Connections, Local Potentials, and Classical Gauge Fields

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Abstract Article 70621 introduced the concept of a **fiber bundle** and its specialization to principal bundles and associated vector bundles. This is the mathematical foundation for the concept of a **gauge field** in classical physics. This article explains how the idea of a gauge field relates to the idea of a principal bundle and how it extends to associated vector bundles.

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1 Motivation

Article 70621 introduced the concept of a *fiber bundle*. Roughly, a fiber bundle consists of a manifold M called the *base space* with a copy of another manifold F called the *fiber* attached to each point of M, in a way that varies smoothly throughout M. The manifold defined by all of these fibers is called the *total space*. Article 70621 also introduced two important specializations of the concept of a fiber bundle: the concept of a *vector bundle*, and the concept of a *principal bundle*.

Many types of field in classical physics may be described as a *section*[1] of a vector bundle. This includes some fields, like spinor fields, that are not normally called "vector fields" (a name that is normally reserved for sections of a tangent bundle).

Gauge fields are different: a classical gauge field is a *connection* on a principal bundle. A connection, which is defined everywhere on the total space, may also be encoded in a collection of *local potentials* that are each defined in a sufficiently small part of the base space. The name **gauge field** may refer either to the connection or to the corresponding local potentials.

Given a principal bundle and a linear representation ρ of G acting on a vector space V, we can define an associated vector bundle with fiber V. Any connection on the principal bundle gives an *induced connection* on the associated vector bundle.

After a brief review of some prerequisites from article 70621, this article introduces the definitions of connection, local potential, and induced connection, along with a few other mathematical concepts that are involved in the study of gauge fields in classical physics.

[1] Most articles in this series use the word *section* for a numbered part of the article. This article uses the word *chapter* instead, like article 70621 does, because *section* is the standard name for a mathematical concept that is prevalent in the study of fiber bundles.

2 Some notation

$$
\begin{aligned}
M &= \text{the base space}\\
E &= \text{the total space}\\
p &= \text{the bundle projection}\\
\tau &= \text{the inverse of a local trivialization}\\
\tau_k &= \text{the inverse of a local trivialization for a chart } U_k\\
\tau_{j\to k} &= \text{transition function from } U_j \text{ to } U_k\\
\sigma &= \text{a section}\\
m \text{ or } u &= \text{a point in } M \text{ or in a chart } U \subset M\\
x &= \text{a point in } E\\
X, Y &= \text{tangent vector fields on } M \text{ or } E \text{ (specified in the text)}
\end{aligned}
$$

Notation for a principal bundle and an associated vector bundle:

$$
\begin{aligned}
G &= \text{the structure group}\\
V &= \text{a vector space}\\
\rho &= \text{a linear representation of } G \text{ on } V\\
E &= \text{the total space of the principal bundle}\\
\hat{E} &= \text{the total space of the associated vector bundle}\\
p &= \text{the principal bundle projection}\\
\hat{p} &= \text{the associated vector bundle projection}\\
\tau \text{ or } \tau_k &= \text{the inverse of a local trivialization of the principal bundle}\\
\hat{\tau} \text{ or } \hat{\tau}_k &= \text{the inverse of a local trivialization of the associated vector bundle}\\
\tau_{j\to k} &= \text{transition function for the principal bundle}\\
\hat{\tau}_{j\to k} &= \text{transition function for the associated vector bundle}\\
\delta &= \text{a gauge transformation of the principal bundle}\\
[\cdot,\cdot] &= \text{a Lie bracket, or the equivalence class defined in chapter 5}
\end{aligned}
$$

3 Fiber bundle: review of the definition

Article 70621 introduced the concept of a (smooth) fiber bundle.[2] This chapter reviews the definition. A **fiber bundle** (E, M, p) involves these data:

- A manifold E called the **total space**,
- A manifold M called the **base space**,
- A map $p : E \to M$ called the **(bundle) projection**.

Those data must satisfy these conditions:

- The **fibers** $p^{-1}(m) \subset E$ are all diffeomorphic[3] to each other, for all $m \in M$.
- Each point $m \in M$ has a neighborhood U whose inverse image $p^{-1}(U) \subset E$ is diffeomorphic to $U \times F$, where the manifold F is diffeomorphic to each fiber. More specifically, a diffeomorphism τ from $U \times F$ to $p^{-1}(U)$ exists with this property: $p(\tau(u, f)) = u$ for all $(u, f) \in U \times F$. The inverse τ^{-1} of such a map is called a **local trivialization**, so τ will be called an **inverse local trivialization**.

Intuitively, the first condition says that E looks like $M \times F$ locally, even if it is not like $M \times F$ globally. The second condition says that this local similarity is compatible with the projection p. Altogether, we can think of the fiber bundle as a base space M with a copy of the fiber F attached to each point of M in a way that varies smoothly throughout M.

[2]For the rest of this article, all manifolds and maps are understood to be smooth (not just topological/continuous).

[3]This means that they are equivalent to each other as smooth manifolds (article 93875).

4 Principal bundle: review of the definition

A **principal bundle** is a special kind of fiber bundle. This chapter reviews the definition.[4]

Compared to the general definition of *fiber bundle*, the definition of a principal bundle involves two additional pieces of data and a few additional conditions. The additional data are:

- A Lie group G called the **structure group**,
- A smooth map $E \times G \to E$ called the **(right) action** of G on the total space E. The image of (x, g) under this map will be denoted xg.

The additional conditions are:

- The bundle projection p satisfies
$$p(xg) = p(x) \qquad \text{for all } g \in G \text{ and all } x \in E, \tag{1}$$
so the fibers over different points of M aren't mixed with each other by $x \to xg$.
- The fiber $\tilde{G}$ is almost the Lie group G, but with no distinguished identity element.[5]
- The Lie group G acts on $\tilde{G}$ from the right as a group of diffeomorphisms, and the local trivializations satisfy the additional condition
$$\tau(u, f)g = \tau(u, fg) \tag{2}$$
for all $g \in G$, with f and u defined as in chapter 3. In words: the maps τ are ***G*-equivariant** in the second argument.

The name **principal *G*-bundle** is often used as a concise way to specify the group G that satisfies these conditions.

[4] Article 70621

[5] Article 70621 explains this in more detail.

5 Associated vector bundle: review of the definition

A **vector bundle** is another special kind of fiber bundle, one whose fiber is a vector space.[6] Given a principal bundle with structure group G, and given a vector space V on which a linear representation ρ of G acts, we can define a vector bundle that is associated to the principal bundle in a particular way, called an **associated vector bundle**. This chapter reviews the definition.[6]

Start with a principal bundle with total space E, base space M, and bundle projection $p : E \to M$. Consider the cartesian product $E \times V$. Each element of $E \times V$ is an ordered pair (x, v) with $x \in E$ and $v \in V$. Let $\sim$ be the equivalence relation defined by declaring $(xg, \rho(g^{-1})v)$ to be equivalent to (x, v) for every $g \in G$. Let $[x, v]$ denote the equivalence class that includes (x, v), so

$$[xg, \rho(g^{-1})v] = [x, v]. \tag{3}$$

The total space of the associated vector bundle is the quotient $\hat{E} \equiv (E \times V)/\!\sim$, also denoted $\hat{E} = E \times_\rho V$. The bundle projection $\hat{p} : \hat{E} \to M$ is defined by

$$\hat{p}\big([x, v]\big) = p(x).$$

This is consistent with the equivalence relation (3), because

$$\hat{p}\Big(\big[xg, \rho(g^{-1})v\big]\Big) = p(xg) = p(x) = \hat{p}\big([x, v]\big).$$

The fiber has the structure of a vector space: within the fiber over the point $m \in M$, addition and scalar multiplication are defined by $[x, v_1] + [x, v_2] \equiv [x, v_1 + v_2]$ and $c[x, v] \equiv [x, cv]$ for any $x \in E$ with $p(x) = m$, where c is a scalar (in $\mathbb{R}$ or $\mathbb{C}$, whichever is the field of scalars for the given vector space V) Altogether, this defines a vector bundle with base space M and fiber V.

[6] Article 70621

6 The concept of a connection

Let (E, M, p) be any fiber bundle. Consider any point $x \in E$, and let $F_x \equiv p^{-1}(p(x))$ be the fiber through that point. The **vertical subspace** at x, denoted V_xE, is the space of vectors that are tangent to F_x at x. This is a subspace of T_xE, the vector space of all vectors tangent to E at x.

A **connection** specifies a **horizontal subspace** H_xE, one for each point $x \in E$, subject to these conditions:[7]

- Every vector in T_xE may be written uniquely as the sum of a vector in V_xE and a vector in H_xE.

- The horizontal subspaces vary smoothly throughout E.[8]

Given a connection, every smooth tangent vector field X on E may be written as the sum of two smooth tangent vector fields, X_V and X_H, where all vectors in X_V are in vertical subspaces and all of the vectors in X_H are in horizontal subspaces.

A fiber bundle may admit many different connections (many different choices of *horizontal*). No one connection is more natural than the others, because the smooth manifold E is not equipped with any definition of the angle between two vectors: linear independence is defined, but angles are not.

Sometimes a connection as defined here is called an **Ehresmann connection**, but often[9] that name is used only for a connection that is *complete* – an additional condition that will be described in chapter 7.

[7] Nakahara (1990), definition 10.2; and Figueroa-O'Farrill (2017), section 5.1, page 37

[8] The vertical subspaces automatically vary smoothly throughout E, because of the required properties of the bundle projection p (chapter 3).

[9] This difference in terminology is acknowledged in the text below definition 3.3.4 in Ryan (2014). Page 81 Kolář *et al* (1993) uses the name *Ehresmann connection* only for complete connections, but section 1 of del Hoyo (2016) uses the name for connections that are not necessarily complete.

7 Horizontal lifts

Let (E, M, p) be any fiber bundle, and let γ be a smooth curve[10] in E. More explicitly: let γ be a smooth map from the interval $[0, 1] \subset \mathbb{R}$ to E. If the curve is not tangent to any fibers, then applying the bundle projection $p : E \to M$ to the curve γ gives a smooth curve $p(\gamma)$ in the base space M, which I'll call the **base curve**. Conversely, the curve γ in E is one way to **lift** the base curve from M to E. It is not the only way, because many different curves in E project to the same base curve in M.

With respect to a given connection, a curve $\gamma \subset E$ is called **horizontal** if all of its tangent vectors are in horizontal subspaces.[11] In this case, the curve γ is called a **horizontal lift** of the corresponding base curve $p(\gamma)$.[12]

The given connection is called **complete** if every curve in the base space M has a horizontal lift.[13] If the fiber is a compact manifold, then every connection is complete.[14] A complete connection is also called an **Ehresmann connection**.[15]

[10] A **curve** is a one-dimensional path. It may be closed or it may have endpoints.

[11] Kobayashi and Nomizu (1963), chapter 2, page 64

[12] Proposition 1.13 in de los Ríos (2020) and proposition 1.2 in chapter II of Kobayashi and Nomizu (1963) address the uniqueness of horizontal lifts in the context of a principal bundle.

[13] Section 2 in del Hoyo (2016) gives an example of a connection that is not complete.

[14] Kolář *et al* (1993), page 81

[15] Sometimes this name is used more generally for connections that are not necessarily complete (footnote 9 in chapter 6).

8 Parallel transport

Let (E, M, p) be a fiber bundle on which a connection (a definition of *horizontal*) is given. Let γ be a curve in the total space E that starts at a point x and ends at a point x'. Applying the bundle projection p to this curve gives a curve $p(\gamma)$ in the base space M that starts at $p(x)$ and ends at $p(x')$. The points x and x' in E belong to the fibers at $p(x)$ and $p(x')$, respectively. If the curve γ is horizontal, then the point $x' \in E$ is said to be obtained from the point $x \in E$ by **parallel transport** along the curve $p(\gamma)$ in M. Different connections give different parallel transports.

In the case of a vector bundle, the fiber is a vector space V, and a point in the fiber over $m \in M$ is a vector at m. In this case, parallel transport is a rule for transporting a vector along a curve in M.

Parallel transport is usually defined in terms of a connection, as in the first paragraph above, but we can also think of parallel transport as the more basic concept.[16,17] One author says it like this:[18]

> If care is taken, it is possible to define a connection via parallel transports. The difficulty comes in identifying the correct statement of how the parallel transport maps should depend smoothly on each of their arguments.

To avoid that difficulty, this article treats the concept of a connection as the more basic concept, which is then used to define parallel transport.

[16] Theorem 5.4 in Morrison (2018) says this in the context of a principal bundle, and the beginning of section 2.2 in Morrison (2018) says this in the context of a vector bundle.

[17] Parallel transport around closed loops is sufficient to determine a a connection modulo gauge transformations. Freed *et al* (2007) explains this in the text around equation (2.3) (in the preprint). Collier *et al* (2015) mentions that parallel transport around closed loops actually determines both the bundle and the connection (modulo gauge transformations). More references are cited in `https://ncatlab.org/nlab/show/parallel+transport`, and some intuition is given in `https://mathoverflow.net/questions/338595/`.

[18] McCarthy (2019), page 43

9 Pushforward and pullback

This chapter reviews two concepts from differential geometry that will be used in the following chapters. Other prerequisite concepts (vector fields, differential forms, and other tensor fields) are reviewed in article 09894.

If M and M' are smooth manifolds and $\delta : M \to M'$ is a diffeomorphism, then:

- Any vector field[19] X on M defines a vector field on $\delta(M) \subset M'$ called the **pushforward** of X, denoted $\delta_*(X)$ or $\delta_* X$.
- Any differential form ω on M' defines a differential form on M called the **pullback** of ω, denoted $\delta^*(\omega)$ or $\delta^*\omega$.

Roughly, the definitions are:

- Given a vector field on M, we can think of the vector at each point as being the "velocity" vector for some curve in M.[20] The diffeomorphism δ maps the curve in M to a curve in M', so it implicitly maps vectors tangent to the original curve in M to vectors tangent to the new curve in M'. That's the pushforward.[21]
- A differential form ω on M' is a (special kind of) map from vector fields on M' to scalar fields on M'. Given a vector field on M, the effect of the differential form $\delta^*\omega$ on that vector field is defined by using the pushforward to get a vector field on M' and then applying ω to that new vector field.[22] In symbols:[23]

$$(\delta^*\omega)(X) = \omega(\delta_* X). \tag{4}$$

[19] Recall (footnote **??** in chapter 1) that the name *vector field* refers specifically to a field of tangent vectors.

[20] Isham (1999), definition 2.4

[21] Tu (2017), section 13.2. This may also be inferred from Lee (2013), proposition 8.19 and the text below it, using proposition 8.16 as the definition of "δ-related" vector fields. Since δ is a diffeomorphism, pushforwards play nicely with vector fields, not just with individual vectors: the pushforward maps each vector field on M to a vector field on M' (Tu (2011), section 14.5).

[22] This is illustrated by example 8.20 on pages 183-184 in Lee (2013).

[23] The objects $\delta^*\omega$ and X live on the manifold M, and the objects ω and $\delta_* X$ live on the manifold M'.

10 Representing a connection as a tensor field

Any connection, as defined in chapter 6, may be represented as a tensor field[24] Φ of type $\binom{1}{1}$ defined on the manifold E. Such a tensor field may be viewed as a device whose input is a vector field on E and whose output is another vector field on E, subject to these conditions[25] (along with the general conditions that every tensor field is required to satisfy):

- All of the vectors in the vector field $\Phi(X)$ belong to vertical subspaces.

- $\Phi\big(\Phi(X)\big) = \Phi(X)$.

In words: Φ projects every vector field X onto its vertical part X_V, as defined in chapter 6.

Given a connection as defined in chapter 6, the corresponding tensor field Φ is the one for which $\Phi(X) = 0$ wherever X is horizontal. Conversely, a tensor field Φ of type $\binom{1}{1}$ satisfying the conditions listed above uniquely determines the horizontal subspaces: they are the subspaces that Φ projects to zero.[26]

[24] Article 09894 reviews the concept of a *tensor field*.

[25] Kolář *et al* (1993), definition 9.3; Koszul (1960), section 3.1, definition 1

[26] Nakahara (1990), equation (10.4), and Bertlmann (1996), text below equation (2.464)

11 The concept of a principal connection

The general definition of *connection* that was given in chapter 6 can be applied to any fiber bundle, but in the context of a principal G-bundle, the definition is usually refined by adding one more condition: the pushforward[27] of the right action of g on the total space defines an action of a group element $g \in G$ on a horizontal subspace $H_x E$, and the result of this action should be another horizontal subspace $H_{xg} E$.[28] A connection satisfying this extra condition is called a **principal connection**.[29] For the rest of this article, this extra condition is understood to be in effect whenever referring to any connection on a principal bundle.

Every principal bundle admits a principal connection.[30]

In terms of the tensor field Φ that was defined in chapter 10, this extra condition is[31]

$$\Phi(\delta_* X) = \delta_* \Phi(X)$$

for any diffeomorphism $\delta : E \to E$ defined by $\delta(x) = xg$ for any fixed $g \in G$. In words: the projection onto the vertical component should commute with the right action of G.

[27] Chapter 9 defined *pushforward*.

[28] Figueroa-O'Farrill (2006a), section 1.3.1

[29] Kolář *et al* (1993), beginning of section 11.1; and Michor (2008), section 19.1

[30] Kolář *et al* (1993), lemma 11.3

[31] Kolář *et al* (1993), definition 11.1, pages 99-100

12 Induced connections on associated bundles

Any connection on a principal bundle (PB) induces a connection on each of its associated vector bundles (AVB). The induced connection may be defined using this sequence of steps:[32]

$$\begin{aligned}\text{connection on PB} &\to \text{parallel transport on PB} && \text{(step 1)}\\ &\to \text{parallel transport on AVB} && \text{(step 2)}\\ &\to \text{connection on AVB.} && \text{(step 3)}\end{aligned}$$

In more detail, consider a principal G-bundle with projection p and an associated vector bundle with projection $\hat{p}$, both with base space M. The steps are:

- **Step 1:** Given a point $x \in p^{-1}(m)$ in the fiber over $m \in M$ and a curve γ from m to m' in M, parallel transport[33] gives a point $x' \in p^{-1}(m')$.
- **Step 2:** Now consider an associated vector bundle defined by a representation ρ of G, and let $\hat{p}$ be its bundle projection. Parallel transport on the associated vector bundle is defined to take the point[34] $[x, v] \in \hat{p}^{-1}(m)$ to the point $[x', v] \in \hat{p}^{-1}(m')$, where x' is obtained from x by parallel transport in the principal bundle (step 1).
- **Step 3:** In the associated bundle, starting with any point $[x, v] \in \hat{p}^{-1}(m)$ in the total space, parallel transport along any given curve in the base space gives a curve in the total space (step 2). Declaring the tangent vectors to these curves to be *horizontal* defines a connection on the associated bundle. This connection is said to be **induced** by the original connection on the principal bundle.

[32] Section 1 in Abbassi and Lakrini (2021) describes this directly in terms of horizontal subspaces instead of via the concept of parallel transport. The description given here (and in Zykoski (2020)) can be recovered from that one by thinking of each horizontal vector as being tangent to a horizontal curve. Section 11.8 in Kolář *et al* (1993) – which is essentially repeated in section 19.8 in Michor (2008) – describes it in a more elegant (but more obtuse) way. Other formulations are given in theorem 3.8 in de los Ríos (2020) and theorem 46 in Haydys (2019).

[33] Chapter 8 defined *parallel transport*.

[34] Chapter 5 defined the notation $[\cdot, \cdot]$.

13 Different ways to describe a Lie algebra

Every Lie group G has a corresponding Lie algebra. This chapter reviews a few different ways of thinking about the corresponding Lie algebra. All three of them will be used in this article.

First, here's a quick review of the concept of the *Lie bracket* of two vector fields. A vector field on any smooth manifold M may be defined as a *derivation*.[35] Given a coordinate system $x_1, x_2, ...$ in a chart $U \subset M$, a vector field (derivation) may be described within U as a first-order partial derivative operator $\sum_k c_k \, \partial/\partial x_k$, with smooth functions $c_k(x_1, x_2, ...)$ as coefficients. Using this description, the commutator of two vector fields in U is well-defined and yields another vector field (another first-order partial derivative operator) in U.[36] This commutator is a local version of the **Lie bracket** $[X, Y]$ of two vector fields X, Y, which is defined on the whole manifold M and always gives another vector field on M, even if the products XY and YX are not individually defined.[37]

A Lie group G is a smooth manifold, so the Lie bracket of vector fields on G is defined. A Lie group is also a group, and for any given $h \in G$, the map $\tilde{G} \to \tilde{G}$ defined by $g \mapsto gh$ (for all $g \in \tilde{G}$) is a diffeomorphism of the manifold $\tilde{G}$. I'm writing $\tilde{G}$ instead of G here to emphasize that the definition of this diffeomorphism doesn't involve the product of elements of $\tilde{G}$ with each other. It only involves the right action of G on $\tilde{G}$, just like the fiber of a principal bundle.[38] A vector field X on $\tilde{G}$ is called **right-invariant** if $h_* X = X$ for all $h \in G$, where $h_* X$ is the pushforward of X by the diffeomorphism defined by $g \mapsto gh$. In words: a right-invariant vector field on $\tilde{G}$ is preserved by the right action of G on $\tilde{G}$.

Now the **Lie algebra** of $\tilde{G}$ may be defined as the set of right-invariant vector fields on $\tilde{G}$, together with the Lie bracket (viewed as an operation that ingests any two such vector fields and returns another one).[39] This is one way to describe the

[35] Article 09894

[36] Lee (2013), proposition 8.26

[37] Lee (2013), lemma 8.25

[38] Chapter 4

[39] Nakahara (1990), definition 5.48

Lie algebra associated with a Lie group G.

The Lie algebra may also be described using only vectors tangent to a given point of G, such as the point that serves as the identity element of G, instead of using vector fields defined everywhere on G. In this view, each element of the Lie algebra is an individual tangent vector at that point. The Lie bracket of two vector fields that was defined above is defined only for vector fields, not for individual tangent vectors, but the Lie bracket between individual vector fields may be defined in a different way (not reviewed here).[40] Then the previous way of thinking about the Lie algebra may be recovered from this one by using the fact that each tangent vector has a unique extension to a right-invariant vector field on all of G.

In the physics literature (and in this article), the Lie group G is usually a matrix group, using ordinary matrix multiplication as the group operation. In this case, any vector tangent to G at the identity element is tangent to a curve of the form $g(t) = e^{\ell t}$ for some matrix ℓ,[41,42] and the matrix ℓ may be used to represent the vector tangent to this curve at $t = 0$. Then the Lie bracket of ℓ and ℓ' is just the commutator $[\ell, \ell'] \equiv \ell\ell' - \ell'\ell$, where $\ell\ell'$ and $\ell'\ell$ are ordinary matrix products.

Altogether, this gives three different ways of thinking about elements of a Lie algebra:

- An element of the Lie algebra of G is a right-invariant vector field on G (or on $\tilde{G}$).
- An element of the Lie algebra of G is an individual vector tangent to G.
- If G is a matrix group, then a vector tangent to G (an element of the Lie algebra) may be described as a matrix.

The rest of this article switches freely between these three perspectives.

[40] Fulton and Harris (1991), section 8.1

[41] Article 18505 defines $e^{\ell t}$ when ℓ is a matrix.

[42] Example: if G is the group of $N \times N$ unitary matrices, then ℓ is an $N \times N$ antihermitian matrix. Another example: if G is the group of $N \times N$ orthogonal matrices, then ℓ is an $N \times N$ antisymmetric matrix.

14 Connection one-form: definition

This chapter introduces a useful way of representing a principal connection on a principal bundle.

In a principal bundle, the fiber is almost a Lie group G, so a vertical vector (a vector tangent to the fiber) at any given point of E corresponds to an element of $\text{Lie}(G)$. Using this correspondence, any vertical vector field may be converted to a Lie-algebra-valued field in a natural way (and conversely). In particular, since each $\ell \in \text{Lie}(G)$ may be viewed as a right-invariant vector field on G, each ℓ naturally defines a vertical vector field X_ℓ on E called the **fundamental vector field** for ℓ.[43] Among vertical vector fields, the fundamental vector fields are special because each fundamental vector field corresponds to the same element of $\text{Lie}(G)$ everywhere on E.

In a principal bundle, the fiber is a Lie group G, so each vertical subspace V_xE is a copy of the Lie algebra $\text{Lie}(G)$ of G. Given a connection on a principal bundle, a Lie-algebra-valued one-form ω on the total space E called the **connection one-form** is defined like this:[44,45]

$$\omega(X) = \begin{cases} \ell & \text{if } X \text{ is the fundamental vector field for } \ell \in \text{Lie}(G), \\ 0 & \text{if } X \text{ is horizontal.} \end{cases} \tag{5}$$

This implies that $\omega(X) = 0$ if and only if X is horizontal, so the connection one-form is another way of encoding a connection on a principal bundle – an alternative to using the tensor field of type $\binom{1}{1}$ described in chapter 10.

To represent a *principal* connection on a principal G-bundle, the connection one-form ω must satisfy an additional condition.[46] Chapter 15 will show how to express that condition when G is a matrix group.

[43] Figueroa-O'Farrill (2006a), section 1.3.1

[44] Figueroa-O'Farrill (2006a), section 1.3.2

[45] For a generic vector field X tangent to E, the output $\omega(X)$ is a function from E to $\text{Lie}(G)$. When the input X is a fundamental vector field, the output $\omega(X)$ is a constant function: it assigns the same ℓ to every point of the total space E.

[46] Chapter 11

15 Connection one-forms and the right action

Consider a principal G-bundle with total space E. Each element $g \in G$ defines a diffeomorphism $\delta : E \to E$ by $\delta(x) = xg$. Given a connection one-form ω, its pullback by δ gives another connection one-form $\delta^*\omega$. Suppose that G is a matrix group, so that elements of G and $\text{Lie}(G)$ may be multiplied with each other. If G is a matrix group and ω is a principal connection, then the new connection one-form is[47,48,49]

$$\delta^*\omega = g^{-1}\omega g. \tag{6}$$

Conversely, if a connection one-form on a principal bundle satisfies the condition (6), then it is a principal connection.[50]

To deduce that a principal connection satisfies (6), start with the fact that $\delta^*\omega$ is defined by

$$(\delta^*\omega)(X) \equiv \omega(\delta_* X) \tag{7}$$

for all vector fields X, where $\delta_* X$ is the pushforward of X by δ. Consider a single point in E, and let v be the vector that X assigns to that point. This vector may be characterized as the tangent vector to a curve γ in E at the point $\gamma(0)$, like this:

$$(vf)(\gamma(0)) = \frac{d}{dt} f(\gamma(t))\bigg|_{t=0} \tag{8}$$

for all functions $f : E \to \mathbb{R}$. To determine the effect of the new connection one-form $\delta^*\omega$ using equation (7), we need to consider the vector $v' \equiv \delta_* v$, which is characterized by an equation like (8) but with $\gamma(t)g$ in place of $\gamma(t)$:

$$(v'f)(\gamma(0)) = \frac{d}{dt} f(\gamma(t)g)\bigg|_{t=0} \tag{9}$$

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[47] Figueroa-O'Farrill (2010a), lemma 5.1; Tu (2017), proposition 27.13; Nakahara (1990) (10.3b')

[48] On the right-hand side of (6), the output of ω at each point of E is an element of $\text{Lie}(G)$, and sandwiching that element of $\text{Lie}(G)$ between g^{-1} and g gives another element of $\text{Lie}(G)$, which is the output of $\delta^*\omega$ at that point of E.

[49] This result will be used in chapter 19 to derive the effect of a gauge transformation on a connection ω, even though the map δ defined here is not necessarily a gauge transformation (chapter 18).

[50] Proof: if a vector field X is horizontal, then $\omega(X) = 0$, which implies $g^{-1}\omega(X)g = 0$, and then (6) gives $0 = (\delta^*\omega)(X)$, which says that $\delta_* X$ is horizontal (using the definition (7)).

If we write v' as a sum of its vertical and horizontal components (with *horizontal* defined by the connection ω), then only the vertical component affects the output of ω.[51] The vector v involves only on the first derivative of $f(\gamma(t))$ at $t=0$, so we can choose the curve γ to have the form

$$\gamma(t) = \gamma_H(t)\exp(\ell t) \tag{10}$$

where $\gamma_H(t)$ is a horizontal curve and $\ell \in \text{Lie}(G)$, so that $\exp(\ell t)$ is an element of G for each $t \in \mathbb{R}$. Then the vertical component of the slope of the curve at $t=0$ is specified by ℓ. Equation (10) implies

$$\gamma(t)g = \gamma_H(t)\exp(\ell t)g = \gamma_H(t)g\exp(g^{-1}\ell g t),$$

so the vertical component of v' corresponds to $\ell_g \equiv g^{-1}\ell g \in \text{Lie}(G)$. Use this in equation (9) to get

$$\begin{aligned}(v'f)(\gamma(0)) &= \frac{d}{dt}f\big(\gamma_H(0)g\exp(\ell_g t)\big)\bigg|_{t=0} + \frac{d}{dt}f\big(\gamma_H(t)g\big)\bigg|_{t=0} \\ &= \frac{d}{dt}f\big(\gamma(0)g\exp(\ell_g t)\big)\bigg|_{t=0} + \frac{d}{dt}f\big(\gamma_H(t)g\big)\bigg|_{t=0}.\end{aligned} \tag{11}$$

The first and second terms on the right-hand side correspond to the vertical and horizontal components of v'. Only the vertical component contributes to $\omega(v')$, so (11) implies

$$\omega(v') = g^{-1}\ell g = g^{-1}\omega(v)g.$$

Combine this with (7) to get (6).

[51] Equation (5)

16 Three ways to represent a connection

In the literature, the name *connection* may refer to any of these closely related concepts:

- a choice of horizontal subspaces as described in chapter 6,
- a tensor field Φ of the type described in chapter 10,
- a Lie-algebra-valued one-form ω of the type described in chapter 14.

The first two concepts make sense on any fiber bundle, but the third one is specific to principal bundles.[52] In the physics literature, the name *connection one-form* typically refers to the third concept, and it always will in the rest of this article.

An advantage of the Lie-algebra-valued version ω is that its exterior derivative is defined. It's defined because the values of ω all belong to the same vector space (the Lie algebra), so ω may be written as $\sum_k \ell_k\omega_k$ where ℓ_k are constant elements of the Lie algebra and ω_k are scalar-valued one-forms. The vertical-vector-valued version Φ can't be written that way, because the values don't all belong to one vector space.[53]

Chapter 21 will introduce something called a **local potential**, which may be viewed as yet another way of encoding a connection.[54]

[52]The distinction between Φ and ω is clearly acknowledged in Koszul (1960), section 3.3, and in Kolář *et al* (1993), section 11.1

[53]This might be why Cohen (2023) page 61 says, "Even though spaces of forms with values in a bundle [like the tangent bundle of the total space] are easy to define, there is no canonical analogue of the exterior derivative."

[54]Figueroa-O'Farrill (2017), section 5.2.4

17 Gauge transformations

Consider a principal G-bundle with total space E, base space M, and bundle projection $p : E \to M$. A **gauge transformation** is a diffeomorphism $\delta : E \to E$ from the total space to itself with these properties:[55]

- It is G-equivariant: $\delta(xg) = \delta(x)g$ for all $x \in E$ and $g \in G$.
- It doesn't affect the base space: $p(\delta(x)) = p(x)$ for all $x \in E$.

Given any function $h : E \to G$ that satisfies

$$h(xg) = g^{-1}h(x)g \tag{12}$$

for all $x \in E$ and $g \in G$, the transformation

$$\delta(x) = xh(x) \tag{13}$$

clearly satisfies the two conditions listed above,[56] so this qualifies as a gauge transformation. Any gauge transformation may be written this way.[57]

The name *gauge transformation* may also refer to other closely related things. It may refer to a change in the choice of local trivialization, or to the relationship between local trivializations in overlapping patches (a transition function), both of which may be described using a local version of a gauge transformation in the sense defined above.[58] A local version of a gauge transformation may also be used to implement a change in the choice of local section.[59] This will be explained in chapter 22.

[55]Husemoller (1966), chapter 7, definition 1.1; Kolář *et al* (1993), definition 10.14; Cohen (2023), definition 4.5 in section 4.3; Maxim (2018), remark 1.10

[56]Equations (12) and (13) imply the first condition. Equations (13) and (1) imply the second condition.

[57]Neeb (2010), proposition 1.6.7

[58]Article 70621.

[59]Choosing a local section in the principal bundle is related to choosing a local trivialization in an associated vector bundle (article 70621).

18 Gauge transformations and the left action

If the group G is abelian, then the condition (12) reduces to $h(xg) = h(x)$. In this case, any transformation of the form $\delta(x) = xh$, with h independent of x, qualifies as a gauge transformation.

In contrast, if the group G is nonabelian and h does not commute with everything in G, then $\delta(x) = xh$ does not qualify as a gauge transformation because it violates the G-equivariance requirement: $\delta(xg) = xgh \neq xhg = \delta(x)g$ for some $g \in G$. This shows that right-multiplication by a fixed element of G is typically not a gauge transformation. The next paragraph shows that left multiplication by a fixed element of G is a gauge transformation, at least locally.

Each point in a principal bundle has a neighborhood that admits a local trivialization: it is equivalent $U \times G$ where U is the corresponding neighborhood in the base space. In this neighborhood, instead of treating the fiber as $\tilde{G}$ (which is only *almost* a Lie group), we can treat it as the Lie group G. Then, in addition to the right action of G that is built into the definition of any principal bundle,[60] a left action of G on the fiber is also defined. The left action commutes with the right action, so for any $g_0 \in G$, the transformation defined by

$$\delta(u, f) = (u, g_0 f) \qquad \text{for } (u, f) \in U \times G \tag{14}$$

is G-equivariant ($\delta(xg) = \delta(x)g$). This shows that, locally, left-multiplication by a fixed element of G is an example of a gauge transformation.

If we take the function h in equation (13) to be $h(u, f) = f^{-1} g_0 f$ for all $u \in U$, then it automatically satisfies the condition (12), and equation (13) becomes equation (14):

$$\delta(u, f) = (u, f)h(u, f) = (u, f)f^{-1}g_0 f = (u, f f^{-1} g_0 f) = (u, g_0 f).$$

[60] Chapter 4

19 Gauge transformations and connections

Let γ be a horizontal curve in the total space E of a principal bundle. In other words, γ is a horizontal lift of a curve $p(\gamma)$ in the base space.[61] *Horizontal* can defined by a connection one-form ω.[62] A gauge transformation $\delta : E \to E$, as defined in chapter 17, changes the curve γ in the total space without changing the curve $p(\gamma)$ in the base space. The new curve $\delta\gamma$ is horizontal with respect to a new connection one-form $\delta^*\omega$ characterized by the condition that the new horizontal subspaces are annihilated by the new connection one-form.[63] This new connection one-form is the pullback[64] of the original connection one-form ω by δ. The gauge transformation δ preserves the vertical subspaces[65] but may change the horizontal subspaces.

Chapter (20) will derive an expression for the new connection one-form $\delta^*\omega$ in terms of ω and the function h that implements the gauge transformation in equation (13), assuming that G is a matrix group. The result is[66]

$$\delta^*\omega = h^{-1}\omega h + h^*\theta, \tag{15}$$

where θ is the **Maurer-Cartan** form on G, whose input is a vector field on G and whose output is the corresponding Lie-algebra-valued field on G.[67,68] Its pullback $h^*\theta$ by the map $h : E \to G$ gives a Lie-algebra-valued one-form on E.

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[61] Chapter 7

[62] Chapter 14

[63] Equation (5)

[64] Chapter 9

[65] Figueroa-O'Farrill (2006a), text above exercise 1.5

[66] This is similar to proposition 1.8 in de los Ríos (2020), but that proposition describes the effect of a gauge transformation on the *local potential* that will be introduced in chapter 21. The derivation is essentially the same, because a local potential is $\omega(\sigma_* X)$ where σ is a local section and X is a vector field on the base space, and any vector tangent to E may be written as a linear combination of vectors of the form $\sigma_* X$ by using various local sections and various choices of X.

[67] Section 2.1 in Collinucci and Wijns (2006) defines the Maurer-Cartan form more carefully and explains how to justify an alternative notation for it that is common in the physics literature.

[68] The natural relationship between tangent vectors on G and elements of $\mathrm{Lie}(G)$ was mentioned in chapter 14.

20 Derivation of (15)

This chapter outlines the derivation of equation (15).

The new connection one-form $\omega' \equiv \delta^*\omega$ is defined in terms of the original one-form ω by the condition[69]

$$\omega'(X) = \omega(\delta_* X),$$

where $\delta_* X$ is the pushforward of the vector field X, so we can study the relationship between ω and ω' by studying the effect of the pushforward on vectors tangent to E. To describe such a vector, let γ be a curve in E, not necessarily horizontal, described as a map from the interval $[-1, 1]$ of real numbers to E, and consider the vector v that is tangent to γ at $\gamma(0)$, so

$$(vf)(\gamma(0)) = \frac{d}{dt} f(\gamma(t))\bigg|_{t=0} \tag{16}$$

for all functions $f : E \to \mathbb{R}$. Define

$$g(t) \equiv h\big(\gamma(t)\big) \tag{17}$$

where h implements the gauge transformation as in equation (13). Then the gauge-transformed curve is

$$\gamma'(t) = \gamma(t) g(t), \tag{18}$$

so the pushforward v' of v is given by

$$\begin{aligned}(v'f)\big(\gamma'(0)\big) &= \frac{d}{dt} f\big(\gamma(t)g(t)\big)\bigg|_{t=0} \\ &= \frac{d}{dt} f\big(\gamma(t)g(0)\big)\bigg|_{t=0} + \frac{d}{dt} f\big(\gamma(0)g(t)\big)\bigg|_{t=0} \end{aligned}\tag{19}$$

In equation (19), the first step used equation (18), and the second step used a familiar property of ordinary derivatives. We can write v' as the sum of two terms,

$$v' = v'_1 + v'_2,$$

[69] Equation (4)

corresponding to the two terms on the last line of equation (19). The term v_1' would be the same as the original vector v except that the curve to which the vector is tangent is modified by right-multiplication by $g(0)$. This implies[70]

$$\omega(v_1') = g(0)^{-1}\omega(v)g(0) \qquad\qquad \text{at the point where } t = 0. \tag{20}$$

Using $\gamma'(0) = \gamma(0)g(0)$, the term v_2' may be written

$$\frac{d}{dt}f\big(\gamma(0)g(t)\big)\bigg|_{t=0} = \frac{d}{dt}f\big(\gamma'(0)g(0)^{-1}g(t)\big)\bigg|_{t=0}. \tag{21}$$

The curve $\gamma'(0)g(0)^{-1}g(t)$ lies within the fiber through $\gamma'(0)$, so this shows that v_2' is tangent to the fiber and therefore corresponds to an element of Lie(G), which may be written

$$g(0)^{-1}\frac{d}{dt}g(t)\bigg|_{t=0} \tag{22}$$

when G is a matrix group. Equation (5) says that $\omega(v_2')$ is the element of Lie(G) corresponding to the vertical vector v_2', but so is the quantity (22), so $\omega(v_2')$ is equal to (22). Equation (17) says that $g(t)$ is the composition of the maps $\gamma : \mathbb{R} \to E$ and $h : E \to G$, and v is tangent to γ at $t = 0$, so the quantity (22) may also be written as $\theta(h_* v)$, where θ is the Maurer-Cartan form.[71] Equation (4) says that this is the same as $(h^*\theta)(v)$, so

$$\omega(v_2') = (h^*\theta)(v). \tag{23}$$

Use this together with (20) on the last line of (19) to get

$$\omega'(v) \equiv \omega(v') = g(0)^{-1}\omega(v)g(0) + (h^*\theta)(v).$$

The fact that this holds for all v gives the result (15).

[70] Equation (6)
[71] Chapter 19

21 The concept of a local potential

A (Lie algebra valued) connection one-form lives in the total space E of a principal bundle. In physics, a gauge field is usually described instead as living in the base space M, which represents spacetime. To define this, choose an open set $U \subset M$ over which a local section σ may be defined. (If the bundle is trivial, then we can take U to be all of M.) A local section σ is[72] a smooth map from U to E with $p(\sigma(u)) = u$ for all $u \in U$. Its pullback[73] σ^* is a map from one-forms on E to one-forms on U. A connection one-form ω on a principal G-bundle is a Lie(G)-valued one-form on the total space E. Applying the pullback σ^* to ω gives a Lie(G)-valued one-form on U called a **local potential**,[74,75] traditionally denoted A:

$$A \equiv \sigma^* \omega. \tag{24}$$

To relate this to the way a local potential is usually written in the physics literature, let[76] $(x^1, x^2, ..., x^K)$ be a coordinate system on the chart U, so that $x(m)$ are the coordinates of a point $m \in U$, and let ∂_k denote the partial derivative with respect to the kth coordinate x^k. The local potential A is a one-form on U, so its input is a vector field on U, such as one of the partial derivatives ∂_k.[77] The quantities $A_k = A(\partial_k)$ are Lie(G)-valued functions on U called the **components** of the local potential in the given coordinate system. The value of A_k at a point $m \in U$ may be denoted $A_k(m)$. For each m and each index k, $A_k(m)$ is an element of Lie(G).[78]

[72] Article 70621

[73] Chapter 9

[74] Nakahara (1990), section 10.1.3; Figueroa-O'Farrill (2017), section 5.2.4

[75] The output of a connection tensor on E can be described as either a vertical vector field or as a Lie-algebra-valued field (chapters 10 and 14), but the output of a local potential on M can only be described as a Lie-algebra-valued field because the "vertical" direction is undefined for the base space M: the base space doesn't have any fibers (Kolář *et al* (1993), beginning of section 11.1; and Mazzoni (2018), text above equation (1.2.1.3)).

[76] The superscript k on x^k is an index, not an exponent.

[77] Partial derivative with respect to coordinates are vector fields (article 09894).

[78] When G is a matrix group, an element of Lie(G) is naturally represented as an antisymmetric matrix, and that's the convention I'm using here. Using symmetric matrices (instead of antisymmetric) is more convenient in some contexts, and article 03838 enforces that convention by inserting a factor of i into the definition (24).

22 Gauge transformations and local sections

Consider a principal G-bundle with total space E, base space M, and bundle projection p. The principal bundle may not have any global sections, but it does have local sections. A local section is a map from a chart $U \subset M$ to the total space E. This chapter shows that any two local sections over the same chart are related to each other by a local version of a gauge transformation – local in the sense that it only needs to be defined in $p^{-1}(U)$, the part of E over that chart.

Consider a diffeomorphism δ from $p^{-1}(U) \subset E$ to itself, satisfying these conditions:

- $\delta(xg) = \delta(x)g$ for all $x \in p^{-1}(U)$ and all $g \in G$
- $p(\delta(x)) = p(x)$.

I'm calling this a *gauge transformation* because these are just like the conditions for a gauge transformation in chapter 17, but here they only need to be defined for the part of E over the chart U.

For any point m of the base space M, the right action of G on the fiber $p^{-1}(m)$ is *free* and *transitive*.[79] This implies that if σ is any local section over a chart $U \subset M$ that admits a local section, then each point in $p^{-1}(U)$ may be written as $\sigma(m)g$ in exactly one way, with $m \in U$ and $g \in G$. Now consider two local sections σ and $\tilde{\sigma}$, and define a gauge transformation δ like this:

- The effect of δ on points in the section σ is defined by $\delta\big(\sigma(m)\big) \equiv \tilde{\sigma}(m)$.
- The effect of δ on points in the shifted section σg is defined by $\delta\big(\sigma(m)g\big) \equiv \tilde{\sigma}(m)g$.

This defines δ on all points in $p^{-1}(U)$, and it satisfies the required conditions by construction. This shows that any two given local sections over the same chart are related to each other by a local version of a gauge transformation.

[79]Article 70621

23 Gauge transformations and local potentials

The definition (24) of the local potential A depends both on a connection ω and on a local section σ defined on a chart U. Choosing a different local section $\tilde{\sigma}$ over the same chart gives a different local potential corresponding to the same connection. As far as the local potential is concerned, chapter 22 showed that keeping the same connection and changing the section is equivalent to keeping the same section and using a (locally defined) gauge transformation to change the connection.[80]

Chapter 24 will use this equivalence to derive how a change of local section affects the local potential. This chapter summarizes the result. Consider the local potentials $A \equiv \sigma^*\omega$ and $\tilde{A} \equiv \tilde{\sigma}^*\omega$ defined by two local sections σ and $\tilde{\sigma}$. The relationship between the two local sections may be written $\tilde{\sigma}(u) = \sigma(u)g(u)$ with $g(u) \in G$ for each $u \in U$. The goal is to express $\tilde{A}$ in terms of A and $g(u)$. When G is a matrix group, as it usually is in physics, the result may be written[81,82]

$$\tilde{A} = g^{-1}Ag + g^{-1}dg. \tag{25}$$

More explicitly,

$$\tilde{A}(X)(u) = \big(g(u)\big)^{-1}A(X)(u)g(u) + \big(g(u)\big)^{-1}dg(u)$$

for all points $u \in U$ and all vector fields X on U. The notation $A(X)(u)$ means "evaluate the one-form A on the vector field X to get a Lie-algebra-valued field on U, and then evaluate that field at the point u." The product is the usual matrix product.[83] The one-form dg is defined by treating each component of the matrix as a scalar field.[84]

[80] Trautman (1979), end of section 2

[81] Nakahara (1990), equation (10.9); Isham (1999), equation (6.1.25); de los Ríos (2020), text before proposition 1.10

[82] If g were replaced by g^{-1}, then this would become $\tilde{A} = gAg^{-1} - (dg)g^{-1}$, as in Figueroa-O'Farrill (2006a).

[83] Example: $A(X)(u)$ is an element of Lie(G), and $g(u)$ is an element of G, so if G is a matrix group (as assumed here), then $A(X)(u)g(u)$ is the ordinary product of two matrices.

[84] Article 09894 defines the one-form $d\phi$ corresponding to a scalar field ϕ.

24 Derivation of equation (25)

This chapter outlines the derivation of equation (25). The strategy will be to combine the equivalence that was described in chapter 22 with equation (15), which describes the effect of a gauge transformation on a connection.[85]

Let δ be a locally-defined gauge transformation that relates the two local sections, as in chapter 22, so

$$\delta\big(\sigma(u)\big) = \tilde{\sigma}(u) \qquad\qquad \text{for all } u \in U. \tag{26}$$

Express δ in terms of a function $h : E \to G$, as in equation (13):

$$\delta(x) = xh(x) \qquad\qquad \text{for all } x \in E. \tag{27}$$

Combine (26) and (27) to get

$$\tilde{\sigma}(u) = \sigma(u)g(u) \tag{28}$$

with

$$g(u) = h\big(\sigma(u)\big). \tag{29}$$

To relate the local potentials $A \equiv \sigma^*\omega$ and $\tilde{A} \equiv \tilde{\sigma}^*\omega$ to each other, start with

$$\delta^*\omega = h^{-1}\omega h + h^*\theta \tag{30}$$

(this is equation (15)) and apply the pullback σ^* to both sides. According to equation (26), the left-hand side becomes

$$\sigma^*\delta^*\omega = \tilde{\sigma}^*\omega = \tilde{A},$$

so equation (30) becomes

$$\tilde{A} = \sigma^*(h^{-1}\omega h + h^*\theta). \tag{31}$$

[85] A faster approach is to apply ω to lemma 10.6 in Nakahara (1990), but the approach used here has the benefit of relating the transformation of the local potential (which is a one-form on a single chart in the base space) to the transformation of the whole connection (which is a one-form on the total space).

The first term on the right-hand side is

$$\sigma^*(h^{-1}\omega h) = g^{-1}(\sigma^*\omega)g = g^{-1}Ag \tag{32}$$

with the function $g : U \to G$ defined by (29). According to equation (29), the second term on the right-hand side of (31) is

$$\sigma^* h^* \theta = g^* \theta. \tag{33}$$

Use (32) and (33) in (31) to get

$$\tilde{A} = g^{-1}Ag + g^*\theta.$$

The one-form $g^*\theta$ takes a vector field on U, uses g to push it forward to a vector field on G, and then uses the Maurer-Cartan one-form θ to convert it to the corresponding Lie(G)-valued function. When G is a matrix group, the net effect may be written as[86]

$$g^*\theta = g^{-1}dg,$$

where dg is the differential of the function $g : U \to G$.[87] This gives the final result (25).

[86] de los Ríos (2020), text before proposition 1.10

[87] Article 09894 defines the differential of a scalar function. When g is a matrix whose components g_{ab} are functions, then dg is defined to be the matrix with components dg_{ab}.

25 Parallel transport and local potentials

Consider a principal G-bundle with total space E, base space M, and bundle projection $p : E \to M$. Let γ be a curve in M that is parameterized by $0 \leq t \leq 1$, starting at $\gamma(0)$. We can lift the curve γ into the total space in various ways, including these:

- If γ is contained in a single chart $U \subset M$ and $\sigma : U \to E$ is a local section, then we can lift the curve γ to the curve γ_σ given by $\gamma_\sigma(t) \equiv \sigma(\gamma(t))$.

- If we choose a connection ω (a definition of *horizontal*), then we can lift γ to a curve γ_ω that starts at the same point $\sigma(\gamma(0))$ as before but is horizontal everywhere.

For each value of t, the points $\gamma_\sigma(t)$ and $\gamma_\omega(t)$ are in the same fiber, so they are related to each other by

$$\gamma_\omega(t) = \gamma_\sigma(t)g(t) \tag{34}$$

with $g(t) \in G$. If we choose the horizontal lift γ_ω to start at the point $\sigma(\gamma(0))$, as indicated above, then $g(0) = 1$.

The local potential $A \equiv \sigma^*\omega$, depends both on the connection ω that was used to define γ_ω and on the local section σ that was used to define γ_σ, so we might expect that the relationship (34) between the two curves may be expressed in terms of A. Such a relationship may be derived by combining equation (34) with the fact that the connection satisfies $\omega(v) = 0$ for every vector v tangent to the horizontal curve γ_ω.[88] This leads to a differential equation for $g(t)$ involving A. If G is a matrix group, then the differential equation may be written[89]

$$\frac{d}{dt}g(t) = -A\big(\dot\gamma(t)\big)g(t), \tag{35}$$

[88] Equation (5)

[89] Nakahara (1990), equation (10.13b); and Figueroa-O'Farrill (2010b), equation (95)

where $\dot{\gamma}(t)$ is the "velocity vector" tangent to the curve γ at t, and the product in the second term is the matrix product.[90] Given the local potential A, we can think of (35) as a differential equation to be solved for $g(t)$.

The result (35) can be understood intuitively. The local potential A is the vertical component of the section's tangent vector, so it measures the rate at which the section deviates from being horizontal along the given curve. The output of the local potential A expresses this vertical component as an element of Lie(G), so integrating this deviation along the curve gives the group element that moves the curve's endpoint from the section to where it would be if the curve were horizontal, which is the inverse[91] of the group element g in equation (34). The differential equation (35) may be viewed as a way of expressing this intuition.

If the quantity $A(\dot{\gamma}(t))$ in equation (35) were constant (independent of t), then the solution of (35) could be written as $g(t) = \exp(-At)g(0)$. Intuitively, any curve may be subdivided into pieces that are small enough to be approximately constant, so the general solution of (35) may be written approximately as

$$g(t) \approx \exp(-A_N\,\epsilon)\cdots\exp(-A_2\epsilon)\exp(-A_1\epsilon)g(0) \tag{36}$$

with $A_n \equiv A(\dot{\gamma}(n\epsilon))$ for some small increment $\epsilon \ll 1$. When the group G is abelian, the quantities A_n all commute with each other, and then (36) may be written $\exp(-\epsilon\sum_n A_n)g(0)$. This doesn't work when the group G is not abelian, but the product (36) is still often abbreviated as

$$g(t) = \left(P\,\exp\left(-\int_0^t ds\;A\big(\dot{\gamma}(s)\big)\right)\right)g(0), \tag{37}$$

where the prefix P stands for *path-ordered*, a reminder that the exponential of the "sum" (integral) should be interpreted as the $N\to\infty$ limit of a product of exponentials, one for each term in the "sum," multiplied in sequential order along the path (curve).

[90]For each value of t, $A(\dot{\gamma}(t))$ is the matrix representing an element of Lie(G), and $g(t)$ is a matrix representing an element of G.

[91]This accounts for the sign in equation (35).

26 Parallel transport on associated bundles

Consider a principal G-bundle (E, M, p) an associated vector bundle $(\hat{E}, M, \hat{p})$ with fiber V. According to equation (3), each point of $\hat{E}$ may be represented in many different ways. Given a chart $U \subset M$, we can use a local section $\sigma : U \to E$ to specify a unique representative of each point of $\hat{p}^{-1}(U) \subset \hat{E}$, namely[92]

$$[\sigma(m), v] \tag{38}$$

with $m \in U$ and $v \in V$.

Now let ω be a connection on the principal bundle, and define the curves γ_σ and γ_ω as in chapter 25. Both of these curves in E project to the same curve γ in M, but one stays on the section σ and the other stays horizontal. Step 2 in chapter 12 says that in the associated vector bundle, the function $[\gamma_\omega(t), v]$ describes parallel transport along the curve γ. To express this in the representation (38), use equation (34) and then (3) to get

$$[\gamma_\omega(t), v] = \Big[\sigma\big(\gamma(t)\big)g(t), v\Big] = \Big[\sigma\big(\gamma(t)\big), \rho\big(g(t)\big)v\Big]. \tag{39}$$

The right-hand side again describes parallel transport along γ, now expressed using the representation (38).

If desired, equation (37) may be used to write the right-hand side of (39) in terms of the local potential $A \equiv \sigma^*\omega$.

[92]Previous sections use the letter m for a generic point in the base space M and used the letter u for a point in a particular chart $U \subset M$. Here, the letter m is used for a point of U, because u looks too much like v – the letter that is being used for an element of the vector space V.

27 Gauge transformations and associated bundles

Consider two sections related by $\tilde{\sigma}(m) = \sigma(m)g(m)$. As stated at the beginning of chapter 23, keeping the same connection and changing the section is equivalent to keeping the same section and using a (locally defined) gauge transformation to change the connection. Equation (3) gives

$$\big[\tilde{\sigma}(m), v\big] = \big[\sigma(m)g(m), v\big] = \big[\sigma(m), \rho\big(g(m)\big)v\big]$$

The function $g(m)$ here is the same as in equation (25).

Example: if $G = U(1)$, then $g(m) = e^{i\theta(m)}$. This gives $\rho(g(m)) = e^{in\theta(m)}$ for some integer n that characterizes the representation ρ of $U(1)$, and equation (25) becomes $\tilde{A} = A + d\theta$.

28 Holonomy

Consider a principal G-bundle with total space E, base space M, and bundle projection $p : E \to M$. Suppose that a connection ω (a definition of *horizontal*) has been chosen. Let γ_ω be a horizontal curve in E, and let $\gamma \equiv p(\gamma_\omega)$ be its projection to the base space M. Even if γ_ω starts and ends on the same fiber (so that its projection γ is a closed curve in M), it might not start and end at the same point in that fiber. In other words, a horizontal lift of a closed curve in M is not necessarily a closed curve in E.[93]

Suppose that the curve γ_ω is parameterized by $0 \leq t \leq 1$, starting at $t = 0$ and ending at $t = 1$. Suppose that the initial point $x \equiv \gamma_\omega(0)$ and final point $x' \equiv \gamma_\omega(1)$ are in the same fiber, so $p(x') = p(x)$. This implies $x' = xh$ for some $h \in G$. The group element h depends on the connection ω and on the curve γ in M. For a given $x \in E$, the group consisting of these hs for all curves that start at x and end on the same fiber is called the **holonomy group** at x,[94] and the group element h for an individual curve is sometimes called a **holonomy**.[95]

The group element h doesn't depend on any local section, but if the corresponding curve γ in M is contained within a single chart $U \subset M$, then we can choose a local section σ and use the approach in chapter 25 to express h in terms of the function $g(t)$ that was defined there. Equation (34) implies that the holonomy h is

$$h = (g(0))^{-1} g(1), \tag{40}$$

because $\gamma_\sigma(1) = \gamma_\sigma(0)$. The function $g(t)$ defined in chapter 25 depends on the local section σ, but the combination (40) does not.[96]

[93] For this reason (McCarthy (2019), text below definition 2.1.25), horizontal sections typically don't exist: in a typical fiber bundle with a typical connection, every section has some tangent vectors that are not horizontal.

[94] Collinucci and Wijns (2006), text below equation (51)

[95] Sengupta (2007), text above equation (5)

[96] To check this, choose another local section σ' and define the function $\beta : U \to G$ by $\sigma = \sigma'\beta$. Then equation (34) may be written either as $\gamma_\omega(t) = \sigma(\gamma(t))g(t)$ or as $\gamma_\omega(t) = \sigma'(\gamma(t))g'(t)$ with $g'(t) = \beta(\gamma(t))g(t)$. Now use $\beta(\gamma(0)) = \beta(\gamma(1))$ to get $(g'(0))^{-1}g'(1) = (g(0))^{-1}g(1)$, which shows that the holonomy is independent of the local section.

29 Holonomy and the Lie bracket

This chapter explains how the Lie bracket[97,98] may be used to formulate an infinitesimal version of holonomy. For motivation, start with these observations:

- Even if a loop γ in the base space is closed, a horizontal lift of γ into the total space (with *horizontal* defined by a given connection) might not be closed.

- Even if $[X, Y] = 0$ for two given vector fields X, Y in the base space, horizontal lifts X_H and Y_H of X and Y into the total space might have a nonzero Lie bracket: $[X_H, Y_H] \neq 0$.

These two facts are related to each other. The relationship uses the concept of the *flow* of a vector field, which is introduced in the next paragraph.[99]

Given a smooth manifold, let $\varphi(t)$ be a one-parameter group of diffeomorphisms from the manifold to itself, one for each real number t, with $\varphi(0)$ the trivial (identity) diffeomorphism that doesn't do anything. For any given point x in the manifold, the function $x(t) \equiv \varphi(t)x$ describes a curve starting at x. The "velocity" vectors tangent to these curves constitute a vector field X_φ on the manifold. The family $\varphi(t)$ of diffeomorphisms is called the **flow** of X_φ. Conversely, given any vector field X on the manifold, each point x in the manifold has a neighborhood in which a corresponding **local flow** $\varphi_X(t)$ exists.[100,101]

Now let X and Y be vector fields in the base space M, and consider a neighborhood $U \subset M$ in which corresponding local flows φ_X and φ_Y exist. The Lie bracket $[X, Y]$ of the vector fields is zero if and only if the flows φ_X and φ_Y commute with

[97] A derivation of this relationship is also outlined in Clarke and Santoro (2012), section 3.4.1.

[98] Chapter 13 reviewed the *Lie bracket* concept.

[99] This is introduced more carefully in Lee (2013), chapter 9, page 214.

[100] Karshon (2011), first page

[101] A vector field X is called **complete** if we can take this neighborhood to be the whole manifold (Gualtieri (2012), definition 4.9). If the manifold is compact, then every smooth vector field is complete (Gualtieri (2012), theorem 4.10).

each other:[102]

$$[X, Y] = 0 \qquad \Leftrightarrow \qquad \varphi_X(t)\varphi_Y(t) = \varphi_Y(t)\varphi_X(t).$$

If the flows commute with each other, then

$$\varphi_Y^{-1}(t)\varphi_X^{-1}(t)\varphi_Y(t)\varphi_X(t)x = x$$

for each point $x \in U$. This implies that the piecewise curve with segments

$$\begin{aligned}
\varphi_Y(t)x \qquad & 0 \le t \le \epsilon \\
\varphi_Y(t)\varphi_Y(\epsilon)x \qquad & 0 \le t \le \epsilon \\
\varphi_X^{-1}(t)\varphi_Y(\epsilon)\varphi_Y(\epsilon)x \qquad & 0 \le t \le \epsilon \\
\varphi_Y^{-1}(t)\varphi_X^{-1}(\epsilon)\varphi_Y(\epsilon)\varphi_Y(\epsilon)x \qquad & 0 \le t \le \epsilon
\end{aligned}$$

is a closed loop. This closed loop shrinks to a point as $\epsilon \to 0$. In this way, the condition $[X, Y] = 0$ at a given point is related to an infinitesimal closed loop with tangent vectors in X and Y.

Now let X_H and Y_H be horizontal lifts of X and Y. The condition $[X, Y] = 0$ doesn't imply that $[X_H, Y_H]$ is zero (the horizontal lift of a closed curve is not necessarily closed), but it does imply that $[X_H, Y_H]$ is vertical.[103] To prove this, use the fact that the pushforward of the Lie bracket of two vectors fields is equal to the Lie bracket of their pushforwards.[104] In particular,

$$p_*[X_H, Y_H] = [p_*X_H, p_*Y_H]$$

where p is the bundle projection. The right-hand side is zero because $p_*X_H = X$, $p_*Y_H = Y$, and $[X, Y] = 0$. This implies that the left-hand side is zero, which says that $[X_H, Y_H]$ is vertical, as claimed.

[102] Lee (2013), theorem 9.44; Gualtieri (2012), theorem 4.16
[103] Clarke and Santoro (2012), text above corollary 2.4.8
[104] Lee (2013), corollary 8.31

In a principal G-bundle, a vertical vector corresponds to an element of the Lie algebra $\mathrm{Lie}(G)$, which is like an infinitesimal version of an element of G. When $[X, Y] = 0$, this says that $[X_H, Y_H]$ assigns an element of $\mathrm{Lie}(G)$ to each point of the total space. This is formalized by the *curvature two-form* that will be introduced in chapter 30.

Altogether, at any given point in the total space, $[X_H, Y_H]$ assigns a vertical vector to that point (if $[X, Y] = 0$ in the base space), and the corresponding element of $\mathrm{Lie}(G)$ is an infinitesimal version of the group element that represents the holonomy around the closed curve in the base space that was described above. This loose statement is made precise by the **Ambrose-Singer theorem**, which says that the identity component[105] of the holonomy group of a point is generated by the elements of $\mathrm{Lie}(G)$ that the curvature two-form assigns to that point, for all possible horizontal vector fields X_H and Y_H.[106]

[105]The **identity component** of a Lie group is its largest connected subgroup.

[106]Ambrose and Singer (1953), theorem 2. What those authors call the *null holonomy group* is the same as the identity component of the holonomy group (Figueroa-O'Farrill (2010b), end of section 7.1).

30 The curvature of a connection

Given a Lie(G)-valued connection one-form ω on a principal bundle with total space E. The curvature Ω of ω is a Lie(G)-valued two-form on E. It is defined by[107]

$$\Omega(X,Y) = (d\omega)(X_H, Y_H) \tag{41}$$

for all vector fields X, Y on E, where X_H and Y_H denote their horizontal components and $d\omega$ is the exterior derivative of ω. The identity

$$(d\omega)(X,Y) = X\omega(Y) - Y\omega(X) - \omega\big([X,Y]\big)$$

holds for any one-form ω.[108] Use this together with (5) in (41) to get

$$\Omega(X,Y) = -\omega\big([X_H, Y_H]\big), \tag{42}$$

which explains the significance of the curvature two-form more intuitively: the curvature detects the vertical component of the Lie bracket of horizontal vectors.[109] The definition (41) turns out to imply[110,111]

$$\Omega(X,Y) = (d\omega)(X,Y) + \big[\omega(X), \omega(Y)\big].$$

This may be written more concisely as

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega], \tag{43}$$

where $[\alpha, \beta]$ is defined for one-forms α and β by[112]

$$[\alpha,\beta](X,Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)].$$

[107] Figueroa-O'Farrill (2006b), section 2.1.2
[108] Nakahara (1990), equation (5.70); Tu (2011), beginning of section 20
[109] Nakahara (1990), section 10.3.3
[110] Figueroa-O'Farrill (2006b), proposition 2.1; Collinucci and Wijns (2006), section 2.4, theorem 7
[111] de los Ríos (2020) uses this as the definition (definition 2.3) and then derives (41) as a result (corollary 2.6).
[112] de los Ríos (2020), text below definition 2.1

31 The field strength

Let ω be a connection on a principal bundle, let Ω be its curvature, and let σ be a local section. The pullback $\sigma^*\Omega$ of the curvature is often called the **field strength**.[113] If X, Y are vector fields on the base space and $\sigma_* X$ and $\sigma_* Y$ are their pushforwards, then $\sigma^*\Omega$ is the two-form on the base space M, defined by[114]

$$(\sigma^*\Omega)(X,Y) \equiv \Omega(\sigma_* X, \sigma_* Y). \tag{44}$$

Pullbacks commute with exterior differentiation,[115] so if we write $A \equiv \sigma^*\omega$ for the local potential, then equation (43) implies

$$(\sigma^*\Omega)(X,Y) = (dA)(X,Y) + \big[A(X), A(Y)\big]. \tag{45}$$

If we choose a coordinate system on the chart over which the section σ is defined, then the partial derivative ∂_k with respect to the kth coordinate is a vector field on the base space, and the **components** of A and $\sigma^*\Omega$ in this coordinate system are defined by $A_k \equiv A(\partial_k)$ and $(\sigma^*\Omega)_{jk} \equiv (\sigma^*\Omega)(\partial_j, \partial_k)$. Equation (45) implies

$$(\sigma^*\Omega)_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k],$$

which is the usual expression for the field strength in the physics literature.[116]

The curvature Ω is defined everywhere on the total space E, but the field strength $\sigma^*\Omega$ is defined on the base space, and only on the chart $U \subset M$ where σ is defined. Chapter 32 will deduce how the field strength is affected when the section is changed. If the new section is written as σg for some function $g : U \to G$, then the result is

$$(\sigma g)^*\Omega = g^{-1}(\sigma^*\Omega)g. \tag{46}$$

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[113] Nakahara (1990), section 10.3.4

[114] Chapter 9

[115] Tu (2011), proposition 19.5

[116] The components of the field strength on the left-hand side are usually denoted F_{jk}, but this article uses the letter F for the fiber instead.

32 Section-dependence of the field strength

This chapter outlines a relatively intuitive way to deduce the result (46),[117] starting with equation (42).

Let X, Y be vector fields on the base space M, and let $(\sigma_* X)_H$ and $(\sigma_* Y)_H$ be the horizontal components of the pushforwards $\sigma_* X$ and $\sigma_* Y$ with respect to a local section σ. Use equation (44) and then (42) to get

$$(\sigma^*\Omega)(X,Y) \equiv \Omega(\sigma_* X, \sigma_* Y) = -\omega\Big(\big[(\sigma_* X)_H,\, (\sigma_* Y)_H\big]\Big). \tag{47}$$

At each point m in the chart U where σ is defined, the vector in X at m has a unique horizontal lift to the point $\sigma(m)$ in the total space. At this point, the horizontal lift is the same as the vector $(\sigma_* X)_H$, because the difference between $\sigma_* X$ and the horizontal lift of X is a vertical vector.[118]

Now consider the effect of replacing the section σ with σg for some $g : U \to G$, and use the abbreviation

$$[X_H,\, Y_H]_\sigma \equiv \big[(\sigma_* X)_H,\, (\sigma_* Y)_H\big].$$

As emphasized above, the vector in X at m has a unique horizontal lift to the point $\sigma(m)$ in the total space, so if $g(m) = 1$ for some $m \in U$, then $[X_H, Y_H]_{\sigma g} = [X_H, Y_H]_\sigma$ at $\sigma(m)$. In words: the vector $[X_H, Y_H]_\sigma$ at $\sigma(m)$ doesn't depend on the slope of the section σ through the point $\sigma(m)$. It only depends on the point $\sigma(m)$ itself.[119] For this reason, the effect of the change $\sigma \to \sigma g$ is just like equation (6) even though the g here may vary over the chart U. This gives

$$\omega\big([X_H, Y_H]_{\sigma g}\big) = g^{-1}\omega\big([X_H, Y_H]_\sigma\big)g,$$

and using this in equation (47) gives the result (46).

[117]The same result can be derived in various ways. Other approaches include page 273 in section 30.2 of Tu (2017), and the text above equation (83) in Collinucci and Wijns (2006).

[118]To see this, use the fact that the bundle projection pushes both $\sigma_* X$ and the horizontal lift of X back to the same vector X in the base manifold.

[119]Contrast this to equations (15) and (25), where a vector's pushforward is not projected onto its horizontal component, so those results do depend on the slope of the function (h or σ) by which the vector is pushed forward. That's why those equations have an extra term that is absent in the result (46).

33 The abelian case

Typically, when G is a nonabelian group, the field strength $\sigma^*\Omega$ is defined only in the chart where the local section σ is defined. Where two charts overlap, with local sections σ and σ', respectively, the corresponding field strengths are related by equation (46) with g defined by $\sigma' = \sigma g$. If the bundle is nontrivial, then at least one of these transformations must involve $g \neq 1$, so a single field-strength two-form cannot be defined everywhere on the base space.

When G is an abelian group, equation (46) says that the field strength $\sigma^*\Omega$ is independent of the local section σ. It still differs from Ω, because Ω is a two-form on the total space E whereas $\sigma^*\Omega$ is a two-form on the base space M, but $\sigma^*\Omega$ is independent of σ (when G is abelian).[120] As a result, the field strength is defined everywhere on the base space (when G is abelian), just like Ω is defined everywhere on the total space, even though the local potential is still defined only in charts.[121]

When G is abelian, the second term on the right-hand side of (45) is zero because all elements of Lie(G) commute with each other. Then the field strength is just dA, the exterior differential of the local potential A. In this case, we can use *Stokes's theorem*[122] to write the **flux** $\int_\Sigma dA$ through a given surface Σ as an integral $\oint_{\partial\Sigma} A$ around the boundary $\partial\Sigma$ of that surface. This assumes that the surface Σ is contained within a part of the base space that is small enough to admit a local section, so that the local potential A is defined everywhere on Σ. If G is abelian, then we can define the integral of the field strength over *any* surface Σ in the base space, because the field strength is defined everywhere even though A is not. In particular, if the principal bundle is nontrivial, then the integral of the field strength over a closed surface Σ may be nonzero, which would not be possible if the field strength were equal to dA with A defined everywhere on that surface. Article 03838 describes a classic example of this phenomenon in detail.

[120] Bertlmann (1996), equation (2.509)

[121] The local potential still depends on the local section, because the second term on the right-hand side of equation (25) is nonzero even when G is abelian.

[122] Article 09894

34 References

Abbassi and Lakrini, 2021. "A note on induced connections" *Hacettepe Journal of Mathematics & Statistics* **50**: 1738 - 1744, https://dergipark.org.tr/en/pub/hujms/issue/66204/829790

Ambrose and Singer, 1953. "A theorem on holonomy" *Trans. Amer. Math. Soc.* **75**: 428-443, https://www.ams.org/journals/tran/1953-075-03/S0002-9947-195

Bertlmann, 1996. *Anomalies in Quantum Field Theory.* Oxford University Press

Clarke and Santoro, 2012. "Holonomy groups in riemannian geometry" https://impa.br/wp-content/uploads/2017/04/PM_39.pdf

Cohen, 2023. "Bundles, Homotopy, and Manifolds" http://virtualmath1.stanford.edu/~ralph/book.pdf

Collier *et al*, 2015. "Parallel transport on bundles over stacks" https://arxiv.org/abs/1509.05000v1

Collinucci and Wijns, 2006. "Topology of Fibre bundles and Global Aspects of Gauge Theories" https://arxiv.org/abs/hep-th/0611201

del Hoyo, 2016. "Complete connections on fiber bundles" *Indagationes Mathematicae* **27**: 985-990, https://www.sciencedirect.com/science/article/pii/S0019357716300283

de los Ríos, 2020. "Connections on principal bundles" https://www.homotopico.com/assets/docs/notes/principal-connections.pdf

Figueroa-O'Farrill, 2006a. "Lecture 1: Connections on principal fibre bundles" https://empg.maths.ed.ac.uk/Activities/GT/Lect1.pdf

Figueroa-O'Farrill, 2006b. "Lecture 2: Curvature" https://empg.maths.ed.ac.uk/Activities/GT/Lect2.pdf

Figueroa-O'Farrill, 2010a. "Lecture 5: Connections on principal and vector bundles" https://empg.maths.ed.ac.uk/Activities/Spin/Lecture5.pdf

Figueroa-O'Farrill, 2010b. "Lecture 7: Holonomy groups" https://empg.maths.ed.ac.uk/Activities/Spin/Lecture7.pdf

Figueroa-O'Farrill, 2017. "Spin Geometry" https://empg.maths.ed.ac.uk/Activities/Spin/SpinNotes.pdf

Freed *et al*, 2007. "Heisenberg Groups and Noncommutative Fluxes" *Annals of Physics* **322**: 236-285, https://arxiv.org/abs/hep-th/0605200v2

Fulton and Harris, 1991. *Representation Theory: A First Course.* Springer

Gualtieri, 2012. "Vector fields" http://www.math.toronto.edu/mgualt/courses/MAT1300F-2012/docs/1300-2012-notes-9.pdf

Haydys, 2019. "Introduction to Gauge Theory" http://haydys.net/misc/IntroGaugeTheory_LectNotes.pdf

Husemoller, 1966. *Fibre Bundles (Third Edition).* Springer-Verlag

Isham, 1999. *Modern Differential Geometry for Physicists (second edition).* World Scientific

Karshon, 2011. "Crash course on flows" http://www.math.toronto.edu/karshon/grad/2010-11/flows.pdf

Kobayashi and Nomizu, 1963. *Foundations of Differential Geometry, volume I.* Interscience

Kolář *et al*, 1993. *Natural Operations in Differential Geometry.* Springer-Verlag, https://www.emis.de/monographs/KSM/index.html

Koszul, 1960. *Lectures On Fibre Bundles and Differential Geometry.* Tata Institute of Fundamental Research

Lee, 2013. *Introduction to Smooth Manifolds (Second Edition).* Springer

Maxim, 2018. "Math 754, Chapter III: Fiber bundles. Classifying spaces. Applications" https://people.math.wisc.edu/~lmaxim/bundle.pdf

Mazzoni, 2018. "A fibre bundle approach to U(1) symmetries in physics" https://amslaurea.unibo.it/16776/1/tesi_mazzoni_finale_1.pdf

McCarthy, 2019. "Notes" https://www.ma.imperial.ac.uk/~jbm18/notes.pdf

Michor, 2008. "Topics in Differential Geometry" https://www.mat.univie.ac.at/~michor/dgbook.pdf

Morrison, 2018. "Connections on principal fibre bundles" https://tqft.net/papers/ConnectionsAndBundles.pdf

Nakahara, 1990. *Geometry, Topology, and Physics.* Adam Hilger

Neeb, 2010. *Differential Topology of Fiber Bundles.* FAU Erlangen-Nuernberg, https://faubox.rrze.uni-erlangen.de/dl/fiDyTGgNQ1aCQHpr9X1C3nCh

Ryan, 2014. "Geometry of horizontal bundles and connections" https://soar.wichita.edu/bitstream/handle/10057/10941/d14012_Ryan.pdf?sequence=1

Sengupta, 2007. "Gauge Theory in Two Dimensions: Topological, Geometric and Probabilistic Aspects" https://www.math.lsu.edu/~sengupta/papers/YMLisbon06Sengup.pdf

Trautman, 1979. "On Gauge Transformations and Symmetries" *Bull. Acad. Pol. Sci.* **XXVII**: 7-13, http://trautman.fuw.edu.pl/publications/Papers-in-pdf/42.pdf

Tu, 2011. *An Introduction to Manifolds (Second Edition).* Springer

Tu, 2017. *Differential Geometry: Connections, Curvature, and Characteristic Classes.* Springer

Zykoski, 2020. "Principal Bundles, Day 5: Vector Bundles" http://www-personal.umich.edu/~zykoskib/day5.pdf

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