Renormalization Group Flow Near the Trivial Fixed Point

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Abstract Article 10142 introduced the idea of the **renormalization group** in quantum field theory. In colloquial terms, the renormalization group describes how a model changes when we "zoom out." Article 22212 derived a system of ordinary differential equations that describe how the coefficients in the action of a scalar quantum field change under the flow of the renormalization group. This article uses those flow equations to study which parts of the action are **relevant** (becoming more important at lower resolution) and which parts are **irrelevant** (becoming less important at lower resolution) with respect to the **trivial fixed point**, where interactions are absent.

Contents

1 Introduction

Article 22212 studied the effect of renormalization group transformations[1] on a family of models of a single scalar quantum field $\phi(x)$ in d-dimensional spacetime. The models were defined using the path integral formulation,[2] with a euclidean action of the form[3]

$$S[\phi] = \int d^dx \; \left(\frac{\big(\partial\phi(x)\big)^2}{2} + c_2\frac{\phi^2(x)}{2!} + c_4\frac{\phi^4(x)}{4!} + c_6\frac{\phi^6(x)}{6!} + \cdots\right). \tag{1}$$

A renormalization group transformation changes the values of the coefficients. In the **local potential approximation**, the action (1) retains its original form but with coefficients c_n whose values depend on the scale factor λ that parameterizes the renormalization group flow. Parts of the action whose coefficients increase as λ increases become more important at lower resolution, and parts of the action whose coefficients decrease as λ increases become less important at lower resolution. These are called **relevant** and **irrelevant** parts of the action, respectively.

Article 22212 derived the λ-dependence of the coefficients c_n using the local potential approximation together with the one-loop approximation. Section 13 will show that treating d as a continuous parameter can reveal indirect evidence of a nontrivial fixed point when $d = 3$ called the **Wilson-Fisher fixed point**. The evidence is not compelling by itself, because that fixed point lies outside the domain where those approximations work well, but taking $d = 4 - \epsilon$ with $0 < \epsilon \ll 1$ puts the fixed point close to the trivial fixed point, where those approximations should work well. Further investigation has produced better evidence for the existence of such a nontrivial fixed point when $d = 3$.[4]

[1] Article 10142 defines these transformations.

[2] Article 63548

[3] To define the models, spacetime was treated as a lattice instead of as a continuum. Those details of how the models were defined are not important here, though, so here the action is written as an integral over x instead of as a sum over x.

[4] Kleinert and Schulte-Frohlinde (2001) reviews some of the methods that have been used for this. As a warning, section 14 describes another example, one that could presumably be debunked by using such better approximations.

2 The system of flow equations

This section reviews the system of flow equations that article 22212 derived. These equations describe the λ-dependence of the action's dimensionless coefficients $\vec{g} \equiv (g_2, g_4, ...)$. Use the abbreviations[5]

$$\omega_d \equiv \frac{\Omega_d}{2(2\pi)^d} \qquad \partial \equiv \frac{\partial}{\partial s} \qquad t \equiv \log \lambda$$

where Ω_d is the "surface area" of the unit sphere in d-dimensional euclidean spacetime. Examples:

$$\Omega_1 = 2 \qquad \Omega_2 = 2\pi \qquad \Omega_3 = 4\pi \qquad \Omega_4 = 2\pi^2.$$

The flow equations are

$$\frac{d}{dt} g_{2k}(t) = \beta_{2k}\big(\vec{g}(t)\big)$$

with the **beta functions** $\beta_{2k}(\vec{g}\,)$ given by

$$\beta_{2k}\big(\vec{g}(t)\big) \approx \big(d + (2-d)k\big) g_{2k}(t) + \omega_d\, \partial^{2k} \log\big(1 + \partial^2 V(s)\big)\Big|_{s=0} \tag{2}$$

for all $k \in \{1, 2, 3, 4, ...\}$, and

$$V(s) = g_2(t)\frac{s^2}{2!} + g_4(t)\frac{s^4}{4!} + g_6(t)\frac{s^6}{6!} + g_8(t)\frac{s^8}{8!} + \cdots \tag{3}$$

The derivatives with respect to s will be calculated in section 3, and those results will be used in section 4 to write the individual flow equations more explicitly.

Article 22212 derived these equations with the help of a few approximations. Most of this article pretends that these equations are exact.

[5] The definition $t \equiv \log \lambda$ saves a little writing, because $d\vec{g}/dt = \lambda\, d\vec{g}/d\lambda$. The variable s is used in equation (2).

3 Expansion in powers of the field

This section evaluates the derivatives with respect to s in equation (2). Use the abbreviation $D \equiv 1 + \partial^2 V(s)$. Then[6]

$$\partial \log(D) = \frac{V_3}{D}$$

$$\partial^2 \log(D) = \frac{V_4}{D} - \frac{V_3^2}{D^2}$$

$$\partial^3 \log(D) = \frac{\partial^5 V}{D} - \frac{3V_4 V_3}{D^2} + \frac{2V_3^3}{D^3}$$

$$\partial^4 \log(D) = \frac{\partial^6 V}{D} - \frac{4V_5 V_3 + 3V_4^2}{D^2} + \frac{12 V_4 V_3^2}{D^3} - \frac{6V_3^4}{D^4}$$

$$\partial^5 \log(D) = \frac{V_7}{D} - \frac{5V_6 V_3 + 10 V_5 V_4}{D^2} + \frac{30 V_4^2 V_3 + 20 V_5 V_3}{D^3} - \frac{60 V_4 V_3^3}{D^4} + \text{terms with} \geq 4 \text{ factors of odd-order derivatives of } V$$

$$\partial^6 \log(D) = \frac{\partial^8 V}{D} - \frac{15 V_6 V_4 + 6 V_7 V_3 + 10 V_5^2}{D^2} + \frac{30 V_6 V_3^2 + 120 V_5 V_4 V_3 + 30 V_4^3}{D^3} - \frac{270 V_4^2 V_3^2}{D^4} + \text{terms with} \geq 3 \text{ factors of odd-order derivatives of } V$$

$$\partial^7 \log(D) = \frac{V_9}{D} - \frac{7V_8 V_3 + 21 V_7 V_4 + 35 V_6 V_5}{D^2} + \frac{210 V_6 V_4 V_3 + 210 V_5 V_4^2}{D^3} - \frac{630 V_4^3 V_3}{D^4} + \text{terms with} \geq 2 \text{ factors of odd-order derivatives of } V$$

$$\partial^8 \log(D) = \frac{V_{10}}{D} - \frac{28 V_8 V_4 + 35 V_6^2}{D^2} + \frac{420 V_6 V_4^2}{D^3} - \frac{630 V_4^4}{D^4} + \text{terms with} \geq 1 \text{ factor of odd-order derivatives of } V$$

[6] The coefficients were cross-checked by calculating them both by computer and by hand. The script that did the symbolic calculations is posted here: https://cphysics.org/extras/79649v.html

4 The system of flow equations, explicit version

Use the equations from section 3 in the flow equation (2) to get these explicit expressions for the individual flow equations:[7]

$$\begin{aligned}
\frac{dg_2}{dt} &= 2g_2 + \omega_d \frac{g_4}{1+g_2} \\
\frac{dg_4}{dt} &= (4-d)g_4 + \omega_d \left(\frac{g_6}{1+g_2} - \frac{3g_4^2}{(1+g_2)^2}\right) \\
\frac{dg_6}{dt} &= (6-2d)g_6 + \omega_d \left(\frac{g_8}{1+g_2} - \frac{15g_6 g_4}{(1+g_2)^2} + \frac{30g_4^3}{(1+g_2)^3}\right) \\
\frac{dg_8}{dt} &= (8-3d)g_8 + \omega_d \left(\frac{g_{10}}{1+g_2} - \frac{28g_8 g_4 + 35g_6^2}{(1+g_2)^2} + \frac{420g_6 g_4^2}{(1+g_2)^3} - \frac{630g_4^4}{(1+g_2)^4}\right)
\end{aligned} \tag{4}$$

and so on. Section 5 will explain how Feynman diagrams can be used to understand the structure of these equations. Section 6 will re-define the symbols β_{2k} and g_{2k} so that the analysis doesn't become cluttered with unenlightening factors of ω_d.

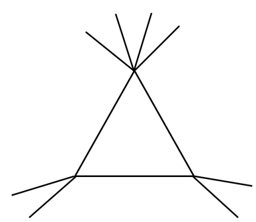
[7] The first three equations agree with equations (5.57) in Skinner (2016), after correcting a minor typo.

5 Relating the structure to Feynman diagrams

The structure of equations (4) is easy to understand when each term is depicted as a Feynman diagram.[8] Each term is represented by a single Feynman diagram. In a given term, each factor of g_n in the numerator is represented by a vertex with n edges emanating from it, and each factor of $1+g_2$ in the denominator is represented by connecting two edges to give a single edge with no free ends. For terms with one factor of ω_d in the original form of the flow equations, the connections are such that the graph is connected and has exactly one loop. This term contributes to β_k, where k is the total number of edges with one loose end. Example: the term

$$\frac{g_6 g_4^2}{(1+g_2)^3}$$

in β_8 is represented by this Feynman diagram:



The top vertex has 6 edges, representing the factor of g_6. The other two vertices each have 4 edges, representing the two factors of g_4. The three sides of the triangle (the lines with no loose ends) represent the three factors of $1+g_2$ in the demoninator. This term contributes to β_8 because 8 edges have one loose end.

For another example, consider the term $g_{10}/(1+g_2)$ in β_8. This is represented by a Feynman diagram with a single vertex having 10 edges, two of which are connected to each other to form a single loop (a single line with no loose ends) that represents the single factor of $1+g_2$ in the denominator. The number of edges with one loose end is 8, so this term contributes to β_8.

[8] Article 22212

6 Absorbing the factors of ω_d

The factors of ω_d in equations (4) can be absorbed into the definitions of g_{2k} and β_{2k}. To do this, multiply the equation for g_{2k} by ω_d^{k-1}, then define

$$g'_{2k} \equiv \omega_d^{k-1} g_{2k} \qquad\qquad \beta'_{2k} \equiv \omega_d^{k-1} \beta_{2k},$$

and then rename

$$g'_{2k} \to g_{2k} \qquad\qquad \beta'_{2k} \to \beta_{2k}.$$

This gives

$$\frac{d}{dt} g_{2k}(t) = \beta_{2k}\big(\vec{g}(t)\big) \tag{5}$$

with

$$\begin{aligned}
\beta_2(\vec{g}\,) &= 2g_2 + \frac{g_4}{1+g_2} \\
\beta_4(\vec{g}\,) &= (4-d)g_4 + \frac{g_6}{1+g_2} - \frac{3g_4^2}{(1+g_2)^2} \\
\beta_6(\vec{g}\,) &= (6-2d)g_6 + \frac{g_8}{1+g_2} - \frac{15g_6g_4}{(1+g_2)^2} + \frac{30g_4^3}{(1+g_2)^3} \\
\beta_8(\vec{g}\,) &= (8-3d)g_8 + \frac{g_{10}}{1+g_2} - \frac{28g_8g_4 + 35g_6^2}{(1+g_2)^2} + \frac{420g_6g_4^2}{(1+g_2)^3} - \frac{630g_4^4}{(1+g_2)^4}.
\end{aligned} \tag{6}$$

The rest of this article uses these new versions of g_{2k} and β_{2k} so that the analysis doesn't become cluttered with unenlightening factors of ω_d.

7 Fixed points and (ir)relevant directions

The system (5) of differential equations is **autonomous**, which means that the independent variable t enters only through the unknown functions $\vec{g}(t)$. A **fixed point**[9] of the system (5) is a set of initial values $\vec{g}(0) = \vec{g}^*$ for which $\vec{\beta}(\vec{g}^*) = 0$, which implies $\vec{g}(t) = \vec{g}^*$ for all t. In the context of renormalization, a fixed point at the origin ($\vec{g}^* = 0$) is called the **trivial** fixed point, because it corresponds to a quantum field model with no interactions. Other fixed points are called **nontrivial**.

If we start at a point $\vec{g} = \vec{g}^* + \delta g$ where δg is an infinitesimal perturbation, then a solution of the flow equations with this initial condition may progress toward or away from the fixed point as a function of increasing t. These perturbations are called *irrelevant* and *relevant*, respectively.[10] In most cases, the (ir)relevance of a perturbation can be diagnosed by linearizing the flow equations around the fixed point:

$$\frac{d\,\delta\vec{g}(t)}{dt} = M\,\delta\vec{g} + O\big((\delta g)^2\big) \qquad\qquad \delta\vec{g}(t) \equiv \vec{g}(t) - \vec{g}^*, \tag{7}$$

where M is the matrix with components $M_{jk} = \partial\beta_j/\partial g_k$ evaluated at the fixed point. In the linear approximation, if ψ is an eigenvector of M so that $M\psi = m\psi$ for some real number m, then the flow equation is satisfied by $\delta g(t) = \exp(mt)\psi$. This shows that negative eigenvalues of M correspond to irrelevant perturbations and that positive eigenvalues of M correspond to relevant perturbations.[11] When $m = 0$, the nonlinear terms must be considered in order to determine the direction of the flow.

[9] In the math literature about differential equations, fixed points are called **critical points** or **equilibrium points**. In physics, the name *critical point* usually refers to a point in model-space where the correlation length diverges in units of the lattice scale (article 10142).

[10] Articles 10142 and 22212

[11] According to the definition given in the text, if δg is a linear combination of an irrelevant perturbation and a relevant perturbation, then δg is a relevant perturbation. This is illustrated by the pictures in section 13. For the study of universality and continuum limits, the important quantity is the number of linearly independent relevant perturbations (modulo the irrelevant ones), because that's the number of parameters needed to parameterize the set of quantum field models that have continuous-spacetime limits in a neighborhood of the given fixed point (article 10142).

8 Linearized flow near the trivial fixed point

This section initiates the study of the flow near the trivial fixed point $\vec{g} = 0$. This section studies the flow using the linear approximation, in which terms of order $\vec{g}^2$ are ignored. This approximation is sufficient when all of the eigenvalues of the matrix M in equation (7) are nonzero. When M has a zero eigenvalue, then higher-order terms must be included in order to determine the direction of the flow. That will be done in section 10. For simplicity, these analyses will ignore all g_{2k} with $2k \geq 10$.

Use the notation

$$\vec{g} \equiv \begin{bmatrix} g_2 \\ g_4 \\ g_6 \\ g_8 \end{bmatrix}$$

so that equations (6) may be written

$$\frac{d}{dt}\vec{g} = M\vec{g} + O(g^2) \qquad \Rightarrow \qquad \vec{g}(t) = e^{Mt}\vec{g}(0) + O(g^2)$$

with

$$M = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 4-d & 1 & 0 \\ 0 & 0 & 6-2d & 1 \\ 0 & 0 & 0 & 8-3d \end{bmatrix} = 2I + (2-d)X$$

where I is the identity matrix and X is defined by

$$X \equiv \begin{bmatrix} 0 & -r & 0 & 0 \\ 0 & 1 & -r & 0 \\ 0 & 0 & 2 & -r \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad\qquad r \equiv \frac{1}{d-2}.$$

This article assumes $d \geq 3$ from now on, so $r > 0$.

9 Solution of the linearized flow equations

The matrix X has these eigenvectors:

$$\psi_2 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \psi_4 \equiv \begin{bmatrix} -r \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \psi_6 \equiv \begin{bmatrix} r^2/2 \\ -r \\ 1 \\ 0 \end{bmatrix} \qquad \psi_8 \equiv \begin{bmatrix} -r^3/3! \\ r^2/2 \\ -r \\ 1 \end{bmatrix}$$

with

$$X\psi_2 = 0 \qquad X\psi_4 = \psi_4 \qquad X\psi_6 = 2\psi_6 \qquad X\psi_8 = 3\psi_8. \tag{8}$$

Implications:

- If $\vec{g}(0) \propto \psi_2$, then $\vec{g}(t) = e^{2t}\vec{g}(0) + O(g^2)$.
- If $\vec{g}(0) \propto \psi_4$, then $\vec{g}(t) = e^{(4-d)t}\vec{g}(0) + O(g^2)$.
- If $\vec{g}(0) \propto \psi_6$, then $\vec{g}(t) = e^{(6-2d)t}\vec{g}(0) + O(g^2)$.
- If $\vec{g}(0) \propto \psi_8$, then $\vec{g}(t) = e^{(8-3d)t}\vec{g}(0) + O(g^2)$.

The structure of the eigenvectors ψ_{2k} should be compared to the structure of the normal-ordered version of the operators ϕ^{2k} described in article 23277.[12]

[12] The value of r might be different, because the value of r depends on exactly how the renormalization group is defined (article 22212).

10 Rewriting the nonlinear flow equations

The next goal is to write the full nonlinear equations in terms of eigenfunctions of the linearized RG equations, so that the (ir)relevance of operators with marginal scaling dimensions can be determined. To do this, define

$$U \equiv \begin{bmatrix} 1 & -r & r^2/2 & -r^3/3! \\ 0 & 1 & -r & r^2/2 \\ 0 & 0 & 1 & -r \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad U^{-1} = \begin{bmatrix} 1 & r & r^2/2 & r^3/3! \\ 0 & 1 & r & r^2/2 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The set of equations (8) may also be written

$$MU = U\Delta \qquad\qquad \Delta \equiv \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & (4-d) & 0 & 0 \\ 0 & 0 & (6-2d) & 0 \\ 0 & 0 & 0 & (8-3d) \end{bmatrix},$$

which implies $U^{-1}MU = \Delta$. Now define $\vec{u}(t)$ by

$$\vec{u}(t) \equiv \begin{bmatrix} u_2(t) \\ u_4(t) \\ u_6(t) \\ u_8(t) \end{bmatrix} \equiv U^{-1}\vec{g}(t), \tag{9}$$

which satisfies

$$\begin{aligned} \frac{d}{dt}\vec{u}(t) &= U^{-1}\frac{d}{dt}\vec{g}(t) \\ &= U^{-1}M\vec{g}(t) + O(g^2) \\ &= (U^{-1}MU)\vec{u}(t) + O(u^2) \\ &= \Delta\vec{u}(t) + O(u^2). \end{aligned}$$

This shows that the components of $\vec{u}(t)$ are the eigenfunctions of the linearized RG equations. To write the nonlinear RG equations in terms of $\vec{u}$, first write them as[13]

$$\frac{d}{dt}\vec{g} = M\vec{g} + \vec{\beta}^{\,\mathrm{NL}}(\vec{g}\,). \tag{10}$$

According to equations (6), the components of $\vec{\beta}^{\,\mathrm{NL}}(\vec{g}\,)$ are[14]

$$\begin{aligned}
\beta_2^{\mathrm{NL}} &= -g_4 g_2 + O(g^3) \\
\beta_4^{\mathrm{NL}} &= -g_6 g_2 - 3g_4^2 + O(g^3) \\
\beta_6^{\mathrm{NL}} &= -g_8 g_2 - 15 g_6 g_4 + O(g^3) \\
\beta_8^{\mathrm{NL}} &= \qquad\quad - 28 g_8 g_4 - 35 g_6^2 + O(g^3).
\end{aligned} \tag{11}$$

All of the quadratic terms are negative, a consequence of the pattern that was shown in section 3. Multiply both sides of (10) by U^{-1} and then substitute

$$\vec{g}(t) = U\vec{u}(t)$$

to get

$$\frac{d}{dt}\vec{u} = \Delta\vec{u} + U^{-1}\vec{\beta}^{\,\mathrm{NL}}(U\vec{u}\,). \tag{12}$$

Instead of working this out explicitly in full generality, the following sections consider cases where a single component of $\vec{u}(0)$ is nonzero, because the goal is to determine whether such a perturbation of the action is (ir)relevant.

[13] The superscript NL stands for *nonlinear*.

[14] To derive these expressions, use $1/(1+g_2) = 1 - g_2 + O(g_2^2)$.

11 Example: all components zero except $u_4(0)$

Section 9 used the linearlized flow equations, which cannot resolve whether the case $\vec{g}(0) \propto \psi_4$ is relevant or irrelevant when $d = 4$. To resolve this, suppose that $u_4(0)$ is the only nonzero component of $\vec{u}(0)$. Then

$$\vec{g}(0) = U\vec{u}(0) = U \begin{bmatrix} 0 \\ u_4(0) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -r \\ 1 \\ 0 \\ 0 \end{bmatrix} u_4(0).$$

Use this in (11) to get

$$\begin{aligned} \beta_2^{\rm NL}\big(U\vec{u}(0)\big) &= ru_4^2(0) + O(u^3) \\ \beta_4^{\rm NL}\big(U\vec{u}(0)\big) &= -3u_4^2(0) + O(u^3) \\ \beta_6^{\rm NL}\big(U\vec{u}(0)\big) &= 0 + O(u^3) \\ \beta_8^{\rm NL}\big(U\vec{u}(0)\big) &= 0 + O(u^3). \end{aligned}$$

Use this in (12) to get

$$\frac{d}{dt}u_4(t)\bigg|_{t=0} = (4-d)u_4(0) - 3u_4^2(0) + O(u^3) \tag{13}$$

When $\vec{u}(0)$ is small, the sign of the right-hand side is determined by the linear term unless its coefficient is zero, in which case the sign is determined by the quadratic term. If $u_4(0)$ is the only nonzero component of $\vec{u}(0)$, then the sign is negative when $d = 4$, so in that case equation (13) says that the corresponding perturbation of the action is *irrelevant*.

Even though this perturbation is irrelevant for all $d \geq 4$, it still induces nonzero values of all of the coefficients g_{2k} with $2k \geq 6$ at t increases. Each of equations (6) has a term involving only g_4 and g_2, so if g_4 and g_2 are the only nonzero gs when $t = 0$, then every g_{2k} has a nonzero initial derivative. That phenomenon was not visible in this section, because this section ignored terms of order g^3.

12 Example: all components zero except $u_6(0)$

Section 9 used the linearlized flow equations, which cannot resolve whether the case $\vec{g}(0) \propto \psi_6$ is relevant or irrelevant when $d = 3$. To resolve this, suppose that $u_6(0)$ is the only nonzero component of $\vec{u}(0)$. Then

$$\vec{g}(0) = U\vec{u}(0) = U \begin{bmatrix} 0 \\ 0 \\ u_6(0) \\ 0 \end{bmatrix} = \begin{bmatrix} r^2/2 \\ -r \\ 1 \\ 0 \end{bmatrix} u_6(0),$$

which gives

$$\begin{aligned} \beta_2^{\text{NL}}\big(U\vec{u}(0)\big) &= \frac{r^3}{2} u_6^2(0) + O(u^3) \\ \beta_4^{\text{NL}}\big(U\vec{u}(0)\big) &= -\frac{3r^2}{2} u_6^2(0) + O(u^3) \\ \beta_6^{\text{NL}}\big(U\vec{u}(0)\big) &= 15 r u_6^2(0) + O(u^3) \\ \beta_8^{\text{NL}}\big(U\vec{u}(0)\big) &= -35 u_6^2(0) + O(u^3). \end{aligned}$$

Use this in (12) to get

$$\left.\frac{d}{dt} u_6(t)\right|_{t=0} = (6 - 2d) u_6(0) - 20 r u_6^2(0) + O(u^3) \tag{14}$$

If $u_6(0)$ is the only nonzero component of $\vec{u}(0)$, then the sign is negative when $d = 3$, so in that case equation (13) says that the corresponding perturbation of the action is *irrelevant*.[15]

Now the phenomenon described at the end of section 11 is already visible at order u^2: the value of β_8^{NL} is nonzero, even though the initial values of g_{2k} are zero for all $2k \geq 8$. This occurs because the equation for β_8 in (6) has a term proportional to g_6^2.

[15] This agrees with the first paragraph on page 32 in Poland *et al* (2019) and also with the text below equation (2.1) in Alvarez-Gaume *et al* (2019).

13 The Wilson-Fisher fixed point

According to standard lore, the quantum field model that led to the flow equations (2) doesn't have any nontrivial fixed points when $d \geq 4$ but does have an isolated nontrivial fixed point when $d = 3$. This isolated nontrivial fixed point is called the **Wilson-Fisher fixed point**.[16] This section reviews the usual textbook evidence (not proof) for the existence of such a fixed point and analyzes its stability.

For this analysis, use a small-g approximation and ignore terms of order g^3 and higher, and consider flows that start at a point where all components of $\vec{g}$ are zero except g_2 and g_4. In that approximation and for that set of initial conditions, the flow equations (5)-(6) reduce to

$$\begin{aligned} \beta_2(\vec{g}\,) &= 2g_2 + g_4 - g_2 g_4 + O(g^3) \\ \beta_4(\vec{g}\,) &= (4-d)g_4 - 3g_4^2 + O(g^3) \\ \beta_{2k}(\vec{g}\,) &= 0 + O(g^3) \text{ for all } 2k \geq 6. \end{aligned} \tag{15}$$

If we pretend that these equations are exact, then the fixed-point condition $\beta_2 = \beta_4 = 0$ has a nonzero solution when $d < 4$:[17]

$$g_2^* = \frac{d-4}{2+d} < 0 \qquad\qquad g_4^* = \frac{4-d}{3} > 0. \tag{16}$$

We should be wary of this prediction, though, because it relies on a small-g approximation, and the values of g_2 and g_4 at this fixed point might not be small enough for that approximation to be valid.

The original quantum field model is defined only for integer values of d (the number of spacetime dimensions), but the flow equations are defined for arbitrary real values of d. If we treat d as a continuous parameter in the flow equations, then we can study the properties of the nontrivial fixed point for $d = 4 - \epsilon$ with

[16] The first two sections Liendo (2017) give a concise introduction.

[17] This is zero when $d = 4$. When $d > 4$, g_4^* is negative, which is not allowed because then the function $V(s)$ defined by equation (3) would not have a lower bound, and then the path integral would be undefined (article 22212).

$0 < \epsilon \ll 1$ and extrapolate the results to $\epsilon = 1$ ($d = 3$), where the underlying quantum field model can be defined. This bold extrapolation turns out to work well, at least after making other improvements to the approximations that led to equation (2).[18]

We can analyze the (in)stability of the nontrivial fixed point just like we did for the trivial fixed point in sections 8-9. With respect to the fixed point (16), the linearized flow equations are given by equation (7) with

$$\delta\vec{g} = \begin{bmatrix}\delta g_2\\ \delta g_4\end{bmatrix}$$

and

$$M = \begin{bmatrix}2 - g_4^* & 1 - g_2^*\\ 0 & 4 - d - 6g_4^*\end{bmatrix} = \begin{bmatrix}(2+d)/3 & (6+d)/(2+d)\\ 0 & d-4\end{bmatrix}.$$

The matrix M has two eigenvectors ψ_2 and ψ_4, with eigenvalues $(2+d)/3$ and $d-4$, respectively, so the flow equations have these solutions:

- If $\vec{g}(0) \propto \psi_2$, then $\vec{g}(t) = e^{(2+d)t/3}\vec{g}(0) + O(\delta g^2)$.
- If $\vec{g}(0) \propto \psi_4$, then $\vec{g}(t) = e^{(d-4)t}\vec{g}(0) + O(\delta g^2)$.

This shows that when $d < 4$, the perturbations corresponding to ψ_2 and ψ_4 are *relevant* and *irrelevant*, respectively.

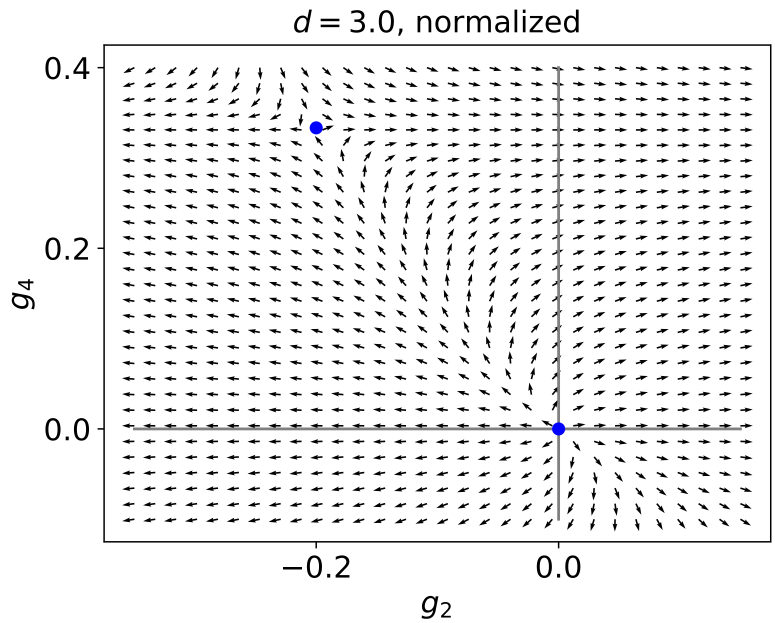
The flow pattern defined by equations (15) is shown in the pictures on the next two pages.[19] Each arrow[20] shows the direction of the vector (β_2, β_4) at that point in the g_2-g_4 plane. When $d = 3$ (first picture), the flow has two fixed points: a nontrivial fixed point (upper left) and the trivial fixed point (lower right). When $d = 4$ (second picture), the flow has only the trivial fixed point. When $d = 3.9$ (third picture, with closeup in the fourth picture), the nontrivial fixed point is

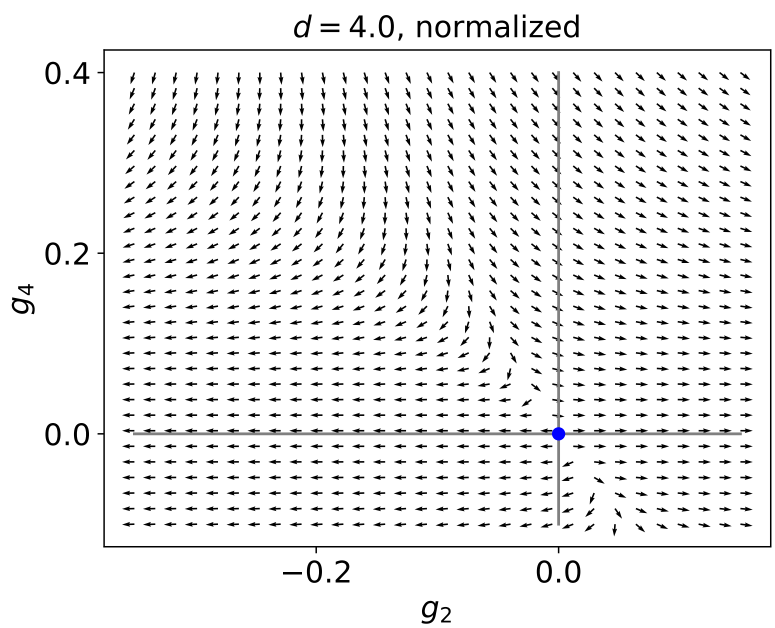
[18] Kleinert and Schulte-Frohlinde (2001) reviews of some of the methods that have been used to study the Wilson-Fisher fixed point in a more general family of models with N scalar fields.

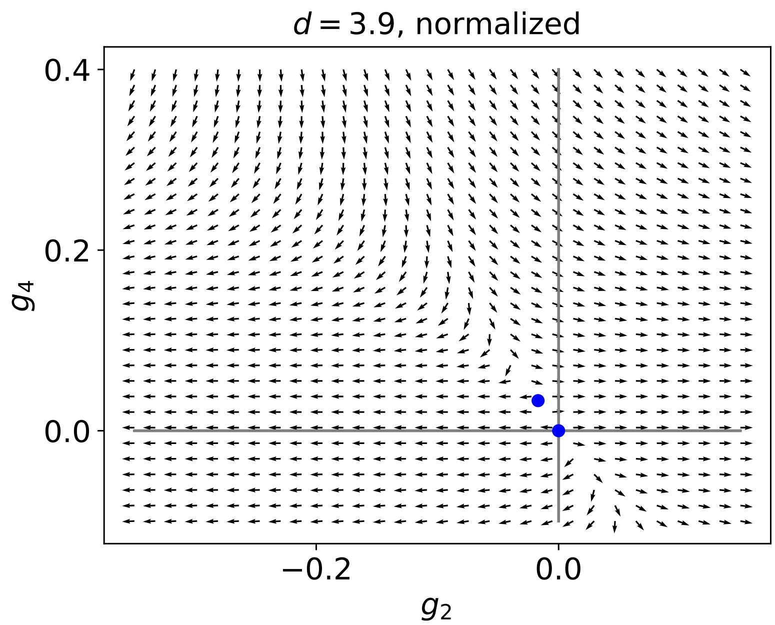
[19] The script that generated these pictures is posted here: https://cphysics.org/extras/79649f.html

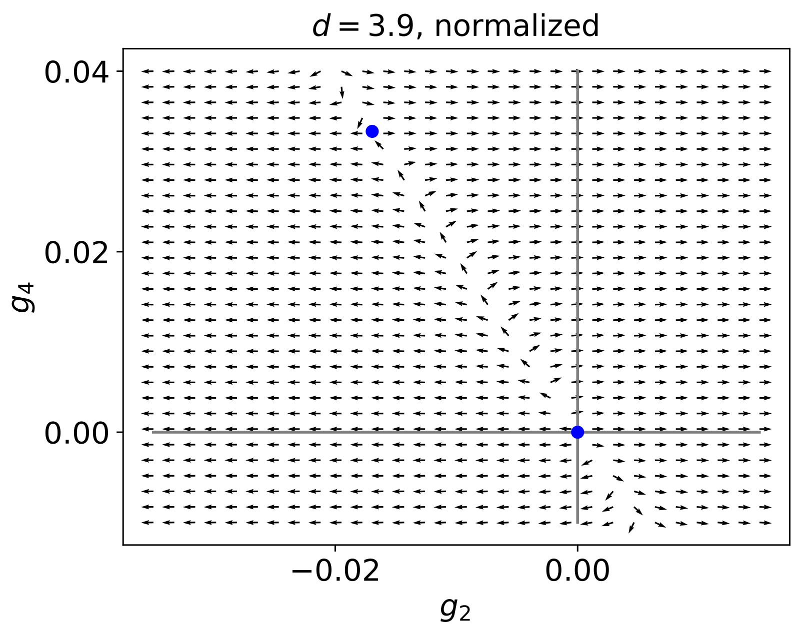
[20] The arrows are all drawn with the same length (this is why the titles say *normalized*), because otherwise the disparity in their relative lengths would make deciphering the picture too difficult.

closer to the trivial fixed point. For $d = 4 - \epsilon$ with arbitrarily small $\epsilon > 0$, the nontrivial fixed point can be made arbitrarily close to the trivial fixed point, where the small-g approximation is valid.









From these pictures, the existence of a flow from the trivial fixed point to the non-trivial one is evident. These results remind us of an important fact: when a given perturbation of the action is characterized as *(ir)relevant*, that characterization refers to a specific fixed point. With respect to the trivial fixed point, two linearly independent relevant perturbations exist in the g_2-g_4 plane (modulo the irrelevant ones). In contrast, with respect to the nontrivial fixed point, the set of infinitesimal perturbations in the g_2-g_4 plane is spanned by one linear and one nonlinear perturbation.

Recall (article 10142) that the number of linearly independent relevant perturbations (modulo the irrelevant ones) is the number of parameters that must be tuned while the continuum limit is being taken in order to obtain a model that differs from the scale-invariant fixed point. A two-parameter family of models has been constructed directly (and nonperturbatively) in continuous three-dimensional spacetime.[21] Those models all live near the trivial fixed point, in the sense that they can be reached from a lattice model (one that treats spacetime as a lattice) by controlling the rate at which the parameters g_2 and g_4 approach zero while the limit is being taken. The resulting family of models in continuous spacetime retains two independent parameters, corresponding to the correlation length (or single-particle mass, which is the inverse of the correlation length) and interaction strength. For most of these models, the interaction strength is nonzero: the models are nontrivial, even though they are closely associated to the trivial fixed point. Within this family of models, only one of them is scale-invariant, and its interaction strength is zero. This is the trivial fixed point.

That two-parameter family of models does not include the quantum field model corresponding to the Wilson-Fisher fixed point. The Wilson-Fisher fixed point corresponds to a quantum field model in continuous spacetime that has a nonzero interaction strength in addition to being scale-invariant. As far as I know, this model has not yet been constructed nonperturbatively, even though we have compelling evidence for its existence.

[21]Several references about the construction of this family of models are cited at the bottom of page 3 in section 2.1 of Dedushenko (2022).

14 A continuum of fixed points?

Equations (5)-(6) were derived using some approximations, including a small-coupling approximation, so they are not guaranteed to be valid except when $\vec{g}$ is infinitesimal. Still, just for fun, we could pretend that equations (5)-(6) are exact and ask whether they have any nontrivial fixed points, even though this doesn't necessarily tell us anything reliable about the original quantum field model.

This section highlights a fun feature of the flow equations (5)-(6), namely the existence of a *continuum* of fixed points. This is presumably[22] only an artifact of the approximations that were used to derive equation (2). If this feature of the approximate system of flow equations (5)-(6) really were a true feature of the underlying quantum field model, then this continuum of fixed points would be called a **conformal manifold**,[23] and a perturbation of the action that keeps the model in this manifold would be called an **(exactly) marginal perturbation**.

A nontrivial fixed point of equations (5)-(6) is a nonzero value of $\vec{g}$ for which $\beta_{2k}(\vec{g}\,) = 0$ for all $k \in \{1, 2, 3, ...\}$. Such values of $\vec{g}$ clearly exist, because a glance at equations (6) shows that for each k, the condition $\beta_{2k}(\vec{g}\,) = 0$ can be solved for g_{2k+2} as a function of the g_n with $n \leq 2k$:

$$\begin{aligned} g_4 &= -2(1+g_2)g_2 \\ g_6 &= -(4-d)(1+g_2)g_4 + \frac{3g_4^2}{1+g_2} \\ g_8 &= -(6-2d)(1+g_2)g_6 + \frac{15g_6g_4}{1+g_2} - \frac{30g_4^3}{(1+g_2)^2} \\ g_{10} &= -(8-3d)(1+g_2)g_8 + \frac{28g_8g_4 + 35g_6^2}{1+g_2} - \frac{420g_6g_4^2}{(1+g_2)^2} + \frac{630g_4^4}{(1+g_2)^3} \end{aligned} \tag{17}$$

and so on. By substitution, we can express each g_{2k+2} as a function of only two variables: $g_{2k} = p_{2k}(g_2, d)$ for all $k \geq 2$. Inspection of equations (17) reveals that

[22] I have not checked this myself, and I don't know of any references that check it, but if the quantum model really did have such a continuum of fixed points, then surely references about this phenomenon would be easy to find!

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these functions are polynomials,[24] so they are finite for all g_2.[25] This shows that the system of flow equations (5)-(6), without any additional constraints, has a continuum of fixed points: one for every value of g_2 except $g_2 = -1$, where the functions (6) are undefined.

Even if the approximations that led to equation (2) were exact, we would still need to check that the function $V(s)$ defined in equation (3) converges at the alleged fixed points. To ensure that the path integral is well-defined (within the approximations that led to (2)), we should also require $V(s)$ to have a finite lower bound. These conditions are easy to check when $\vec{g}$ is small enough to justify the linear approximation, as long as $d \geq 5$ so that the coefficients of the linear terms in (17) are all nonzero.[26] The linearized version of equations (17) is

$$g_{2k+2} = \big((d-2)k - d\big)g_{2k} + O(g^2). \tag{18}$$

for all $k \geq 1$. When $d \geq 5$, the coefficients g_{2k+2} with $k \geq 2$ all have the same sign, so we can choose the sign of g_2 to make them all positive, which ensures that $V(s)$ has a finite lower bound. To check convergence, we can use the **ratio test** for the coefficients in (3). Equation (18) gives

$$\frac{g_{2k+2}/(2k+2)!}{g_{2k}/(2k)!} \approx \frac{(d-2)k}{(2k+2)(2k+1)}$$

when k is large. This ratio goes to zero as $k \to \infty$, indicating that $V(s)$ is finite for all s, as required.[27] A similar conclusion holds for all real values of d except $d \in \{3, 4\}$, in which case the ratio test is affected by the nonlinear terms in (18).[26]

Again, this continuum of fixed points is presumably an artifact of the approximations that led to equation (2). The original quantum field model does not have any such continuum of fixed points, as far as I know.

[24] A Python script that uses the SymPy library to derive these polynomials symbolically is posted here: https://cphysics.org/extras/79649s.html

[25] The polynomial p_{2k} all go to zero when $g_2 \to -1$, but one of the approximations that was used to derive the beta functions (6) assumes $g_2 > -1$, and the beta functions (6) are undefined when $g_2 = -1$.

[26] When $d = 3$ or $d = 4$, one of the coefficients is zero, and then the linear approximation is no longer sufficient even when $\vec{g}$ is small, as explained at the beginning of section 8.

[27] This implication follows from the first theorem quoted in Cruz-Uribe (1997).

15 References

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