Infinitesimal Diffeomorphisms and the Metric Tensor

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Abstract This short article derives how an infinitesimal diffeomorphism affects the metric tensor (via pullback) and how the result may be expressed in terms of covariant derivatives.

Contents

1 Infinitesimal diffeomorphism

A diffeomorphism of the spacetime manifold corresponds in a natural way (through pullbacks and pushforwards) to a transformation of the tensor fields that live on that manifold.[1] For an infinitesimal diffeomorphism

$$x^a \to x^a + \theta^a(x),$$

the next section shows that the corresponding transformation[2] of the metric field is

$$\boxed{\delta g_{ab} = \theta^c \partial_c g_{ab} + g_{ca}\partial_b \theta^c + g_{cb}\partial_a \theta^c.} \tag{1}$$

The lagrangian of pure general relativity (with no fields other than the metric field) is invariant under this gauge transformation. Section 3 shows how the right-hand side of (1) can be expressed using only covariant derivatives (specifically the Levi-Civita connection).

[1] Article 93875 reviews the concepts of smooth manifolds and diffeomorphisms, article 09894 reviews the concept of a tensor field, and article 00418 defines the corresponding transformations of the fields. Pullbacks and pushforwards are defined in Lee (2013).

[2] By "corresponding transformation," I mean the pullback, as in Lee (2013), chapter 13.

2 Derivation of (1)

To derive (1), start by applying the infinitesimal diffeomorphism to the line element:

$$g_{ab}(x)\,dx^a\,dx^b \to g_{ab}(x+\theta)\Big(dx^a + d\theta^a\Big)\Big(dx^b + d\theta^b\Big) \equiv \bar{g}_{ab}(x)\,dx^a\,dx^b$$

where $\bar{g}_{ab}(x)$ is the transformed metric. To first order in θ, we have

$$g_{ab}(x+\theta) = g_{ab}(x) + \theta^c\partial_c g_{ab}(x)$$

and

$$d\theta^a = (\partial_c\theta^a)dx^c.$$

Use these in the preceding equation to get

$$\bar{g}_{ab} = g_{ab} + \theta^c\partial_c g_{ab} + g_{ac}\partial_b\theta^c + g_{cb}\partial_a\theta^c.$$

This gives (1).

3 Expression using covariant derivatives

General relativity uses a special covariant derivative called the Levi-Civita connection (article 03519). The result (1) may be written more concisely in terms of the Levi-Civita connection ∇ like this:

$$\boxed{\delta g_{ab} = \nabla_a \theta_b + \nabla_b \theta_a.} \tag{2}$$

To verify this, use

$$\theta_b = g_{bc}\theta^c \qquad\qquad \nabla_a g_{bc} \cdots = g_{bc}\nabla_a \cdots$$

in (2) to get

$$\delta g_{ab} = g_{cb}\nabla_a \theta^c + g_{ca}\nabla_b \theta^c. \tag{3}$$

Then use

$$\nabla_a \theta^c = \partial_a \theta^c + \Gamma^c_{ae}\theta^e$$
$$g_{cb}\Gamma^c_{ae} = \frac{1}{2}(\partial_a g_{eb} + \partial_e g_{ab} - \partial_b g_{ae}).$$

to get (1).

4 References

Lee, 2013. *Introduction to Smooth Manifolds (Second Edition)*. Springer

5 References in this series

Article **00418** (https://cphysics.org/article/00418):
"Diffeomorphisms, Tensor Fields, and General Covariance" (version 2022-02-20)

Article **03519** (https://cphysics.org/article/03519):
"Covariant Derivatives and Curvature" (version 2022-02-06)

Article **09894** (https://cphysics.org/article/09894):
"Tensor Fields on Smooth Manifolds" (version 2022-03-02)

Article **93875** (https://cphysics.org/article/93875):
"From Topological Spaces to Smooth Manifolds" (version 2022-02-05)