Linear Operators on a Hilbert Space

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Abstract The general principles of quantum theory are expressed in terms of observables (measurable things), which are represented by linear operators on a Hilbert space. Hilbert spaces are introduced in article 90771. Each closed subspace of a Hilbert space has an associated projection operator, a special kind of linear operator. In quantum theory, projection operators are used to represent possible outcomes of a measurement. This article reviews some basic definitions and constructions associated with linear operators, with emphasis on projection operators.

Contents

1 Linear operators

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$ of complex numbers.[1] An **operator** A on $\mathcal{H}$ is something that replaces each vector $|a\rangle$ in $\mathcal{H}$ with another (possibly same) vector $A|a\rangle$ in $\mathcal{H}$. An operator A is called **linear** if

$$A\big(|a\rangle + |b\rangle\big) = A|a\rangle + A|b\rangle \qquad\qquad A\big(z\,|a\rangle\big) = z\big(A|a\rangle\big)$$

for all vectors $|a\rangle$ and $|b\rangle$ and all complex numbers z. The inner product of $|a\rangle$ and $A|b\rangle$ is a complex number denoted

$$\langle a|A|b\rangle.$$

Convention:

> For the rest of this article, "operator" means *linear* operator.

Two operators have special names:

- The **identity operator** 1 is the (unique) operator satisfying $1|a\rangle = |a\rangle$ for all vectors $|a\rangle$. This article uses same symbol 1 for two different things: the unit complex number, and the identity operator.

- The **zero operator** 0 is the (unique) operator satisfying $0|a\rangle = 0$ for all vectors $|a\rangle$. This article uses same symbol 0 for three different things: the zero complex number, the zero vector, and the zero operator.

[1] Hilbert space is introduced in article 90771. All Hilbert spaces in this article are over $\mathbb{C}$. Elements of a Hilbert space will be called **vectors**.

2 The domain of definition

Recall[2] that the **norm** of a vector $|a\rangle$ is defined by

$$\big\|\, |a\rangle \,\big\| \equiv \sqrt{\langle a|a\rangle}.$$

An operator A is called **bounded**[3] if a finite real number r exists such that

$$\big\|\, A|a\rangle \,\big\| \leq r \big\|\, |a\rangle \,\big\|$$

for all vectors $|a\rangle$.

This article considers only operators that are defined on the whole Hilbert space. That's an important thing to keep in mind when reading some of the definitions and assertions, so I'll highlight it:

> This article considers only operators that are defined on the whole Hilbert space.

Every such operator is bounded. Conversely, if an operator is defined on a subset of the Hilbert space and is bounded with respect to all vectors $|a\rangle$ in that subset, then the operator can be defined on the whole Hilbert space.[4]

Using unbounded operators is often convenient. An unbounded operator can only be defined part of the Hilbert space.[5] This article doesn't consider unbounded operators.[6]

[2] Article 90771

[3] Debnath and Mikusinski (2005), section 4.2, page 146

[4] Debnath and Mikusinski (2005), section 4.11, page 202. If the operator was originally defined on a dense subset of the Hilbert space, then its extension to the whole Hilbert space is unique (Debnath and Mikusinski (2005), section 4.11, page 202).

[5] Debnath and Mikusinski (2005) uses the sneaky terminology "defined *in* a Hilbert space" for an operator that is not necessarily defined *on* the whole Hilbert space (section 4.11, pages 202)

[6] The general principles of quantum theory can be expressed using only bounded operators, even though unbounded operators are often used as a tool for constructing specific models that satisfy those general principles.

3 The adjoint of an operator

Every complex number z has a complex conjugate z^*. Similarly, every operator A has an **adjoint** A^*, which is defined using the inner product. The adjoint of an operator A is the unique operator A^* that satisfies[7]

$$\boxed{\langle a|A^*|b\rangle = \langle b|A|a\rangle^*} \tag{1}$$

for all vectors $|a\rangle$ and $|b\rangle$. Notice that the order of the vectors is reversed on the right-hand side. The asterisk on the right-hand side is the ordinary complex conjugate. This definition implies that the adjoint reverses the order of multplication:

$$\boxed{(AB)^* = B^*A^*.} \tag{2}$$

To prove this, let $|A^*a\rangle$ and $|Bb\rangle$ denote the vectors defined by $A^*|a\rangle$ and $B|b\rangle$, respectively. Then

$$\begin{aligned}\langle b|(AB)^*|a\rangle &= \langle a|AB|b\rangle^* = \langle a|A|Bb\rangle^* = \langle Bb|A^*|a\rangle = \langle Bb|A^*a\rangle \\ &= \langle A^*a|Bb\rangle^* = \langle A^*a|B|b\rangle^* = \langle b|B^*|A^*a\rangle = \langle b|B^*A^*|a\rangle.\end{aligned}$$

This is true for all vectors, so it implies (2).

Using A^* to denote the adjoint of A is standard in the math literature,[8] because the adjoint is a natural generalization of complex conjugation. In the physics literature, the adjoint of A is commonly denoted $A^\dagger$ instead. Section 5 explains why.

[7] Debnath and Mikusinski (2005), definition 4.4.1. This definition assumes that the operator is defined on the whole Hilbert space. Debnath and Mikusinski (2005), definition 4.11.5 gives a more general definition that can be used for operators that are only defined on a dense subset of the Hilbert space.

[8] Debnath and Mikusinski (2005), definition 4.4.1

4 Example: finite-dimensional Hilbert space

Consider an N-dimensional Hilbert space $\mathcal{H}$, with finite N, and let $|a_k\rangle$ with $k \in \{1, 2, ..., N\}$ be a list of orthonormal vectors. Each (linear) operator A on $\mathcal{H}$ can be represented as an $N \times N$ matrix of complex numbers, with components A_{jk} defined by

$$A_{jk} \equiv \langle a_j|A|a_k\rangle.$$

Then, for any vector $|a\rangle = \sum_k z_k|a_k\rangle$,

$$A|a\rangle = \sum_j |a_j\rangle\,\langle a_j|A|a\rangle = \sum_{j,k} |a_j\rangle A_{jk} z_k = \sum_j w_j|a_j\rangle$$

with $w_j \equiv \sum_k A_{jk} z_k$. The vector $|a\rangle$ can be represented by its list of coefficients z_k, and if these are regarded as the components of an $N \times 1$ matrix z, then the effect of the operator A can be written as a matrix product: $w = Az$, where now A denotes the *matrix* representing the original operator A.

5 Two different notations for the adjoint

In the matrix representation described in the previous section, the adjoint of an operator is the hermitian conjugate of the matrix. If the adjoint of A is denoted A^*, which is the standard notation in the mathematical literature, then components of the matrix representing the adjoint of A are

$$(A^*)_{jk} = (A_{kj})^*. \tag{3}$$

Notice that the subscripts on the right-hand side are reversed. This conflicts with another standard notation: when A is a *matrix*, the notation A^* often refers instead to the matrix whose components are the complex conjugates of the original components, *without* transposing the matrix:

$$(A^*)_{jk} = (A_{jk})^*. \tag{4}$$

To avoid this conflict, the adjoint of an operator A is typically denoted by a different symbol in the physics literature, namely the symbol $A^\dagger$.

On the other hand, especially in the context of quantum field theory, we often want to uses matrices for a different purpose: instead of using a matrix to represent an individual operator (like in the previous section), we often consider a matrix M whose individual components are operators, and we often want to consider the matrix M^* whose components are the adjoints of those operators, without taking a transpose. In that case, using A^* to denote the adjoint of an operator A is more convenient than using $A^\dagger$.

No single notation is optimal in all contexts. Different notations have different advantages. In this article, the adjoint of an operator A is denoted A^*.

6 Operators with special names

Some kinds of operators have special names:

- An operator A is called **self-adjoint**[9] if $A^* = A$. If A is self-adjoint, then $\langle a|A|a\rangle$ is a real number for every vector $|a\rangle$.

- An operator P is called a **projection operator**[10] if it is self-adjoint and if applying it twice is the same as applying it once: $P^* = P = P^2$.

- An operator A is called **positive** if $\langle a|A|a\rangle \geq 0$ for every vector $|a\rangle$. Equivalently,[11] an operator A is called positive if $A = B^*B$ for some operator B.

- An operator A is called **strictly positive** or **positive definite**[12] if $\langle a|A|a\rangle > 0$ for every non-zero vector $|a\rangle$.

- An operator U is called **unitary** if $U^*U = UU^* = 1$. Unitary operators preserve inner products: If $|a'\rangle = U|a\rangle$ and $|b'\rangle = U|b\rangle$, then $\langle a'|b'\rangle = \langle a|b\rangle$. If U is a unitary operator on a Hilbert space $\mathcal{H}$, then $\mathcal{H} \to U\mathcal{H}$ is an automorphism (article 90771)

- An operator N is called **normal**[13] if it commutes with its adjoint: $N^*N = NN^*$. All of the operators named in this section are normal. Most operators are not.

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[9] Debnath and Mikusinski (2005), definition 4.4.3. In physics, a self-adjoint operator is often called **hermitian**, but sometimes that word is reserved for a self-adjoint matrix (Debnath and Mikusinski (2005), example 4.4.4).

[10] This language is standard in the physics literature and is also commonly used in the math literature (Murphy (1990), section 2.1, page 36). In the math literature, this is sometimes called an *orthogonal projection operator* (Debnath and Mikusinski (2005), definition 4.7.1) so that the name *projection operator* can be used for any **idempotent** operator – that is, for any operator that equals its own square, whether or not it is self-adjoint (Debnath and Mikusinski (2005), definition 4.7.5). Kadison and Ringrose (1997) says, "In the context of Hilbert space theory, *projection* refers to an orthogonal projection unless there is an explicit statement to the contrary" (from section 2.5, page 110).

[11] Debnath and Mikusinski (2005) definition 4.6.4 and theorems 4.6.6 and 4.6.14

[12] Debnath and Mikusinski (2005) definition 4.6.15

[13] Debnath and Mikusinski (2005), definition 4.5.7

7 The inverse of an operator

An operator A is called **invertible** if an operator B exists such that $AB = BA = 1$. If such an operator B exists, then it is unique.[14] It is called the **inverse** of A, denoted A^{-1}. Most operators are not invertible.

Here's an equivalent definition:[15] an operator A is invertible if $A\mathcal{H} = \mathcal{H}$ and if $A|a\rangle \neq 0$ for all $|a\rangle \neq 0$. Section 9 shows examples of operators that satisfy the second condition but not the first condition, so the second condition by itself is not sufficient.

Remember that this article only considers operators that are defined on the whole Hilbert space. Debnath and Mikusinski (2005) uses a more general definition of *invertible* to accommodate operators that are not necessarily defined on the whole Hilbert space.[16]

[14] To prove that A cannot have more than one inverse, suppose that B and C are two inverses of A, so that $AB = 1$ and $CA = 1$. Multiply the first equation on the left by C to get $CAB = C$, and multiply the second equation on the right by B to get $CAB = B$. This shows that B and C are both equal to CAB, so they are equal to each other.

[15] The equivalence is proven on page 287 below definition 10.18 in Axler (2021).

[16] Debnath and Mikusinski (2005), definition 4.5.1. That's why page 164 in that book says "the inverse of a bounded operator is not necessarily bounded." Even though the definition of an operator can be extended to the whole Hilbert space if the operator is bounded for all vectors in its original domain, the extension is not necessarily invertible even if the original (unextended) operator is invertible according to the more general definition in that book.

8 Spectrum and eigenvalues

The **spectrum** of A is the set of all complex numbers λ for which the operator $A - \lambda$ fails to be invertible.[17] Examples:

- If λ is in the spectrum of a projection operator, then λ is either 0 or 1.[18]
- If λ is in the spectrum of a self-adjoint operator, then λ is a real number.[19]
- If λ is in the spectrum of a unitary operator, then $|\lambda| = 1$.[20]

If $|a\rangle$ is a nonzero vector satisfying $A|a\rangle \propto |a\rangle$, then $|a\rangle$ is called an **eigenvector** of A, and the proportionality factor is called the **eigenvalue**. In a finite-dimensional Hilbert space, the spectrum of A is the set of eigenvalues of A. In an infinite-dimensional Hilbert space, the specturm of A may include some numbers that are not eigenvalues of A. Section 9 shows an example.

Beware that much of the physics literature uses the word *eigenvalue* for any element of the *spectrum*, even though only a true eigenvalue has a corresponding eigenvector within the Hilbert space. This dialect is a common source of easily-preventable confusion in quantum physics.

[17] Axler (2021), definition 10.32. The expression $A - \lambda$ is understood to mean $A - \lambda 1$, where 1 is the identity operator.

[18] Proof: if $\lambda^2 \neq \lambda$, then $P - \lambda$ has an inverse, namely $\big((1-\lambda)^{-1}P - 1\big)/\lambda$.

[19] Axler (2021), theorem 10.49

[20] Axler (2021), theorem 10.62

9 Example: a spectrum with no eigenvalues

Consider the Hilbert space $\mathcal{H}$ of square-integrable complex-valued functions of a single real variable x. Write $f(x)$ for such a function, and write $|f\rangle \in \mathcal{H}$ for the vector that it represents.[21] Define an operator A by

$$A|f\rangle = |g\rangle \qquad \text{with} \qquad g(x) = \frac{x}{\sqrt{1+x^2}}\, f(x).$$

This example highlights the distinction between the *eigenvalues* of an operator and the *spectrum* of an operator:

- **Fact 1:** A does not have any eigenvalues. In other words, $(A-\lambda)|f\rangle \neq 0$ whenever $|f\rangle \neq 0$, for any complex number λ.

- **Fact 2:** The spectrum of A includes all real numbers λ with $|\lambda| < 1$.

To prove Fact 1, suppose $(x/\sqrt{1+x^2})f(x) = \lambda f(x)$ for all x. This implies that $f(x) = 0$ everywhere except at $x = x_0 \equiv \lambda/\sqrt{1-\lambda^2}$. Even if $|\lambda| < 1$ so that such a value of x exists, the function still has zero norm: the integral of $|f(x)|^2$ is zero.[22] Such a function represents the zero vector, but eigenvectors must be nonzero, so A does not have any eigenvectors (or eigenvalues).

To prove Fact 2, define x_0 as before. Then $(x/\sqrt{1+x^2}) - \lambda$ goes to zero as $x \to x_0$. Functions representing elements of $\mathcal{H}$ must be square-integrable, so if $|g\rangle = A|f\rangle$, then $g(x)$ must go to zero fast enough (modulo a function with zero norm, as explained in article 90771) to be consistent with the square-integrability of $f(x)$. Many functions that represent vectors in $\mathcal{H}$ don't go to zero at all as $x \to x_0$, much less fast enough, so $(A-\lambda)\mathcal{H}$ does not include all vectors in $\mathcal{H}$. The definition of invertibility requires $(A-\lambda)\mathcal{H} = \mathcal{H}$, so $A - \lambda$ cannot be invertible.

[21] Article 90771 describes this construction in more detail.

[22] The physics literature often considers "functions" like the **Dirac delta "function"** $\delta(x-x_0)$, which is zero everywhere except at $x = x_0$ but still satisfies $\int dx\; \delta(x-x_0) = 1$. Despite its popular name, this is *not a function*. (Mathematicians call it a **distribution**.) It has a legitimate definition (inside integrals) and useful applications, but it's *not a function*, and it does not represent any vector in the Hilbert space.

10 Operator algebra

If A and B are two operators, then their sum $A+B$ is another operator, and their product AB is another operator, defined like this:[23]

$$(A+B)|\psi\rangle = A|\psi\rangle + B|\psi\rangle \qquad\qquad (AB)|\psi\rangle = A\big(B|\psi\rangle\big).$$

If z is a complex number and A is an operator, then their product zA is defined by

$$(zA)|\psi\rangle = z\big(A|\psi\rangle\big).$$

Equation (1) implies $(zA)^* = z^*A^*$. The product of the identity operator with a complex number z is usually written as just z, because the meaning is usually clear from the context. The product $(-1)A$ is abbreviated $-A$, and the sum $A+(-B)$ is abbreviated $A-B$.

Any collection of operators that is self-contained with respect to linear combinations and products is called an **algebra**. If it is also self-contained with respect to adjoints (so that A^* belongs to the algebra whenever A does), then it is called **$*$-algebra** (pronounced *star algebra*). One example of a $*$-algebra is the one consisting of all (linear) operators on a given Hilbert space.

This section introduced the concepts using operators that act on a Hilbert space, but the general definitions of *algebra* and *$*$-algebra* are expressed using abstract operators that don't necessarily "operate on" anything. Even if we are ultimately interested in a **representation** of the algebra in terms of operators on a Hilbert space, sometimes the abstract approach is a better place to start. Murphy (1990) gives a good introduction to the abstract approach.[24]

[23] The expression $A\big(B|\psi\rangle\big)$ means first apply B to $|\psi\rangle$, and then apply A to the result.

[24] Here, I'm using the word *abstract* to mean that the algebra is defined and studied using only its intrinsic properties, without representing its elements as operators on a Hilbert space. This can also be called the **algebra-first approach**.

11 Isomorphism of operator algebras

Two algebras are called *isomorphic* to each other if they are the same as far as their algebraic structure is concerned, even if they are represented differently. More precisely:

- A **homomorphism**[25] σ is a map from one algebra $\mathcal{A}_1$ to another algebra $\mathcal{A}_2$ such that

$$\sigma(A+B) = (\sigma A) + (\sigma B) \qquad \sigma(zA) = z(\sigma A)$$
$$\sigma(AB) = (\sigma A)(\sigma B)$$

  for all operators $A, B \in \mathcal{A}_1$.

- A bijective[26] homomorphism is called an **isomorphism**.

- A homomorphism from one $*$-algebra $\mathcal{A}_1$ to another $*$-algebra $\mathcal{A}_2$ is called a **$*$-homomorphism** σ if it also satisfies[27]

$$\sigma(A^*) = (\sigma A)^*$$

  for all operators $A \in \mathcal{A}_1$.

- A bijective $*$-homomorphism is called a **$*$-isomorphism**. In a context where the adjoint is clearly one of the structures that we intend to preserve, the word **isomorphism** may be used as an abbreviation for $*$-isomorphism.[28]

- An ($*$-)isomorphism from a $*$-algebra to itself is called an **($*$-)automorphism**.

The homomorphisms defined above (and their iso- and auto- specializations) are specifically **linear** homomorphisms. *Linear* is implied unless specified otherwise.

[25] Murphy (1990), section 1.1, page 5

[26] A homomorphism σ is called **bijective** if another homomorphism σ^{-1} exists for which the compositions $\sigma\sigma^{-1}$ and $\sigma^{-1}\sigma$ are both the identity morphism.

[27] Murphy (1990), section 2.1, page 36

[28] This usage of the word *isomorphism* is consistent with its usage in category theory.

12 Antilinear and anti-isomorphism

These definitions are also useful:

- An **antilinear** homomorphism is like a homomorphism except that the requirement $\sigma(zA) = z(\sigma A)$ is replaced by

$$\sigma(zA) = z^*(\sigma A). \tag{5}$$

- An **antihomomorphism** is like a homomorphism except that it reverses the order of multiplication:

$$\sigma(AB) = (\sigma B)(\sigma A).$$

  A bijective antihomomorphism is called an **anti-automorphism**.

Examples:

- If U is a unitary operator, then the map that replaces every operator A with U^*AU is a linear automorphism.

- The map that replaces every operator A with its adjoint A^* is an antilinear anti-automorphism.

- To construct an example of an antilinear automorphism (one that doesn't reverse the order of multiplication), let Ω be a set of linearly independent operators in terms of which all other operators can be expressed using linear combinations and products. Define $\sigma(A) = A$ whenever $A \in \Omega$, and define σ on other operators using $\sigma(A + B) = (\sigma A) + (\sigma B)$ and $\sigma(AB) = \sigma(A)\sigma(B)$ and (5).

In quantum field theory, the **CPT theorem** says that a relativistic model in flat spacetime always has a special type of symmetry called **CPT symmetry**, which is an antilinear automorphism.[29]

[29] The CPT theorem for four-dimensional spacetime is studied in Streater and Wightman (1980) and Greenberg (2003). Witten (2015) and appendix A in Freed and Hopkins (2021) give a more general perspective.

13 von Neumann algebras

Two operators A, B are said to **commute** with each other if $AB = BA$. The quantity

$$[A, B] \equiv AB - BA$$

is called the **commutator** of A and B. It is zero if and only if A and B commute with each other. Most operators do not commute with each other.

Let $\mathcal{H}$ be a Hilbert space, and let Ω be any set of operators[30] on $\mathcal{H}$. Notation:

- Let Ω' denote the **commutant** of Ω. This is set of all operators that commute with everything in Ω.

- Let Ω'' denote the **double commutant** of Ω. This is the set of all operators that commute with everything in Ω'. Clearly $\Omega \subset \Omega''$.

A **von Neumann algebra** $\mathcal{M}$ is[31] a set of operators that is self-contained with respect to adjoints (so that $A^* \in \mathcal{M}$ whenever $A \in \mathcal{M}$) and that is equal to its own double commutant: $\mathcal{M} = \mathcal{M}''$. This implies that a von Neumann algebra really is an *algebra* (section 10), as its name indicates.

If a set Ω of operators is self-contained with respect to adjoints, then $\mathcal{M} \equiv \Omega'$ is a von Neumann algebra.[32] To prove this, we need to show that $\mathcal{M} = \mathcal{M}''$. We already recognized above that $\mathcal{M} \subset \mathcal{M}''$, so we just need to show that $\mathcal{M}$ includes all of $\mathcal{M}''$. Suppose $A \in \mathcal{M}''$, so that A commutes with everything in $\mathcal{M}'$. This implies that A commutes with everything in Ω (because $\Omega \subset \Omega''$), so $A \in \Omega'$. This can also be written $A \in \mathcal{M}$, which is what we wanted to show.[33]

If a set Ω of operators is self-contained with respect to adjoints, then Ω'' is called the von Neumann algebra **generated** by Ω. The fact that Ω'' really is a von Neumann algebra follows from $\Omega'' = (\Omega'')''$, which can be proved just like before.

[30] Recall that in this article, *operator* means *linear operator*.

[31] The definition is often expressed differently (referring to a topology), but the definition shown here is equivalent, thanks to the famous **double commutant theorem** (Fillmore (1996), section 4.3, page 60).

[32] Example: if $\Omega = \{1\}$, then Ω' is the algebra of all operators on the Hilbert space. This is the simplest type of von Neumann algebra.

[33] Fewster and Rejzner (2019), section 6, exercise 29 on page 27

14 Projection operators and subspaces

Recall (section 6) that a projection operator satisfies $P^* = P = P^2$. Projection operators play a special role in quantum theory: each of the possible outcomes of a measurement is represented by a projection operator.[34] The rest of this article is focused on projection operators.

Recall[35] that a **closed subspace** of $\mathcal{H}$ is a subset of $\mathcal{H}$ that qualifies as a Hilbert space all by itself. For any projection operator P on $\mathcal{H}$, let $P\mathcal{H}$ denote the set of all vectors of the form $P|a\rangle$ with $|a\rangle \in \mathcal{H}$. Closed subspaces are closely related to projection operators:

- $P\mathcal{H}$ is a closed subspace of $\mathcal{H}$. To prove this, use the fact that $P\mathcal{H}$ is orthogonal to $(1-P)\mathcal{H}$, combined with the fact that the orthogonal complement of any subset of $\mathcal{H}$ is a closed subspace.[35]

- Every closed subspace of $\mathcal{H}$ is equal to $P\mathcal{H}$ for some projection operator P. To prove this, recall[35] that if $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, then every vector in $\mathcal{H}$ can be uniquely written as the sum of a vector in $\mathcal{S}$ and a vector in $\mathcal{S}^\perp$. For any vector $|a\rangle$, define $P|a\rangle$ to be the component of $|a\rangle$ in $\mathcal{S}$. Then P is clearly a projection operator, and $\mathcal{S} = P\mathcal{H}$.

[34] This is an idealization. Real measurement have limited resolution, and their possible outcomes can't always be described by a finite list of crisply-defined possibilities. In principle, quantum theory can describe such realistic measurements by regarding the measurement of interest as a physical process that can be probed later using other (idealized) measurements. This doesn't require using **positive operator valued measures (POVMs)**, even though using POVMs can be convenient in practice.

[35] Article 90771

15 Relationships between projection operators

Projection operators P and Q are called **orthogonal** to each other if $PQ = 0$. In this case,

- P and Q commute with each other.[36]
- Every vector in $P\mathcal{H}$ is orthogonal to every vector in $Q\mathcal{H}$.
- $P + Q$ is a projection operator.

The projection operator $1 - P$ is called the **(orthogonal) complement** of P. Clearly, P and $1 - P$ are orthogonal to each other.

A projection operator P is called a **subprojection** of a projection operator Q if $PQ = P$. If P is a subprojection of Q, then:

- P and Q commute with each other.[36]
- $P\mathcal{H} \subset Q\mathcal{H}$.
- $Q - P$ is a projection operator orthogonal to P.

If $P_1, P_2, ...$ is any list of mutually orthogonal projection operators on a Hilbert space $\mathcal{H}$, then the partial sums $Q_n \equiv P_1 + P_2 + \cdots + P_n$ are projection operators satisfying $Q_j\mathcal{H} < Q_k\mathcal{H}$ whenever $j < k$. The next section refers to $Q_1, Q_2, ...$ as a **nested** sequence of projection operators. The original projection operators can be recovered from the Q_ns using $P_n = Q_n - Q_{n-1}$ with $Q_0 \equiv 0$.

A set of mutually orthogonal subspaces of a separable Hilbert space must be countable. This follows from the definition of *separable* (article 90771). In contrast, a sequence of nested subspaces[37] can be uncountable,[38] even if the Hilbert space is separable. The next section expresses this in terms of projection operators.

[36] More generally, if PQ is self-adjoint, then $[P, Q] = 0$. Proof: $QP = Q^*P^* = (PQ)^* = PQ$.

[37] This is called a **nest** of subspaces (Arias and Farmer (1992)).

[38] Regarding the application of the word *sequence* to an uncountable set, see https://math.stackexchange.com/questions/724486/

16 Spectral measure

A nested sequence of projection operators (countable or not) is sometimes called a **resolution of the identity**.[39] This is closely related to[40] the concept of a spectral measure. To define this, let X be any set, such as the set $\mathbb{R}$ of real numbers. A **σ-algebra** (pronounced *sigma* algebra) is a collection S of subsets $s \subset X$ with these properties:[41]

- S includes the empty set $\varnothing$.
- If $s \in S$, then the complement of s (denoted $X \setminus s$) is also in S. In particular, X itself is an element of S.
- If $s_1, s_2, ...$ are all in S, then their union is also in S.

A **spectral measure** is a collection of projection operators $P(s)$, one for each $s \in S$, with these properties:[42]

- $P(\varnothing) = 0$ and $P(X) = 1$.
- If $s_1, s_2 \in S$, then $P(s_1 \cap s_2) = P(s_1)P(s_2)$.
- If $s_1, s_2, ...$ is a sequence of pairwise disjoint sets from S, then

$$P\left(s_1 \cup s_2 \cup \cdots\right) = P(s_1) + P(s_2) + \cdots .$$

The **spectral theorem** associates a spectral measure with any normal operator.[43] The adjective *spectral* in these names alludes to the operator's spectrum (section 8).

[39] https://encyclopediaofmath.org/wiki/Resolution_of_the_identity

[40] von Neumann (1955), section II-7, page 119. Murphy (1990), in the text below theorem 2.5.6, uses the term *resolution of the identity* for the spectral measure associated with a specified normal operator via the spectral theorem. If the normal operator is self-adjoint, so that its spectrum is real, then this coincides with the concept of a nested sequence (possibly continuous) of projection operators.

[41] Axler (2021), definition 2.23

[42] Conway (2000), definition 9.1

[43] Murphy (1990), theorem 2.5.6

17 Inferring orthogonality from a sum

If $P_1, P_2, P_3, \ldots$ are projection operators whose sum

$$Q \equiv P_1 + P_2 + P_3 + \cdots \tag{6}$$

is also a projection operator, then the projection operators P_n must be orthogonal to each other.

To prove this,[44] start with the fact that if P is any projection operator and $|a\rangle$ is any vector, then[45]

$$\langle a|a\rangle \geq \langle a|P|a\rangle \geq 0. \tag{7}$$

Use this to get[46]

$$\langle a|a\rangle \geq \langle a|Q|a\rangle = \sum_n \langle a|P_n|a\rangle \geq \langle a|P_1|a\rangle + \langle a|P_2|a\rangle,$$

which holds for any vector $|a\rangle$. Altogether,

$$\langle a|a\rangle \geq \langle a|P_1|a\rangle + \langle a|P_2|a\rangle.$$

Set $|a\rangle = P_1|b\rangle$ to see that the preceding inequality implies

$$\langle a|a\rangle \geq \langle a|a\rangle + \langle c|c\rangle$$

with $|c\rangle = P_2P_1|b\rangle$. This inequality implies $|c\rangle = 0$, and since this is true for all $|b\rangle$, this implies $P_2P_1 = 0$, as claimed.

[44] Akhiezer and Glazman (1993), section 32, theorem 2

[45] Proof: $\langle a|a\rangle = \langle a|P|a\rangle + \langle a|(1-P)|a\rangle = \langle b|b\rangle + \langle c|c\rangle$ with $|b\rangle = P|a\rangle$ and $|c\rangle = (1-P)|a\rangle$. Use the inequalities $\langle b|b\rangle \geq 0$ and $\langle c|c\rangle \geq 0$ to get (7).

[46] This assumes that the sum has more than one term. If it doesn't, then the theorem is trivial.

18 Packaging projection operators: finite case

Let A be an operator of the form

$$A = z_1 P_1 + z_2 P_2 + z_3 P_3 + \cdots + z_N P_N \tag{8}$$

where the z_n are distinct complex numbers and the P_n are projection operators satisfying

$$\sum_n P_n = 1. \tag{9}$$

Section 17 showed that the condition (9) implies

$$P_j P_k = 0 \text{ if } j \neq k. \tag{10}$$

This section proves that the double commutant $A'' \equiv \{A\}''$ consists of all linear combinations of the projection operators P_n. More elegantly:

$$\boxed{A'' = \{P_1, ..., P_N\}''.} \tag{11}$$

In particular, this shows that the set of projection operators P_n is uniquely determined by the single normal[47] operator (8). This is a convenient way of packaging a set of projection operators that sum to 1.[48,49]

To prove (11), let Ω denote the set of all operators that can be expressed as polynomials in A. Any operator that commutes with A clearly also commutes with any power of A, so $A' = \Omega'$, which in turn implies $A'' = \Omega''$. Since $\Omega \subset \Omega''$, this implies that A'' includes all polynomials in A. The remaining tasks are to prove that each of the projection operators P_n can be expressed as a polynomial in A, and that A'' doesn't include any operators that can't be written as a linear combination of the P_ns.

[47] Recall that an operator is called *normal* if it commutes with its adjoint. Equation (10) implies that (8) is normal.

[48] This is a special case of the spectral theorem (section 16).

[49] We can also recover the coefficients z_n in (8) as the proportionality factors in the identities $AP_n \propto P_n$.

To prove that each of the projection operators P_n can be expressed as a polynomial in A, consider the polynomial

$$A_n \equiv \prod_{k\neq n}(A - z_k).$$

The orthogonality condition (10) implies $AP_n = z_nP_n$, which can also be written $(A - z_n)P_n = 0$. This implies

$$A_nP_k = 0 \qquad \text{if } k \neq n.$$

Use this to get

$$A_n = A_n\sum_k P_k = A_nP_n = \left(\prod_{k\neq n}(z_n - z_k)\right)P_n.$$

The condition that the coefficients z_n all be distinct implies that the quantity in large parentheses is non-zero, so this shows that P_n is proportional to A_n, which is a polynomial in A. This implies $P_n \in A''$.[50]

To prove that A'' contains *only* linear combinations of the P_ns, let $\mathcal{H}$ denote the Hilbert space, and recall that the subspace $P_n\mathcal{H}$ is a Hilbert space all by itself. Let B be any operator that maps each of these subspaces to itself (does not mix them with each other). This implies that B commutes with all of the P_ns and therefore with A, so $B \in A'$. Now suppose $C \in A''$, which means that C must commute with all $B \in A'$. In other words, C must commute with every operator that does not mix the subspaces $P_n\mathcal{H}$ with each other. This includes *every* operator on any one of the Hilbert spaces $P_n\mathcal{H}$ that acts as the identity operator on the other Hilbert spaces $P_k\mathcal{H}$. The operator C must commute with *every* operator on a Hilbert space $P_n\mathcal{H}$, so C must be proportional to the identity operator on each of the subspaces $P_n\mathcal{H}$. This implies that C is a linear combination of the P_ns.

[50]Corollary: if A and B are both linear combinations of (possibly different) projection operators that sum to 1, and if A and B commute with each other, then their associated projection operators also commute with each other.

19 Packaging projection operators: general case

The previous section showed how a single normal operator can be used to package a *finite* list of mutually orthogonal projection operators. If the Hilbert space is finite-dimensional, then every normal operator has this form.[51] If the Hilbert space is not finite-dimensional, then A'' might not be generated by any countable (much less finite) list of projection operators, but the set of projection operators in A'' still has nice properties.

Let $\pi(A)$ denote the set of all projection operators in A'', also called the **spectral projections**[52] for A. Nice properties of $\pi(A)$ include:

- $\pi(A)$ always includes the trivial projection operators 0 and 1.
- A'' is an algebra, so if P and Q are both in $\pi(A)$, then so is their product and their complements, and if $PQ = 0$ (so that $P + Q$ is a projection operator), then so is their sum $P + Q$.
- All operators in A'' commute with A and with each other.[53] In particular, all of the projection operators in A'' commute with each other.
- If A and B commute with each other, then everything in A'' commutes with everything in B''.[54] In particular, the spectral projection operators of A and B commute with each other.
- Any von Neumann algebra $\mathcal{M}$ is generated by its projection operators. More explicitly,[52] $\mathcal{M} = \pi(\mathcal{M})''$. In particular, $A'' = \pi(A)''$.

[51] Equivalently: every matrix that commutes with its adjoint is diagonalizable (Murphy (1990), theorem 2.4.4)

[52] Törnquist and Lupino (2012), exercise 2.12

[53] Proof: if $B \in A''$, then B commutes with everything in A' (by definition of A''). But $A \in A'$, so this implies $[B, A] = 0$. That implies $B \in A'$, which implies $A'' \subset B'$, so everything in A'' commutes with B.

[54] Proof: suppose $C \in A''$. Then C commutes with everything in A'. We assumed $B \in A'$, so $[C, B] = 0$, so $C \in B'$, so $B'' \subset C'$, which says that everything in B'' commutes with C.

20 References

(Open-access items include links.)

Akhiezer and Glazman, 1993. *Theory of Linear Operators on Hilbert Space.* Dover

Arias and Farmer, 1992. "Nests of subspaces in Banach space and their order types" *Transactions of the American Mathematical Society* **331**: 113-130, https://www.ams.org/journals/tran/1992-331-01/S0002-9947-1992-1050084-5

Axler, 2021. "Measure, Integration & Real Analysis" https://measure.axler.net/

Conway, 2000. *A Course in Operator Theory.* American Mathematical Society

Debnath and Mikusinski, 2005. *Introduction to Hilbert Spaces with Applications (Third Edition).* Academic Press

Fewster and Rejzner, 2019. "Algebraic Quantum Field Theory – an introduction" https://arxiv.org/abs/1904.04051

Fillmore, 1996. *A User's Guide to Operator Algebras.* John Wiley and Sons

Freed and Hopkins, 2021. "Reflection positivity and invertible topological phases" *Geom. Topol.* **25**: 1165-1330, https://arxiv.org/abs/1604.06527

Greenberg, 2003. "Why is CPT fundamental?" *Found. Phys.* **36**: 1535-1553, https://arxiv.org/abs/hep-ph/0309309

Kadison and Ringrose, 1997. *Fundamentals of the Theory of Operator Algebras, Volume I: Elementary Theory.* American Mathematical Society

Murphy, 1990. *C*-Algebras and Operator Theory.* Academic Press

Streater and Wightman, 1980. *PCT, Spin and Statistics, and All That.* Princeton University Press

Törnquist and Lupino, 2012. "Set theory and von Neumann algebras" https://www.math.cmu.edu/~eschimme/Appalachian/AsgerMartino.pdf

von Neumann, 1955. *Mathematical Foundations of Quantum Mechanics.* Princeton University Press

Witten, 2015. "Fermion Path Integrals And Topological Phases" *Rev. Mod. Phys.* **88**: 35001, https://arxiv.org/abs/1508.04715

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