Principal Bundles and Associated Vector Bundles

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Abstract The concept of a **gauge field** in classical physics is based on the mathematical concept of a **principal bundle**. Some other classical fields that interact with gauge fields are based on the concept of **vector bundles** that are **associated** with the principal bundle. This article introduces principal bundles and vector bundles as specializations of the general concept of a **fiber bundle**. This is a prerequisite for article 76708, which explains how these things relate to the concept of a gauge field.

Contents

1 Motivation

Roughly, a *fiber bundle* consists of a manifold M called the *base space* with a copy of another manifold F called the *fiber* attached to each point of M, in a way that varies smoothly throughout M. In classical field theory, fields are often described as functions on a smooth manifold M representing space or spacetime. When the manifold M has nontrivial topology, a different kind of description is sometimes required. The concept of a fiber bundle provides the foundation for this different kind of description.

In a *vector bundle*, the fiber is a vector space V. For some types of classical fields, a configuration of the field may be described as a smooth *section* of a vector bundle,[1] with spacetime as the base space M and with V as the fiber.[2] If the fiber bundle is trivial (equivalent to a cartesian product $M \times V$), then a section is the same as a V-valued function of M. More generally, that equivalence holds only locally, not globally (not everywhere on M).

In a *principal bundle*, the fiber is almost[3] a Lie group G. This type of fiber bundle is the mathematical foundation for the concept of a classical *gauge field*, but a gange field is not a section.[4] A gauge field corresponds instead to a *connection* on a principal bundle. Interactions between gauge fields and other types of fields may be described using *associated vector bundles* – vector bundles that are associated to the principal bundle in a particular way.

This article introduces fiber bundles, vector bundles, principal bundles, sections, and associated vector bundles. A separate article (article 76708) will introduce the concept of a connection.

[1] Most articles in this series use the word *section* for a numbered part of the article. This article uses the word *chapter* instead, like Taubes (2011) does, reserving the word *section* for the mathematical concept previewed here.

[2] In this article, the name **vector field** is reserved for a section of a manifold's tangent bundle, as in Tu (2017) and in most of the physics literature. Tangent bundles are important examples vector bundles, but many vector bundles are not tangent bundles. Example: a classical spinor field is a section of a vector bundle that isn't a tangent bundle, so it won't be called a *vector field*. In a generic vector bundle, the number of dimensions of the fiber V may be unrelated to the number of dimensions of the base space M.

[3] Chapter 13 will explain what *almost* means here.

[4] A nontrivial principal bundle doesn't even have any (global) sections.

2 Some notation

$$
\begin{aligned}
M &= \text{the base space} \\
E &= \text{the total space} \\
p &= \text{the bundle projection} \\
U &= \text{a chart (neighborhood of a point in } M) \\
\tau &= \text{the inverse of a local trivialization} \\
\tau_k &= \text{the inverse of a local trivialization for a chart } U_k \\
\tau_{j\to k} &= \text{transition function from } U_j \text{ to } U_k \\
\sigma &= \text{a section} \\
m \text{ or } u &= \text{a point in } M \text{ or in a chart } U \subset M \\
x &= \text{a point in } E \\
X, Y &= \text{tangent vector fields on } M \text{ or } E \text{ (specified in the text)}
\end{aligned}
$$

Notation for a principal bundle and an associated vector bundle:

$$
\begin{aligned}
G &= \text{the structure group} \\
V &= \text{a vector space} \\
\rho &= \text{a linear representation of } G \text{ on } V \\
E &= \text{the total space of the principal bundle} \\
\hat{E} &= \text{the total space of the associated vector bundle} \\
p &= \text{the principal bundle projection} \\
\hat{p} &= \text{the associated vector bundle projection} \\
\tau \text{ or } \tau_k &= \text{the inverse of a local trivialization of the principal bundle} \\
\hat{\tau} \text{ or } \hat{\tau}_k &= \text{the inverse of a local trivialization of the associated vector bundle} \\
\tau_{j\to k} &= \text{transition function for the principal bundle} \\
\hat{\tau}_{j\to k} &= \text{transition function for the associated vector bundle}
\end{aligned}
$$

3 The concept of a fiber bundle

Intuitively, a (smooth) fiber bundle consists of two independent smooth manifolds, M and F, whose dimensions are not necessarily the same, and a third smooth manifold E made by attaching a copy of F to each point of M in a way that varies smoothly throughout M. The cartesian product $E = M \times F$ is a trivial example, but nontrivial examples also exist. This chapter reviews the full definition.

A **(smooth) fiber bundle** (E, M, p) consists of three pieces of data:[5]

- A smooth manifold E called the **total space**,
- A smooth manifold M called the **base space**,
- A smooth map $p : E \to M$ called the **(bundle) projection**.

Those data must satisfy these conditions:[6]

- The **fibers** $p^{-1}(m) \subset E$ are all diffeomorphic[7] to each other, for all $m \in M$.
- Each point $m \in M$ has a neighborhood U whose inverse image $p^{-1}(U) \subset E$ is diffeomorphic to $U \times F$, where the manifold F is diffeomorphic to each fiber. More specifically, a diffeomorphism τ from $U \times F$ to $p^{-1}(U)$ exists with this property: $p(\tau(u, f)) = u$ for all $(u, f) \in U \times F$.

The name **local trivialization** traditionally refers to the inverse τ^{-1} of such a map, but working with τ instead of τ^{-1} will be more convenient in this article. To be consistent with standard terminology, τ will be called an **inverse local trivialization**.

A fiber bundle may be called ***F*-bundle over *M*** to indicate that each fiber is diffeomorphic to F and that the base space is M.

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[5] Distinguishing between *data* and *conditions* can help clarify a definition. Freed and Hopkins (2016) use this device to help clarify a particular detail in quantum field theory (they call them *structure* and *conditions*).

[6] Kolář *et al* (1993), definition 9.1, and Cohen (2023), definition 2.1

[7] This means that they are equivalent to each other as smooth manifolds (article 93875).

4 Generalizations not considered in this article

The definition that was given in chapter 3 is general enough for many purposes, but sometimes it is generalized further:

- Sometimes smoothness is not required, and then the words *diffeomorphic* and *diffeomorphism* in chapter 3 would be replaced by *homeomorphic* and *homeomorphism*.[8] This article considers only smooth fiber bundles,[9] as in Trautman (1979). The word *smooth* will usually be omitted from now on, but all manifolds in this article are understood to be smooth (not just topological), and all maps are understood to be diffeomorphisms (not just homeomorphisms).

- Sometimes the definition is generalized so that the existence of local trivializations is not required,[10,11] and then a fiber bundle that does satisfy that extra condition is called a **locally trivial** fiber bundle.[12]

- The definition can be generalized to a more general class of topological spaces, not restricted to finite-dimensional manifolds.[13] This generalization is important in physics.[14]

[8]Article 93875 reviews the definitions of those words.

[9]The usual definition of *smooth* implies *paracompact* (`https://math.stackexchange.com/questions/98105`), so a smooth fiber bundle is automatically *numerable* in the sense defined in Mitchell (2011), section 12 (also `https://ncatlab.org/nlab/show/numerable+fiber+bundle`).

[10]One example of this generalization is described in Cohen (2023), chapter 2, pages 20-21. Another is described in `https://mathoverflow.net/questions/248116/`.

[11]`https://math.stackexchange.com/questions/4583674` helps explain why the existence of local trivializions is usually required.

[12]Husemoller (1966), chapter 4, definition 6.2

[13]I didn't carefully check which of the results reviewed in this article would still hold in such a generalization. Remark 3.7 in Wendl (2007b) and one of remarks 10.4 in Kolář *et al* (1993) issue warnings about this, one of which will be highlighted in footnote 60 in chapter 18. Michor (1991), Michor (2016), and Wendl (2007a) provide more background for infinite-dimensional cases.

[14]Example: it is important in the study of **'t Hooft anomalies**.

5 Equivalence of two fiber bundles

Consider two fiber bundles (E, M, p) and (E', M, p') with the same base space. They are called **equivalent** to each other if a diffeomorphism $\delta : E \to E'$ exists that satisfies these conditions:[15]

- δ maps each individual fiber of the first bundle to some individual fiber of the second bundle,

- $p'(\delta(x)) = p(x)$ for all $x \in E$. In words: δ doesn't do anything to the base space M.

The definition of *equivalence* does not refer to local trivializations. That's why the definition shown in chapter 3 treats the existence of local trivializations as a condition that the projection p must satisfy[16] instead of treating local trivializations as part of the data that must be specified.

This generalization of *equivalence* may be generalized to *isomorphism*, in which the base spaces are merely diffeomorphic to each other instead of equal to each other,[17] but that generalization won't be needed in this article.

[15]Nakahara (1990), sections 9.2.3-9.2.4

[16]Examples of other authors who use this philosophy include Lee (2013) (pages 249-250 and 268 in chapter 10) and Wendl (2007a) (definitions 2.8, 2.74, and 2.87).

[17]Cohen (2023), definition 2.2; Maxim (2018), definition 1.9; Neeb (2010), definition 1.3.1

6 Trivial bundles

Given any two manifolds M and F, we can make a fiber bundle by taking the total space to be $E = M \times F$ and taking the bundle projection p to be the usual projection onto the product's first factor, M. A fiber bundle is called **trivial** if it's equivalent to one like this. Here are a few examples of trivial fiber bundles:

- Take $E = M$, and take $p(x) = x$ for all points x. In this example, the fiber is a single point.

- Every bundle with a contractible base space is trivial.[18] This implies that every fiber bundle over $M = \mathbb{R}^n$ (which is the usual model of n-dimensional flat spacetime) is trivial.[19]

- Here's an example of a trivial fiber bundle with a non-contractible base space: Take E to be the torus $S^1 \times S^1$ (the cartesian product of two circles), and define $p(m, f) = m$. In this example, the fiber is a circle, and the base space is also a circle.

[18] Cohen (2023), corollary 4.6; Maxim (2018), corollary 1.18; Nakahara (1990), corollary 9.5

[19] Nontrivial fiber bundles are still important in physics, because physics often considers spaces or spacetimes that are not contractible.

7 Examples of nontrivial bundles

Here are two easy examples of nontrivial fiber bundles:

- Think of a circle as the set of complex numbers z with magnitude $|z| = 1$. Take the total space E to be a circle, take the base space M to be a circle, too, and define the bundle projection by $p(z) = z^n$ for some positive integer n. Then the fiber is a set of n points. When $n \geq 2$, this bundle is nontrivial.[20]

- For another example,[21] take the total space E to be the **Klein bottle**, which is the two-dimensional manifold consisting of points (m, f) with $0 \leq m \leq 2\pi$ and $0 \leq f \leq 2\pi$ and with its would-be boundary segments glued together by the equivalence relations $(m, 2\pi) \sim (m, 0)$ and $(2\pi, f) \sim (0, 2\pi - f)$. Define the bundle projection by $p(m, f) = (m, 0)$. Then the fiber is a circle, and so is the base space. Using the language introduced at the end of chapter 3, we can call this a circle bundle over a circle. Chapter 6 mentioned another example of a circle bundle over a circle, but that one was a trivial bundle and this one is not.

[20] Chapter 10 will study the case $n = 2$ in more detail.

[21] Figueroa-O'Farrill (2006), example 1.1

8 Constructing a fiber bundle from trivial patches

According to the definition in chapter 3, a fiber bundle must be locally trivial – it must look locally (but maybe not globally) like the cartesian product of M and F. We can turn this into a method for constructing fiber bundles from trivial patches.

Start with the two manifolds, a base space M and a fiber F, and let $U_1, U_2, U_3, ...$ be a covering of M by open sets, each called a **chart**. A total space E may be constructed from the trivial **patches** $E_k \equiv U_k \times F$, as explained below.

To motivate the idea, suppose that we already have a fiber bundle $p : E \to M$ as defined in chapter 3, and let τ_k be the inverse local trivialization that maps $U_k \times F$ to the corresponding part of E. If U_j and U_k overlap, then for each point $u \in U_j \cap U_k$, we can define a **transition function** $\tau_{j\to k}(u) : F \to F$ by the condition[22]

$$\tau_k^{-1}(\tau_j(u, f)) = \big(u,\, \tau_{j\to k}(u)f\big). \tag{1}$$

Equation (1) clearly implies

$$\begin{aligned} &\tau_{j\to j}(u) = 1 && \text{for all } u \in U_j, && (2)\\ &\tau_{k\to \ell}(u)\tau_{j\to k}(u)f = \tau_{j\to \ell}(u)f && \text{for all } u \in U_j \cap U_k \cap U_\ell. && (3) \end{aligned}$$

Conversely, suppose we start with the patches $E_k \equiv U_k \times F$ and a set of transition functions $\tau_{j\to k} : U_j \cap U_k \to G$ where G is a Lie group of diffeomorphisms of F, and suppose that these functions $\tau_{j\to k}$ satisfy the conditions (3). Then we can define a fiber bundle like this: convert the disjoint union of the patches E_k to a single manifold E by equating $(u, f) \in E_j$ with $\big(u, \tau_{j\to k}(u)f\big) \in E_k$ for $u \in U_j \cap U_k$, and define a projection $p : E \to M$ by taking $p(u, f) = u$ whenever $(u, f) \in E_k$. Intuitively: the transition functions $\tau_{j\to k}$ describe how the fibers over points of $U_j \times F$ should be matched with the fibers over points of $U_k \times F$ where U_j and U_k overlap. The result turns out to be a fiber bundle as defined in chapter 3.[23,24]

[22] Notation: the result of applying the diffeomorphism $\tau_{j\to k}(u)$ to $f \in F$ is written $\tau_{j\to k}(u)f \in F$.

[23] Nakahara (1990), section 9.2.2; Kolář *et al* (1993), above lemma 9.2; Sengupta (2007), text around equation 1

[24] This is sometimes called the **fiber bundle reconstruction theorem**. Some sources use this to define the concept of a fiber bundle.

9 Transition functions for a trivial bundle

A fiber bundle is trivial if and only if the transition functions $\tau_{j\to k}$ may all be written[25]

$$\tau_{j\to k}(u) = \phi_j^{-1}(u)\phi_k(u) \tag{4}$$

for smooth maps $\phi_j : U_j \to G$, where G is the same Lie group of diffeomorphisms of F to which the transition functions are restricted (section 8). To prove the *if* part, define $\tilde{\tau}_j$ by

$$\tilde{\tau}_j(u,f) \equiv \tau_j\big(u, \phi_j(u)f\big) \tag{5}$$

for all j, where τ_j is an inverse local trivialization. Then

$$\begin{aligned}
\tilde{\tau}_k^{-1}\big(\tilde{\tau}_j(u,f)\big) &= \tilde{\tau}_k^{-1}\Big(\tau_j\big(u, \phi_j(u)f\big)\Big) && \text{(equation (5))}\\
&= \tau_k^{-1}\Big(\tau_j\big(u, \phi_k^{-1}(u)\phi_j(u)f\big)\Big) && \text{(equation (5))}\\
&= \big(u, \tau_{j\to k}(u)\phi_k^{-1}(u)\phi_j(u)f\big) && \text{(equation (1))}\\
&= (u,f). && \text{(equation (4))}
\end{aligned}$$

This shows that if (4) holds, then we can construct new local trivializations $\tilde{\tau}$ that make all of the transition functions equal to the identity function, which implies that the bundle is trivial. To prove the *only if* part, use the fact that if $\tau_{j\to k}(u) = 1$, then (4) is satisfied by choosing $\phi_j(u) = 1$ for all j and all $u \in U_j$.

Equation (1) does not imply (4).[26] Equation (1) can be used to define transition functions for any fiber bundle, not just for trivial bundles. Chapter 10 will describe an example of a nontrivial bundle with transition functions given by equation (1).

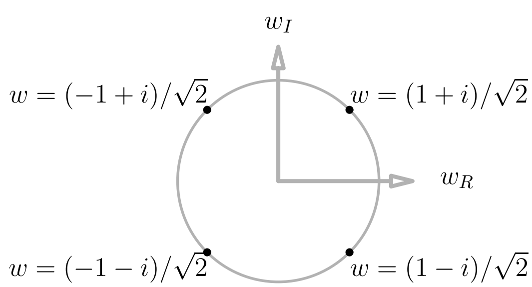
[25] Bertlmann (1996), page 99

[26] Each map ϕ_j in equation (4) must be defined throughout the corresponding chart U_j. Requiring ϕ_j to be smooth throughout U_j restricts the possible behaviors of ϕ_j in $U_j \cap U_k$.

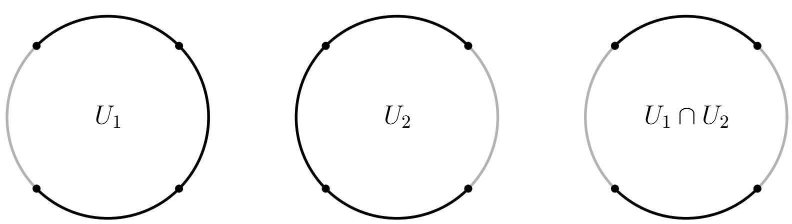
10 Example of a patchwise construction

This chapter uses the approach in chapter 8 to construct one of the fiber bundles that was described in chapter 7: the total space E is the set of complex numbers z with magnitude $|z| = 1$, the base space is M is also the set of complex numbers with magnitude 1, and the bundle projection is defined by $p(z) = z^2$ so that the fiber is a pair of points. This will be called the $\boldsymbol{z^2}$ **bundle**. This example shows how to construct the z^2 bundle using only two charts, U_1 and U_2, both contractible. Since M is a circle, the intersection $U_1 \cap U_2$ necessarily has two parts that aren't connected to each other, and one of this example's messages is that the construction still works even though $U_1 \cap U_2$ is not connected.

Let M be the set of complex numbers w with magnitude $|w| = 1$, which is the unit circle in the complex plane. Any such w may be written $w = w_R + iw_I$ with $w_R = \cos\theta$, $w_I = \sin\theta$, and $\theta \in \mathbb{R}$. Four such points around the unit circle are are labelled here:



The unit circle may be covered with two charts. This example will use use one chart U_1 defined by $w_R > -1/\sqrt{2}$ and another chart U_2 defined by $w_R < 1/\sqrt{2}$, as shown here:



The intersection $U_1 \cap U_2$ consists of two separate regions: the upper region $w_I > 1/\sqrt{2}$, and the lower region $w_I < -1/\sqrt{2}$.

Using these two charts, we can construct a fiber bundle whose base space M is the unit circle. Take the fiber to be $F = \{1, -1\}$, and take the transition function to be

$$\tau_{1\to 2}(w)f = \begin{cases} f & \text{in the upper region,} \\ -f & \text{in the lower region.} \end{cases} \tag{6}$$

Using the recipe described in chapter 8, the total space E is constructed by equating $(w, f) \in U_1 \times F$ with $(w, \tau_{1\to 2}(w)f) \in U_2 \times F$ and taking the projection to be $p(w, f) = w$.

The goal is to demonstrate that the resulting fiber bundle is equivalent to the z^2 bundle that was described in the first paragraph of this chapter. To demonstrate this, use these properties of the z^2 bundle:

- U_1 may be described as $-3\pi/4 < \theta < 3\pi/4$, and then the function $\tau_1 : U_1 \times F \to E$ defined by $\tau_1(e^{i\theta}, \pm 1) = \pm e^{i\theta/2}$ is a local trivialization for U_1.

- U_2 may be described as $\pi/4 < \theta < 7\pi/4$. and then the function $\tau_2 : U_2 \times F \to E$ defined by $\tau_2(e^{i\theta}, \pm 1) = \pm e^{i\theta/2}$ is a local trivialization for U_2.

- In the upper region, these local trivializations satisfy $\tau_1(e^{i\theta}, \pm 1) = \tau_2(e^{i\theta}, \pm 1)$, because the range of θ in this region is $\pi/4 < \theta < 3\pi/4$ for both τ_1 and τ_2.

- In the lower region, the local trivializations satisfy $\tau_1(e^{i\theta}, \pm 1) = \tau_2(e^{i\theta}, \mp 1)$ instead, because the range of θ in this region is $5\pi/4 < \theta < 7\pi/4$ when the local trivialization τ_1 is used but is $-3\pi/4 < \theta < -\pi/4$ when the local trivialization τ_2 is used.[27]

This shows that the transition function (6) may be written as in (1), so the fiber bundle constructed above from two patches is the same as the z^2 bundle that was described in the first paragraph.

[27]These two intervals of θ both describe the same part of the unit circle (the lower region), but the definitions τ_1 and τ_2 refer to the square root $e^{i\theta/2}$ of $e^{i\theta}$, and this square root depends on how that part of the unit circle is described in terms of θ. The two square roots differ by the factor $e^{i2\pi/2} = e^{i\pi} = -1$.

11 The concept of a vector bundle

A **vector bundle** is a fiber bundle whose fiber is a vector space V and whose local trivializations are linear transformations at each point of the base space.[28] Here are a few examples:

- Trivial vector bundles: If M is any smooth manifold and V is any vector space, then their cartesian product $M \times V$ is the total space of a trivial[29] vector bundle with projection defined by $p(m, v) = m$ for all $m \in M$ and $v \in V$.

- The **Möbius bundle**: Consider the two-dimensional manifold E consisting of pairs (θ, v) modulo the equivalence relation $(\theta + 2\pi, v) \sim (\theta, -v)$ with $\theta, v \in \mathbb{R}$. In words, E is a Möbius band with infinite width (no boundary). The projection defined by $p(\theta, v) = e^{i\theta}$ makes it a vector bundle with fiber $\mathbb{R}$ (regarded as a one-dimensional vector space) and base space S^1, where the circle S^1 is regarded as the set of complex numbers with magnitude 1. This vector bundle is nontrivial.[30]

- **Tangent bundles**: If M is a smooth n-dimensional manifold, then it has an n-dimensional space of tangent vectors at each point $m \in M$. The $2n$-dimensional smooth manifold TM whose points are these tangent vectors[31] is the total space of the **tangent bundle** with base space M. The fiber over a point $m \in M$ is the n-dimensional space of tangent vectors at m, and the bundle projection p projects each tangent vector to the point of M to which it is tangent.

[28] Tu (2017), definition 7.1

[29] *Trivial* was defined in chapter 6.

[30] If v is restricted to the two values $\{-1, 1\}$ instead of ranging over all of $\mathbb{R}$, then the Möbius bundle reduces to the z^2 bundle. Chapter 10 described a patchwise construction of the z^2 bundle, and chapter 20 will use a patchwise construction of the Möbius bundle to illustrate another important insight.

[31] In this definition, the tangent spaces at different points of M are disjoint: by definition, they do not share any vectors with each other, not even if M is $\mathbb{R}^n$.

12 Nontriviality of the tangent bundle of S^2

A tangent bundle may or may not be trivial. The tangent bundle with base space S^{n-1} is trivial if and only if $n \in \{1, 2, 4, 8\}$.[32] This chapter uses geometric intuition to demonstrate that the tangent bundle of S^2 is nontrivial (not isomorphic to $S^2 \times \mathbb{R}^2$).

To demonstrate this, think of S^2 as the surface of the unit sphere in three-dimensional euclidean space so that the length of each vector is defined, and consider the **unit tangent bundle** of S^2, which is what's left of the tangent bundle after discarding all vectors whose lengths are not equal to 1. At each point of S^2, the set of unit tangent vectors at that point is a circle, S^1, so if the unit tangent bundle were trivial, then its total space would be diffeomorphic to $S^2 \times S^1$. This chapter will show that:[33]

- The total space E of the unit tangent bundle of S^2 is diffeomorphic to $SO(3)$, the Lie group of orientation-preserving rotations.
- $SO(3)$ is diffeomorphic to the three-dimensional real projective space $\mathbb{R}\mathrm{P}^3$.

This shows that E is not diffeomorphic to $S^2 \times S^1$, so the unit tangent bundle of S^2 must be nontrivial. This implies that the full tangent bundle is nontrivial, too.

To show that E is diffeomorphic to $SO(3)$, start with the fact that E may be regarded as the set of all possible ordered pairs of mutually orthogonal unit vectors: the first vector in the pair may be used to specify a point $m \in S^2$ (because S^2 may be viewed as the set of unit vectors in three-dimensional euclidean space), and then the second vector in the pair may be used to specify a tangent direction at m. Given any one ordered pair of mutually orthogonal unit vectors, any other ordered pair may be obtained from it by a unique orientation-preserving rotation[34] about the origin. This shows that E is diffeomorphic to $SO(3)$.

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[32] Milnor (1958), theorem 2 (reviewed in Hatcher (2001), section 4.B, page 428), combined with the fact that "a manifold is called **parallelizable** if its tangent bundle is trivial" (Cohen (2023), end of section 3.2.2, page 40).

[33] The argument used in this chapter is essentially the proof of lemma 1 in Klingenberg and Sasaki (1975).

[34] An orientation-preserving rotation is a composition of an even number of reflections, not an individual reflection.

To show that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$, recall that $\mathbb{R}P^3$ is defined to be the set of lines through the origin of four-dimensional euclidean space, each line representing a single point of $\mathbb{R}P^3$. This may also be described as one hemisphere of the unit sphere S^3 with opposite points on the hemisphere's boundary (which is diffeomorphic to S^2) identified, because each line through the origin either intersects that hemisphere in exactly one point or else intersects its boundary in two points. One hemisphere of S^3 is diffeomorphic to a three-dimensional ball, so $\mathbb{R}P^3$ is diffeomorphic to a three-dimensional ball with opposite points on its boundary identified. Now, take this ball to have radius π. For each point x in this ball, define a corresponding orientation-preserving rotation like this:

- The rotation axis is the line through x.
- The magnitude of the rotation angle is the distance from the origin to x.
- The rotation is clockwise when looking from the origin toward the point x.

This correspondence is consistent with the fact that opposite points of the ball's boundary are identified, because the effect of clockwise and counter-clockwise rotations are identical when the rotation angle is π, so this establishes a smooth one-to-one correspondence between points of $\mathbb{R}P^3$ and orientation-preserving rotations. In other words, $\mathbb{R}P^3$ is diffeomorphic to $SO(3)$.

13 The fiber of a principal bundle

Chapter 14 will introduce a special kind of fiber bundle called a *principal bundle*. This chapter explains something about the nature of a principal bundle's fiber.

The fiber of a principal bundle is almost a Lie group, in this sense: the fiber is diffeomorphic to a Lie group G as a smooth manifold,[35] and multiplication by any element of G gives the same diffeomorphism of the fiber that it does of G, but we never multiply elements of the fiber by each other, so the fiber doesn't have any distinguished identity element like a group would have.[36,37] As a reminder of this, the fiber will be denoted $\tilde{G}$, where G is the Lie group that the fiber would become if we designated one of its elements to serve as the identity element.

More formally, $\tilde{G}$ may be described as a smooth manifold with a *free* and *transitive* right G-action.[38] A **right action** of G on $\tilde{G}$ is a map from $\tilde{G} \times G$ to $\tilde{G}$, called *multiplication* and denoted $(f, g) \mapsto fg$ with $f \in \tilde{G}$ and $g \in G$, satisfying these conditions:

$$(fg)g' = f(gg') \qquad\qquad fI = f$$

where $g, g' \in G$, and I is the identity element of G. The action is called **free** if every non-identity element of G moves every point of $\tilde{G}$ (in other words, $fg \neq f$ for all f and for all g except the identity element). The action is called **transitive** if, for every pair $f, f' \in \tilde{G}$, a group element $g \in G$ exists for which $fg = f'$. A smooth manifold $\tilde{G}$ with a free and transitive right G-action must be diffeomorphic to G.[39,40]

[35] A **Lie group** has the structure of a smooth manifold and the structure of a group. Introductions to Lie groups include chapter 7 in Lee (2013), appendix A in Harlow and Ooguri (2021), and definition 4.1 in Isham (1999).

[36] Wendl (2007a), text below proposition 2.88; McCarthy (2019), text above proposition 1.3.4

[37] If the principal bundle is nontrivial, then the fiber cannot have any such distinguished element, at least not one that varies smoothly throughout M (chapter 19).

[38] Wendl (2007a), text below definition 2.87

[39] Wendl (2007a), text above definition 2.87

[40] $\tilde{G}$ is a ***G*-torsor**, also called a **principal homogeneous space** for G (https://en.wikipedia.org/wiki/Principal_homogeneous_space).

14 The concept of a principal bundle

Compared to the general definition of *fiber bundle* in chapter 3, the definition of **principal $\boldsymbol{G}$-bundle** involves two additional pieces of data and a few additional conditions. The additional data are:

- A Lie group G called the **structure group**,[41,42]
- A smooth map $E \times G \to E$ called the **(right) action**[43] of G on the total space E. The image of (x, g) under this map will be denoted xg.

The additional conditions are:[44]

- The bundle projection p satisfies $p(xg) = p(x)$ for all $g \in G$ and all $x \in E$, so the fibers over different points of M aren't mixed with each other by $x \mapsto xg$.
- The fiber is $\tilde{G}$, as defined in chapter 13.
- The group G acts on $\tilde{G}$ as described in chapter 13, and the inverse local trivializations satisfy the additional condition[45]

  $$\tau(u, f)g = \tau(u, fg) \tag{7}$$

  for all $g \in G$, with f and u defined as in chapter 3. In words: the maps τ are **$\boldsymbol{G}$-equivariant**[46] in the second argument.

Two principal bundles are **equivalent** as principal bundles if they are equivalent as fiber bundles and if the diffeomorphism δ in that equivalence (chapter 5) is G-equivariant:[47] $\delta(x)g = \delta(xg)$.

[41] Isham (1999), section 5.2.1

[42] The name *structure group* is also used for something else (chapter 15).

[43] Chapter 18 will clarify the role of the *left* action.

[44] Mitchell (2011)

[45] Sengupta (2007), page 2

[46] Tu (2017), section 27.1

[47] Wendl (2007a), definition 2.89

15 Two meanings of "structure group"

The name *structure group* is used for two related-but-different things. Chapter 14 mentioned one of them. This chapter acknowledges the other one and clarifies their relationship.

When a fiber bundle is constructed as described in chapter 8, the term *structure group* sometimes refers to any group to which the transition functions belong.[48] Examples:

- A trivial bundle can be constructed using transition functions that are all equal to 1, in which case we could say that the structure group (denoted G in section 8) has only one element, even if the trivial bundle is a principal bundle whose structure group G as defined in section 14 has more than one element. This shows that the two meanings of *structure group* are not equivalent.

- Again, a trivial bundle can be constructed using transition functions that are all equal to 1, but the same trivial bundle may also be constructed using transition functions that are not equal to 1. This shows that the group generated by the transition functions is not an intrinsic property of the fiber bundle.

Even though the two same-named concepts are different, they are related. Their relationship is the subject of a proposition in Kobayashi and Nomizu (1963).[49] Roughly, that proposition says that if two principal bundles have the same base space but the first one has a smaller structure group (as defined in chapter 14), then the first bundle can be imbedded[50] in the second bundle if and only if the second bundle can be constructed (as described in chapter 8) using transition functions that all belong to the smaller structure group (as defined in chapter 14).

[48]Cohen (2023), section 2.1.3

[49]Kobayashi and Nomizu (1963), chapter 1, proposition 5.3

[50]This is defined in the paragraph before the proposition.

16 Examples of principal bundles

Article 03838 describes a nontrivial principal bundle called the **Hopf bundle** or the **Hopf fibration**. The structure group is $U(1)$, which is topologically a circle, the total space is S^3, and the base space is S^2. The Hopf bundle is one of the few fiber bundles in which the fiber, the total space, and the base space are all spheres.[51,52]

For an example of a nontrivial principal bundle with a nonabelian structure group, consider the set of ordered triple of complex numbers: (z_1, z_2, z_3). If we think of this as a space with six real dimensions, then the condition $|z_1|^2+|z_2|^2+|z_3|^2 = 1$ defines a sphere S^5. The group $SU(3)$ acts on these triples as a group of linear transformations that preserve that condition and that have determinant 1. Given any point of S^5, the subgroup of $SU(3)$ that leaves that point fixed is isomorphic to $SU(2)$.[53] We can define a projection $p : SU(3) \to S^5$ by taking $p(x)$ to be the unique[54] point of S^5 that is not affected by the given element x of $SU(3)$. This defines a principal bundle with structure group $SU(2)$, total space $SU(3)$, and base space S^5.[55]

[51] Hatcher (2001), section 4.B, page 428

[52] The unit tangent bundle of S^2 also has a circle as its fiber, but its total space is not a sphere: the total space of the unit tangent bundle of S^2 is $\mathbb{R}P^3$, which may be constructed from S^3 by identifying antipodal points with each other (chapter 12).

[53] To see this, consider the point $(z_1, z_2, z_3) = (1, 0, 0)$.

[54] To see that each element of $SU(3)$ leaves a unique point of S^5 fixed, consider all elements of $SU(3)$ that leave the point $(1, 0, 0)$ fixed, and observe that none of them leaves any other point fixed.

[55] More information about this fiber bundle is given in `https://mathoverflow.net/questions/145482/` and `https://mathoverflow.net/questions/69352/`

17 The role of the left action

The inverse local trivializations introduced in chapter 3 are maps τ from $U \times \tilde{G}$ to $p^{-1}(U)$. The cartesian product $U \times \tilde{G}$ is a trivial bundle, so we can alternatively think of the maps τ as maps from $U \times G$ to $p^{-1}(U)$ by choosing an arbitrary element of $\tilde{G}$ to serve as the identity element.[56]

Now let U_j and U_k be two overlapping charts, at let τ_j and τ_k be corresponding inverse local trivializations. Equation (1) defines a transition function $\tau_{j\to k}$ by the condition

$$\tau_j(u, f) = \tau_k\big(u,\, \tau_{j\to k}(u)f\big) \tag{8}$$

for all $f \in G$ and $u \in U_j \cap U_k$, and then the (right) G-equivariance condition (7) implies that the transition functions are also (right) G-equivariant:

$$\tau_{j\to k}(u)(fg) = \big(\tau_{j\to k}(u)f\big)g.$$

A right G-equivariant function from G to itself is necessarily equivalent to left multiplication by an element of G,[57] so the effect of the diffeomorphism $\tau_{j\to k}(u)$ may be written[58]

$$\tau_{j\to k}(u)f = h_{j\to k}(u)f, \tag{9}$$

with $h_{j\to k}(u) \in G$. The right-hand side is the product of two elements of G.

This shows that when a principal bundle is constructed patchwise as described in chapter 8, the transition functions use a left action of G on the locally-trivialized patches. This left action doesn't interfere with the right action, because the left and right actions commute with each other.[59]

[56]For any fiber bundle, choosing a local trivialization always involves making arbitrary choices.

[57]Tu (2017), lemma 27.7. Proof: let β be a function from G to G satisfying the right G-equivariance condition $\beta(fg) = \beta(f)g$. The quantity $\beta(g)$ may also be written $\beta(Ig)$, where I is the identity element of the group, so the right G-equivariance condition implies $\beta(g) = \beta(I)g$. This shows that applying the function β to G is the same as multiplying on the left by $\beta(I) \in G$.

[58]Tu (2017), equation (27.2)

[59]Proof: group multiplication is associative, so $(g_1 f)g_2 = g_1(fg_2)$ with $g_1, f, g_2 \in G$. This says that multiplying f first by g_1 on the left and then by g_2 on the right is the same as multiplying f first by g_2 on the right and then by g_1 on the left.

18 How to construct a principal bundle

A principal bundle is a special kind of fiber bundle, so the approach described in chapter 8 may be used to construct examples of principal bundles.[60] To enforce the additional conditions that principal bundles must satisfy (chapter 14), the transition functions must also satisfy an additional condition. Consistency with equations (1) and (7) requires

$$\tau_{j\to k}(u)(fg) = \big(\tau_{j\to k}(u)f\big)g$$

for all $f \in F_{x,j}$ and all $g \in G$. In words: the effect of the diffeomorphism $\tau_{j\to k}(u)$ on the fiber should commute with the right action of the group G on the fiber. In fancier words: the transition functions should be G-equivariant.

[60] Definitions 10.1-10.2 in Kolář *et al* (1993) use a patchwise construction like this as the definition of *principal bundle*, and one of remarks 10.4 in Kolář *et al* (1993) suggests that this makes it applicable to infinite-dimensional cases (footnote 13 in chapter 4). This might be one disadvantage the philosophy that was mentioned in footnote 16 in chapter 5.

19 The concept of a section

Let (E, M, p) be a fiber bundle. A **cross section** or just **section**[61] is a smooth map $\sigma : M \to E$ such that $p(\sigma(m)) = m$ for all $m \in M$. Here are a few basic facts about sections:

- Every vector bundle has a section.[62]
- A vector bundle whose fiber is a one-dimensional vector space[63] is called a **line bundle**. Any section σ of a nontrivial line bundle must be zero somewhere.[64]
- More generally, a vector bundle with n-dimensional fiber is trivial if and only if it has n sections that are linearly independent everywhere.[65]
- A principal bundle has a section if and only if the principal bundle is trivial.[66]

The definition of *section* requires that it be defined everywhere on M. Sometimes, the name **global section** is used to emphasize this.

The concept of a **local section**, a smooth map from a chart $U \subset M$ to E, is also useful.[67] Some fiber bundles don't have any (global) sections, but every fiber bundle has local sections. The first example in chapter 7 doesn't have any global sections when $n = 2$, because no matter which of the two possible values we choose for $\sigma(m)$ at one point $m \in M$, the requirement for σ to be smooth everywhere must be violated somewhere. On the other hand, if we consider a chart U that includes all of M except one point, then (exactly two) local sections over U do exist.

[61] This is why this article uses the word *chapter* for parts of the article (footnote 1 in chapter 1).

[62] Example: take $\sigma(m) = 0$ for all $m \in M$, where 0 is the zero vector in the fiber. This is always a smooth section, thanks to the requirement that the local trivializations of a vector bundle should be linear transformations at each point of the base space (chapter 11).

[63] The vector space may be one-dimensional over $\mathbb{R}$ or $\mathbb{C}$. If the field of coefficients is $\mathbb{C}$, then a line bundle is sometimes called a **complex line bundle**, but it is often just called a *line bundle*.

[64] Wendl (2007a), exercise 2.13

[65] Collinucci and Wijns (2006), section 1.4, theorem 1

[66] Husemoller (1966), chapter 4, corollary 8.3

[67] A special case of this concept is defined in Steenrod (1951), section 7.4, page 30.

20 Sections compared to fiber-valued functions

A section of a trivial fiber bundle $M \times F$ is the same thing as a smooth function from the base space M to the fiber F: given a function $\sigma(m) = (m, f) \in M \times F$, projecting $\sigma(m)$ onto its second factor f gives a function from M to F.

In contrast, a nontrivial fiber bundle does not have a projection onto the fiber. A section is always a smooth[68] function from the base space to the total space, but it cannot always be viewed as a smooth function from the base space to the fiber. This is illustrated below using the Möbius bundle that was described in chapter 11.

The base space M of the Möbius bundle is a circle S^1, which may be parameterized by $e^{i\theta}$ with $\theta \in \mathbb{R}$. The fiber F is $\mathbb{R}$. Define two charts U_1 and U_2 as in chapter 10. For this example, the Möbius bundle will be constructed from two patches, $U_1 \times F$ and $U_2 \times F$. These two patches are sewn together using the transition function

$$\tau_{1\to 2}(m)f = \begin{cases} f & \text{for } m \text{ in the upper region,} \\ -f & \text{for } m \text{ in the lower region,} \end{cases} \tag{10}$$

where *upper region* and *lower region* refer to the two parts of $U_1 \cap U_2$ that were highlighted in chapter 10. Here, f can be any real number, because the fiber is $\mathbb{R}$. To construct a fiber bundle from this data, convert the disjoint union of the patches $U_1 \times F$ and $U_2 \times F$ to a single manifold E by equating each (m, f) in $U_1 \times F$ with $\big(m, \tau_{1\to 2}(m)f\big)$ in $U_2 \times F$ whenever $m \in U_1 \cap U_2$. After defining bundle projection as in chapter 8, this gives the Möbius bundle that was introduced in chapter 11.

A section σ of the Möbius bundle may be specified using a pair of functions, $\sigma_1 : U_1 \to F$ and $\sigma_2 : U_2 \to F$, that satisfy the consistency condition

$$\sigma_2(m) = \tau_{1\to 2}(m)\sigma_1(m) \tag{11}$$

for all $m \in U_1 \cap U_2$. If the functions σ_1 and σ_2 are smooth, then this gives a smooth section σ of the Möbius bundle. The question is whether this section may

[68]Chapter 4.

be described as a smooth F-valued function of the whole base space. It could if $\sigma_2(m) = \sigma_1(m)$ for all $m \in U_1 \cap U_2$, but for this to be compatible with (11), we must have $\sigma_1(m) = \sigma_2(m) = 0$ wherever $\tau_{1\to 2}(m) \neq 1$. We can devise sections that do satisfy this condition, but most sections don't, so most sections cannot be described as a single smooth F-valued function of all $m \in M$.

One example of a smooth section on this bundle is the one defined by

$$\sigma_1(e^{i\theta}) = \sin(\theta/2) \text{ for } -3\pi/4 < \theta < 3\pi/4, \tag{12}$$

$$\sigma_2(e^{i\theta}) = \sin(\theta/2) \text{ for } \pi/4 < \theta < 7\pi/4. \tag{13}$$

This pair of functions satisfies the condition (11) with $\tau_{1\to 2}$ given by (10). This is illustrated in figure 1 on the next page. Explicitly:

- The upper region (where $\tau_{1\to 2} = 1$) is $\pi/4 < \theta < 3\pi/4$, so σ_1 and σ_2 are equal to each other in that region.

- The lower region (where $\tau_{1\to 2} = -1$) can be described either as $-3\pi/4 < \theta < -\pi/4$ or as $5\pi/4 < \theta < 7\pi/4$, so σ_1 and σ_2 are each other's negatives in that region because $\sin((\theta + 2\pi)/2) = -\sin(\theta/2)$.

The functions σ_1 and σ_2 are not equal to each other everywhere in $U_1 \cap U_2$, so this section does not correspond to any single smooth F-valued function everywhere on M,[69] even though it does define a smooth section (a smooth E-valued function) everywhere on M.

[69]We could contrive a F-valued function everywhere on M by arbitrarily choosing which points $m \in U_1 \cap U_2$ should use σ_1 and which ones should use σ_2, but the resulting function would not be smooth everywhere.

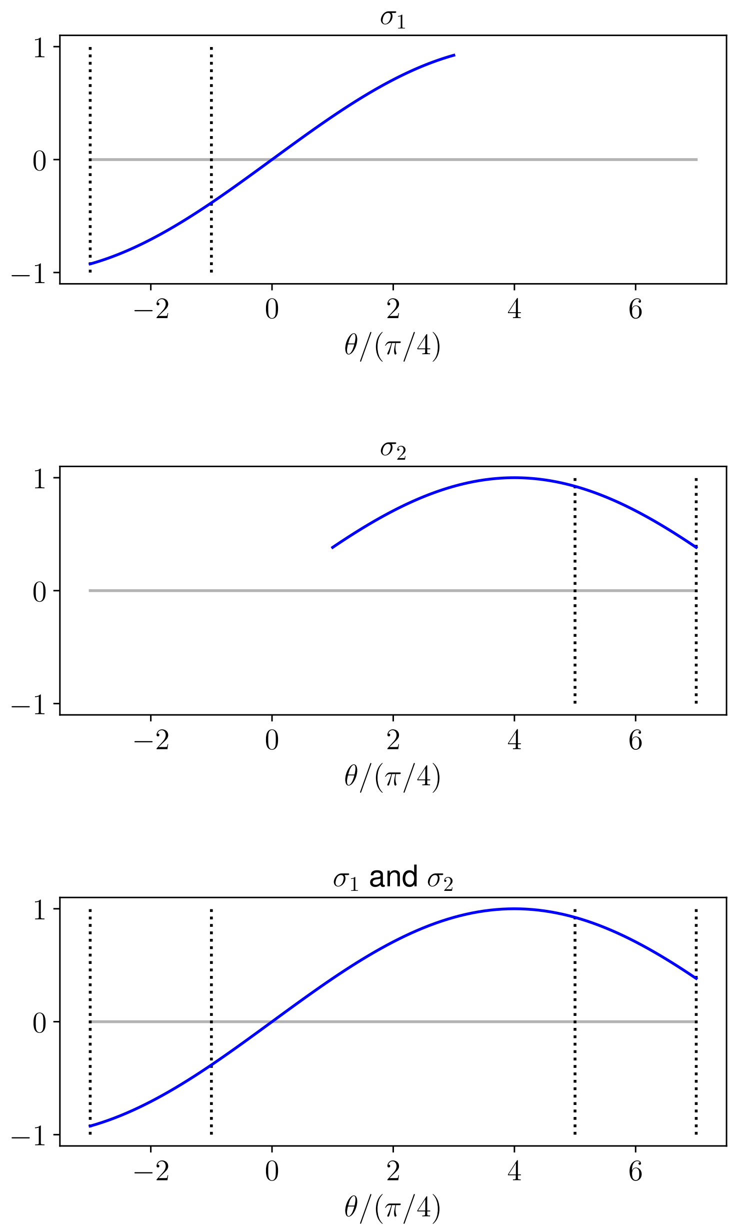

Figure 1 – The top graph shows the function σ_1 defined by equation (12). Similarly, the middle graph shows the function σ_2 defined by equation (13). The bottom graph shows σ_1 and σ_2 overlaid. The dashed lines indicate the part of $U_1 \cap U_2$ in which the transition function equals -1. This confirms that $\sigma_1 = \sigma_2$ where their domains overlap each other, so they define a single smooth global section, as claimed in the text.

21 Vector bundles associated to a principal bundle: preview

The proposition reviewed at the end of chapter 15 implies that a principal G-bundle may be constructed patchwise using transition functions that all belong to G. If a principal G-bundle over a base space M is constructed using transition functions $\tau_{j\to k}$, and if we choose a representation ρ of the group G as a group of linear transformations of a vector space V, then we can use $\rho(\tau_{j\to k})$ as transition functions to construct a vector bundle with the same base space M but whose fiber is V instead of $\tilde{G}$. This vector bundle is said to be *associated* with the principal bundle, because they can both be constructed using the "same" transition functions – same except that the vector bundle uses $\rho(\tau_{j\to k})$ instead of $\tau_{j\to k}$.[70]

If a principal G-bundle is trivial, then any associated vector bundle is also trivial,[71] but the converse depends on the representation. Examples:

- If the principal bundle is nontrivial and the representation of G is isomorphic to G itself, then the associated vector bundle is nontrivial.[72]

- If the representation is trivial (that is, if $\rho(g) = 1$ for all $g \in G$), then the associated vector bundle is trivial even if the principal bundle is not.[73]

Chapter 22 will define the associated vector bundle more directly, without relying on the idea of assembling the bundle from patches using transition functions, and chapter 25 will explain how the more direct definition relates to the one that was previewed here.

[70]Figueroa-O'Farrill (2006), section 1.4
[71]Tu (2017), text above proposition 31.1
[72]Nakahara (1990), definition 10.2
[73]Tu (2017), text below lemma 31.4

22 Vector bundles associated to a principal bundle

Let $p : E \to M$ be a principal G-bundle, and let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) on which a linear representation ρ of G acts, so that $\rho(g) : V \to V$ is a linear transformation for each $g \in G$. This chapter uses that data to define an **associated vector bundle**. Unlike the preview in chapter 21, this definition given here does not rely on a patchwise construction.

Consider the cartesian product $E \times V$. Each element of $E \times V$ is an ordered pair (x, v) with $x \in E$ and $v \in V$. Let $\sim$ be the equivalence relation defined by declaring $(xg, \rho(g^{-1})v)$ to be equivalent to (x, v) for every $g \in G$. Let $[x, v]$ denote the equivalence class that includes (x, v), so

$$[xg, \rho(g^{-1})v] = [x, v]. \tag{14}$$

The total space of the associated vector bundle is the quotient[74] $\hat{E} \equiv (E \times V)/\sim$, also denoted $\hat{E} = E \times_\rho V$. The bundle projection $\hat{p} : \hat{E} \to M$ is defined by

$$\hat{p}\big([x, v]\big) = p(x).$$

This is consistent with the equivalence relation, because

$$\hat{p}\Big(\big[xg, \rho(g^{-1})v\big]\Big) = p(xg) = p(x) = \hat{p}\big([x, v]\big).$$

The fiber has the structure of a vector space: within the fiber over the point $m \in M$, addition and scalar multiplication are defined by $[x, v_1] + [x, v_2] \equiv [x, v_1 + v_2]$ and $c[x, v] \equiv [x, cv]$ for any $x \in E$ with $p(x) = m$, where c is a scalar (in $\mathbb{R}$ or $\mathbb{C}$).[75] Altogether, this defines a vector bundle with base space M and fiber V.[76]

[74] Tu (2017), section 31.1; Abbassi and Lakrini (2021), section 1; and Haydys (2019), definition 31. As a simple check, let n_M, n_G, and n_V be the number of dimensions of M, G, and V, respectively. Then the number of dimensions of $E \times V$ is $n_M + n_G + n_V$, and the number of dimensions of $(E \times V)/\sim$ is $n_M + n_V$ because the equivalence relation removes G parameters. The number $n_M + n_V$ matches the number of dimensions of a vector bundle with base space M and fiber V.

[75] These definitions are consistent with the equivalence relation because ρ is a linear representation.

[76] Michor (2008), theorem 18.7; and Figueroa-O'Farrill (2006), section 1.4

23 Local trivializations for an associated bundle

This chapter explains how to construct local trivializations for the associated vector bundle that was defined in chapter 22. The same notation will be used again here.

Let $U \subset M$ be a chart for which the principal bundle admits a local section $\sigma : U \to E$. We can use this local section to construct a local trivialization $\hat{\tau} : U \times V \to \hat{E}$ for the corresponding part of the associated vector bundle, like this:[77]

$$\hat{\tau}(m, v) \equiv \big[\sigma(m), v\big] \qquad\qquad \text{for all } m \in U. \tag{15}$$

This construction refers to an arbitrary local section σ. This is natural because equation (14) says that any given point in the associated vector bundle may be represented by many different pairs (x, v). Choosing the local section σ is a way of choosing a single pair $\big(\sigma(m), v\big)$ to represent each point in the patch $\hat{\tau}(U \times V) \subset \hat{E}$. Chapter 24 will show that the arbitrariness in the choice of σ is equivalent to the expected arbitrariness in any local trivialization of the associated vector bundle.

[77]Previous sections use the letter m for a generic point in the base space M and used the letter u for a point in a particular chart $U \subset M$. Here, the letter m is used for a point of U, because u looks too much like v – the letter that is being used for an element of the vector space V.

24 Arbitrariness of the local trivializations

This chapter shows that the arbitrariness in the choice of local section used to construct the local trivializations (15) matches the expected degree of arbitrariness in any local trivialization of the vector bundle.

Any other local section $\sigma'(m)$ of $U \times \tilde{G}$ is related to the original one by

$$\sigma'(m) = \sigma(m)g(m) \qquad \text{for all } m \in U, \tag{16}$$

with $g(m) \in G$. If we construct a corresponding local trivialization $\hat{\tau}'$ as in (15), then

$$\begin{aligned} \hat{\tau}'(m, v) &\equiv \big[\sigma'(m), v\big] && \text{(like equation (15))} \\ &= \big[\sigma(m)g(m), v\big] && \text{(equation (16))} \\ &= \Big[\sigma(m), \rho\big(g(m)\big)v\Big] && \text{(equation (14))} \\ &= \hat{\tau}\Big(m, \rho\big(g(m)\big)v\Big) && \text{(equation (15)).} \end{aligned}$$

This shows that changing which local section we use in equation (15) is equivalent to applying an m-dependent linear transformation $\rho(g(m))$ to the vector factor in $U \times V$, which is exactly the freedom we expect to have when choosing a local trivializations of the vector bundle.

25 Transition functions for an associated bundle

This section uses the local trivializations that were described in chapter 23 to construct transition functions between overlapping patches of the associated vector bundle.

Consider two charts, U_j and U_k, and corresponding local sections $\sigma_j : U_j \to E$ and $\sigma_k : U_k \to E$ of the principal bundle. For each point m in the intersection $U_j \cap U_k$, let $g_{j\to k}(m)$ be the element of G that relates the two local sections:[78]

$$\sigma_j(m) g_{j\to k}(m) = \sigma_k(m). \tag{17}$$

Let $\hat{\tau}_{j\to k}$ be the transition function between the jth and kth local trivializations $\hat{\tau}_j$ and $\hat{\tau}_k$ of the associated vector bundle, constructed from σ_j and σ_k as in chapter 23. According to equation (1), the transition functions should satisfy

$$\hat{\tau}_j(m, v) = \hat{\tau}_k\big(m, \hat{\tau}_{j\to k}(m)v\big). \tag{18}$$

The requirement (18) is satisfied by[79]

$$\hat{\tau}_{j\to k}(m)v = \rho\big(g^{-1}_{j\to k}(m)\big)v. \tag{19}$$

Proof:

$$\begin{aligned}
\hat{\tau}_k\big(m, \hat{\tau}_{j\to k}(m)v\big) &= \hat{\tau}_k\Big(m, \rho\big(g^{-1}_{j\to k}(m)\big)v\Big) && \text{(equation (19))} \\
&= \Big[\sigma_k(m), \rho\big(g^{-1}_{j\to k}(m)\big)v\Big] && \text{(equation (15))} \\
&= \Big[\sigma_k(m) g^{-1}_{j\to k}(m), v\Big] && \text{(equation (14))} \\
&= \Big[\sigma_j(m), v\Big] && \text{(equation (17))} \\
&= \hat{\tau}_j(m, v) && \text{(equation (15)).}
\end{aligned}$$

[78] This uses the *right* action of G on the principal bundle. Contrast this to equation (9), which expresses the transition function using a *left* action.

[79] Text below equation (4) in de los Ríos (2020), section 3.2

26 Recovering the preview in chapter 21

This chapter shows how to relate transition functions $\rho(g_{j\to k})$ of the vector bundle to the transition functions (9) of the associated principal bundle defined in chapter 22. The fiber $\tilde{G}$ will be treated as the group G, as in chapter 9, so that left and right actions of G on the fiber are both defined.

Use the same setup that chapter 25 used, and define functions $f_j : U_j \to G$ and $f_k : U_k \to G$ by

$$\tau_j\big(m, f_j(m)\big) = \sigma_j(m) \qquad\qquad \tau_k\big(m, f_k(m)\big) = \sigma_k(m). \tag{20}$$

Then, for all $m \in U_j \cap U_k$,

$$\begin{aligned}
\sigma_k(m) &= \sigma_j(m) g_{j\to k}(m) && \text{(equation (17))}\\
&= \tau_j\big(m, f_j(m)\big) g_{j\to k}(m) && \text{(equation (20))}\\
&= \tau_k\big(m, \tau_{j\to k}(m) f_j(m)\big) g_{j\to k}(m) && \text{(equation (8))}\\
&= \tau_k\big(m, h_{j\to k}(m) f_j(m)\big) g_{j\to k}(m) && \text{(equation (9))}\\
&= \tau_k\big(m, h_{j\to k}(m) f_j(m) g_{j\to k}(m)\big). && \text{(equation (7))}
\end{aligned}$$

Compare this to the second equation in (20) to deduce

$$h_{j\to k} f_j(m) g_{j\to k}(m) = f_k(m). \tag{21}$$

We're treating the fiber $\tilde{G}$ as the group G, so we can take both $f_j(m)$ and $f_k(m)$ to be the identity element I of the group. With that special choice, equation (21) reduces to $h_{j\to k} g_{j\to k}(m) = I$, and then the transition functions (19) for the vector bundle may be written as

$$\hat{\tau}_{j\to k}(m) v = \rho(h_{j\to k}(m)) v. \tag{22}$$

Given transition functions $h_{j\to k}$ for the principal bundle (equation (9)), this shows that we can construct the associated vector bundle from patches using $\rho(h_{j\to k})$ as its transition functions, as anticipated in chapter 21.

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