\Omega'} T_z(\ell)\right) \Psi[u] && \text{(definition (17))}\\
&= \left(\prod_{\ell\in\Omega'}\prod_{\square\in\Omega_\ell} T_z(\square^{-1})\right) \int [du]\; \Psi[u] e^{iS[u]} && \text{(equation (16))}\\
&= \left(\prod_{\ell\in\Omega} T_z(\square)\right) \int [du]\; \Psi[u] e^{iS[u]} && \text{(equation (18))}\\
&= T_z(\Sigma) \int [du]\; e^{iS[u]}\Psi[u] && \text{(definition (10)).}
\end{aligned}$$

More concisely,

$$\boxed{\int [du]\; e^{iS[u]} T_z(\Sigma')\Psi[u] = T_z(\Sigma) \int [du]\; e^{iS[u]}\Psi[u].}$$

The operator on the left side is defined in equation (17), and the operator on the right side is defined in equation (10). The manifolds Σ' and Σ differ from each other by less than one time step dt, so they become indistinguishable in the continuous-time limit. This relates the formulation used in article 53519 to the formulation used in this article.



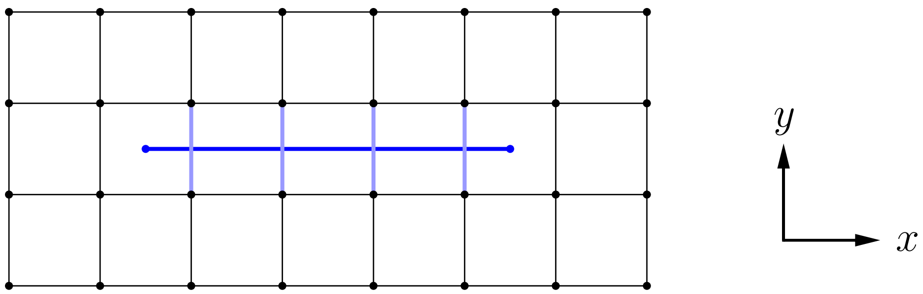


Figure 5 – In the first picture, the bottom layer of grid points sits at the initial time, and the top layer of grid points is one time step into the future. Spacetime is 3-dimensional, so the manifold Σ is 1-dimensional. Its boundary $\partial\Sigma$ is a pair of points situated slightly (less than one time step) before the initial time. The manifold Σ intersects two magnetic plaquettes at the initial time and four electric plaquettes immediately after the initial time. The second picture shows the corresponding arrangement in the canonical formulation in 2-dimensional space at the initial time, where Σ' intersects four links. These links correspond to the four electric plaquettes highlighted in the first picture. If Ω is the set of (electric and magnetic) plaquettes intersected by Σ in the first picture and Ω' is the set of links intersected by Σ' in the second picture with an appropriate choice of orientations, then shifting the values of the traced plaquette variables in Ω in the action is equivalent to shifting the values of the link variables in Ω' in the initial state, as explained in section 14.

15 Relationship between the one-form symmetries

Article 09181 explains, in general terms, how one-form symmetries described in the path integral formulation can also be described in the canonical formulation. This section uses the one-form symmetry (15) to illustrate that general relationship.

Define M_{d-1}, Σ', and Σ as in section 14. Let $W(C)$ be a Wilson operator localized on a closed curve C made of links in the spatial manifold M_{d-1}. The intersection number $\eta(C, M_{d-1})$ is defined because $\dim C + \dim\Sigma' = \dim M_{d-1}$. This is the intersection number that article 53519 uses to describe the one-form symmetry at a single time.

Now imagine that the path integral was originally given over a slightly longer time interval, starting at a slightly earlier time, and define $\hat{\Sigma}$ by pushing the interior of Σ' one half time-step into the past, just like Σ was defined by pushing it into the future. Then $\Sigma^{-1}\cup\hat{\Sigma}$ is the boundary of a $(d-1)$-dimensional submanifold V in d-dimensional spacetime, and the intersection number $\eta(C, V)$ is defined because $\dim C + \dim V = \dim$. Equation (15) describes the one-form symmetry in the path integral formulation, and the identity

$$\eta(C, V) = \eta(C, \Sigma')$$

relates it to the description in article 53519 using the canonical formulation.

16 Shift invariance in the temporal gauge

The **temporal gauge** is useful for recovering the canonical formulation from the path integral formulation.[48] Section 9 defined the shift in terms of plaquette variables, so it remains well-defined when the gauge is fixed, but if the gauge is fixed then the proof of the shift-invariance property needs to be adjusted. This section explains how to adjust the proof when the temporal gauge is used.

Let ℓ be a timelike link. We want to find a transformation of the link variables whose effect is to shift all the traced plaquette variables in the bouquet around ℓ by z without changing any other traced plaquette variables. Section 9 achieved this by shifting the one link variable $u(\ell)$, but if ℓ is timelike, then we can't do that without violating the temporal gauge condition $u(\ell) = 1$. To implement the desired shift of the traced plaquette variables, we need to use a different strategy. The bouquet around a timelike link ℓ involves only electric plaquettes,[49] so the shift may be implemented like this: let x be the earlier (in time) endpoint of ℓ, and replace the spacelike link variables $u(x', y')$ with $zu(x', y')$ for all points x' whose spatial locations are the same as x and whose times are no later than x. This doesn't change any traced plaquette variables except those in the bouquet around ℓ, as desired. Now, as before, we can use (12) to show that the path integral is unchanged.

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[48] Article 89053

[49] Section 14 will define *electric plaquettes* and *magnetic plaquettes*.

17 A condition for the topological property

The derivation of the topological property of $T(\Sigma)$ in section 11 assumes that the model does not include any matter fields charged under the gauged group. If it did, then the dependence of the action on the link variables would not be only through plaquette variables, so the operator (11) would not be equivalent to $T_z(\ell)$. The 't Hooft operator $T(\Sigma)$ would still be well-defined, but it would no longer be a topological operator because the shift-invariance property derived in section 9 would not hold.

Article 53519 derived the topological property of $T(\Sigma)$ using a different formulation. That derivation also assumes that the model does not include any matter fields charged under the gauged group. In that article, the topological property is presented as a consequence of the fact that states are invariant under gauge transformations, assuming that gauge transformations affect only the link variables. If charged matter fields were present, then gauge transformations would affect them, too, and $T(\Sigma)$ would no longer be a topological operator.

The derivation in section 11 effectively uses the model's equations of motion, because the equations of motion are expressions of the invariance of the path integral under shifts of the integration variables.[50] The derivation in article 53519 uses gauge invariance instead. These two derivations are consistent with each other, because the statement that states are gauge invariant is equivalent to Gauss's law,[51] which is one of the equations of motion – the one that doesn't involve time derivatives of any traced plaquette variables.

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[50] Article 63548

[51] Section 9

18 References

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