# Scalar Quantum Fields on a Spacetime Lattice

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**Abstract** This article introduces the **path integral** formulation of quantum field theory for scalar quantum fields, treating spacetime as a (very fine) lattice to make the whole construction mathematically unambiguous. In this formulation, time evolution is expressed in terms of an action (the integral of a lagrangian) that has a manifestly Lorentz symmetric continuum limit. This can be viewed as the quantum version of the action principle. Starting with the path integral formulation, this article explains how to recover the hamiltonian formulation that was used in article 52890, including the equal-time commutation relations.

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## Contents

# 1 Introduction

Different ways of formulating the same thing can lead to different insights. Article 52890 introduced one way of constructing models in quantum field theory (QFT), one in which time-evolution is implemented by a hamiltonian. I'll call that the **hamiltonian** formulation. This article introduces of a different approach called the **path integral** formulation and explains its relationship to the hamiltonian formulation. As in article 52890, this article considers only simple models of scalar quantum fields, but the approach can be extended to other kinds of models, too.

Article 52890 treats space as a (very fine) lattice. This article treats space*time* as a (very fine) lattice. For that reason, the two formulations are not equivalent to each other, but they are effectively equivalent to each other for predictions whose time-resolution is much coarser than the discrete lattice time-increment. That's sufficient for applications, because the lattice is artificial anyway.

This article also introduces **Wick rotation**. Conceptually, Wick rotation converts the signature of the spacetime metric from lorentzian to euclidean, so Lorentz symmetry in $d$-dimensional spacetime becomes ordinary rotation symmetry in $d$-dimensional euclidean space. This is a different way of encoding the same information: the original encoding in lorentzian spacetime can be recovered by un-doing the transformation. One advantage of the euclidean version is that it makes QFT resemble classical statistical mechanics, so insights and methods from that subject area can be used to help improve our understanding of QFT.

Another advantage of the euclidean encoding is that it provides an explicit expression for vacuum expectation values that doens't require knowing which element of the Hilbert space represents the vacuum (lowest-energy) state. For this reason, the euclidean version is often used as the model's definition, from which the lorentzian version is recovered through Wick rotation.[1] This article does the opposite: it uses the lorentzian version as the definition, and then the euclidean version is introduced as a tool for streamlining calculations.

[1]For this to work, the euclidean version must satisfy a condition called **(Osterwalder-Schrader) reflection positivity** (explained in Montvay and Münster (1997), section 1.3, and in Simmons-Duffin (2016), section 7.1).

## 2 Preview

In quantum theory, states can be represented by vectors in a Hilbert space. In the Schrödinger picture,[2] evolution from an initial state-vector $|\Psi_i\rangle$ at time $t_i$ to a final state-vector $|\Psi_f\rangle$ at time $t_f$ is implemented by a unitary operator $U$:

$$|\Psi_f\rangle = U|\Psi_i\rangle. \tag{1}$$

Article 52890 described one approach to constructing models of a scalar quantum field, treating space as a (very fine) lattice to make everything mathematically unambiguous. In that approach, time is still continuous, so the unitary operator $U$ may be written as $U = e^{-iH\,\Delta t}$, where $H$ is the **hamiltonian** and $\Delta t \equiv t_f - t_i$.

This article introduces a different approach in which both space and time are discrete. One real variable $\phi(x)$, called a **field variable**, is associated with each point $x$ in spacetime. If $[\phi]_R$ denotes the set of all $\phi(x)$ with $x \in R$, then equation (1) can be formulated like this:

$$\Psi_f[\phi]_C \propto \int [d\phi]_{A\cup B}\ \exp\left(iS[\phi]_{A\cup B\cup C}\right) \Psi_i[\phi]_A \tag{2}$$

with these definitions of the sets $A, B, C$:

$A$ is the set of spacetime points at time $t_i$,
$B$ is the set of spacetime points with times in the range $t_i < t < t_f$,
$C$ is the set of spacetime points at time $t_f$.

The **action** $S$ is the lattice version of the integral over $x$ of a Lorentz-invariant lagrangian, so the emergence of Lorentz symmetry at low resolution (compared to the lattice scale) is more intuitive than in the hamiltonian formulation.

An integral over an enormous number of field variables, like the integral (2), is often called a **path integral**.[3] Sections 3-7 introduce this approach in more detail, and sections 10-12 recover the hamiltonian formulation from it.

[2] Article 22871

[3] This name made more sense when applied to single-particle quantum mechanics (Feynman (1948)), but now it's used more generally (Weinberg (1995)). It's also called a **functional integral** (Peskin and Schroeder (1995)).

# 3 The lattice and the Hilbert space

Spacetime will be treated as a very fine lattice of indefinite extent along the time axis and finite extent along each of the spatial axes. This notation will be used:

- The number of spatial dimensions will be denoted $D$, so that spacetime is $D+1$-dimensional.
- The number of points along each spatial axis will be denoted $K$, so that the total number of points at any given time is $K^D$.
- A point in spacetime will be denoted $x = (t, \mathbf{x})$, where $t$ is the time coordinate and $\mathbf{x}$ is the list of spatial coordinates.

A state[4] is represented by a complex-valued function $\Psi[\phi]$ of the $K^D$ field variables $\phi(\mathbf{x})$ associated with all points $\mathbf{x}$ in space at a single time. Given two states $\Psi_1[\phi]$ and $\Psi_2[\phi]$, their inner product is

$$\begin{aligned} \langle \Psi_1 | \Psi_2 \rangle &\equiv \int [d\phi] \ \Psi_1^*[\phi] \Psi_2[\phi] \\ &\equiv \int \left( \prod_{\mathbf{x}} d\phi(\mathbf{x}) \right) \ \Psi_1^*[\phi] \Psi_2[\phi]. \end{aligned} \tag{3}$$

The integral is over the full range $-\infty < \phi(\mathbf{x}) < \infty$ of each of the real variables $\phi(\mathbf{x})$.[5] Only functions $\Psi[\phi]$ for which $\langle \Psi | \Psi \rangle$ is finite can be used to represent states.[6]

[4] The rest of this article uses the word *state* to mean a state-vector in the Hilbert space.

[5] This representation of the Hilbert space is the same as the one used in article 52890, but with a different notation: here, the independent variables are denoted $\phi$.

[6] Article 03431

# 4 The action

For motivation, temporarily let $\phi(x)$ denote an ordinary real-valued function of spacetime, and consider the Lorentz-invariant action functional

$$S[\phi] = \int d^{D+1}x \left( \frac{\eta^{ab}\big(\partial_a\phi(x)\big)\big(\partial_b\phi(x)\big)}{2} - V\big(\phi(x)\big) \right), \tag{4}$$

where $\eta^{ab}$ are the components of the Minkowski metric and $V$ is a real-valued function of a single real variable. This action could be used to define a model of a classical scalar field,[7] but here we will use it to define a model of a quantum scalar field instead.[8] To do this, we'll think of the field $\phi(x)$ as a collection of independent real variables, one for each point $x$ in spacetime. To make this unambiguous, spacetime will be treated as a discrete lattice, one whose extent in the spatial dimensions is finite, as described in section 3. Then the number of field variables $\phi(x)$ is finite when limited to points $x$ in at any given finite interval of time.

Writing $\dot{\phi}$ for the derivative of $\phi$ with respect to the time coordinate and $\nabla\phi$ for the gradient of $\phi$ with respect to the spatial coordinates, the action (4) may also be written

$$S[\phi] = \int d^{D+1}x \left( \frac{\dot{\phi}^2(x) - \big(\nabla\phi(x)\big)^2}{2} - V\big(\phi(x)\big) \right).$$

When spacetime is treated as a lattice instead of a continuum, the integrals become sums:

$$\int d^{D+1}x \ \cdots \qquad \to \qquad dt\,\epsilon^D \sum_x \ \cdots, \tag{5}$$

[7] Article 49705

[8] Long-standing tradition refers to the quantum model as a *quantization* of the classical model with the same action, as though the logical sequence were *action* → *classical model* → *quantum model*. In hindsight, a more generally helpful perspective is simply that starting with a Lorentz-invariant action is a good way to construct a model that has Lorentz symmetry (in the continuum limit), whether it's classical or quantum. Then the logical sequence is *action* → *classical* on one hand, and *action* → *quantum* on the other hand, without any "classical → quantum" progression at all. And by the way, just because a quantum model starts with the same action as some classical model doesn't ensure that the classical model is a good approximation to the quantum one.

where $dt$ is the lattice spacing in the time direction and $\epsilon$ is the spacing in each spatial direction. The derivative become finite differences, like this:[9]

$$\dot{\phi}(t,\mathbf{x}) \equiv \frac{\phi(t+dt,\mathbf{x}) - \phi(t,\mathbf{x})}{dt}. \tag{6}$$

An explicit expression for the lattice version of $\nabla\phi$ won't be needed in this article. Altogether, the action in discrete spacetime is

$$S[\phi] \equiv \sum_{t\in\mathcal{I}} \Big(s_1\big[\phi(t+dt),\phi(t)\big] + s_0\big[\phi(t)\big]\Big) \tag{7}$$

where $\mathcal{I}$ is a specified interval of time, and $\phi(t)$ is an abbreviation for the collection of variables $\phi(t,\mathbf{x})$ for all $\mathbf{x}$, and[10]

$$\begin{aligned} s_1[\phi',\phi] &\equiv \left(\epsilon^D \sum_{\mathbf{x}} \frac{\big(\phi'(\mathbf{x}) - \phi(\mathbf{x})\big)^2}{2\,dt}\right) \\ s_0[\phi] &\equiv -dt\,\epsilon^D \sum_{\mathbf{x}} \left(\frac{\big(\nabla\phi(\mathbf{x})\big)^2}{2} + V\big(\phi(\mathbf{x})\big)\right). \end{aligned} \tag{8}$$

In these last two equations, $\phi$ denotes a set of field variables consisting of one variable $\phi(\mathbf{x})$ per point in space, just like in section 3. Similarly, $\phi'$ denotes another set of field variables consisting of one variable $\phi'(\mathbf{x})$ per point in space. The time coordinate doesn't need to be specified in these equations, because it's specified in equation (7) instead.

[9] Many different finite differences reduce to the same derivative in the continuum limit. The choice (6) keeps the math relatively simple.

[10] In $s_1$, the overall factor of $dt$ has been cancelled by one of the factors of $dt$ in the denominator of $\dot{\phi}^2$.

# 5 The time-evolution equation

Section 2 previewed the time-evolution equation for states the Schrödinger picture.[11] This section describes it more explicitly, using the Hilbert space that was defined in section 3 and the lattice action that was defined in section 4. Let $\Psi_t[\phi]$ denote the state at time $t$, as a function of the field variables $\phi$ associated with that one time. The time-evolution equation for a single time-step is

$$\Psi_{t+dt}[\phi'] = \int [d\phi]\ T[\phi', \phi]\Psi_t[\phi] \tag{9}$$

where $T$ is the **transfer matrix** defined by

$$T[\phi', \phi] = \mathcal{N} \exp\big(is_1[\phi', \phi] + is_0[\phi]\big) \tag{10}$$

with $i^2 = -1$, where $\mathcal{N}$ is a normalization factor that will be determined in section 6. It's called a *matrix* because the integral in (9) is analogous to a sum over the continuous "index" $\phi$. Equations (9)-(10) define $T$ as an operator on the Hilbert space, so we can also write equation (9) as

$$|\Psi_{t+dt}\rangle = T|\Psi_t\rangle.$$

Evolution through $M$ time-steps is given by applying $M$ factors of $T$:

$$|\Psi_{t+M\,dt}\rangle = T^M|\Psi_t\rangle. \tag{11}$$

According to equations (7), (9), and (10), this may also be written

$$\boxed{\Psi_{t+M\,dt}\big[\phi(t + M\,dt)\big] = \mathcal{N}^M \int \prod_{m=0}^{M-1} \big[d\phi(t + m\,dt)\big]\ \exp\Big(iS[\phi]\Big)\Psi_t\big[\phi(t)\big]} \tag{12}$$

with $S$ defined by equation (7) for the time interval $\mathcal{I}$ that starts with $t$ and ends with $t + (M - 1)dt$. Equation (12) is the more explicit version of equation (2).

[11] The time-evolution equation for states the Schrödinger picture is often called the **Schrödinger equation**. This generalizes the name from models of strictly nonrelativistic particles to arbitrary models.

# 6 Unitarity

In quantum theory, time evolution should be unitary, as reviewed at the beginning of section 2. In the path integral formulation, time is discretized so that the construction is mathematically unambiguous, but we're really only interested in the continuous-time limit, so requiring time evolution to be unitary in the continuous-time limit would be good enough.

When the time-derivative term in the action has the form shown in equation (8), the transfer matrix defined by equations (9)-(10) is already unitary even without taking the continuous-time limit.[12] The proof is an application of the identity[13]

$$f(r) = \int_\infty^\infty \frac{dp}{2\pi} \int_\infty^\infty ds\ e^{i(r-s)p} f(s),$$

which holds for all square-integrable[14] complex-valued functions $f(r)$ of a single real variable $r$. This identity is often abbreviated

$$\int_\infty^\infty \frac{dp}{2\pi}\ e^{i(r-s)p} = \delta(r-s), \tag{13}$$

where $\delta$ is the Dirac delta distribution.[13] If two state-vectors $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are represented by the functions $\Psi_1[\phi]$ and $\Psi_2[\phi]$, then equations (3) and (9) say that the inner product of $T|\Psi_1\rangle$ with $T|\Psi_2\rangle$ is

$$\langle\Psi_1|T^\dagger T|\Psi_2\rangle = \int [d\phi'][d\phi'']\ \Psi_1^*[\phi']\Omega[\phi',\phi'']\Psi_2[\phi''] \tag{14}$$

with

$$\Omega[\phi',\phi''] \equiv \int [d\phi]\ T^*[\phi,\phi']T[\phi,\phi''].$$

[12] More generally, in many other models, the transfer matrix is only invertible, not unitary, but time-evolution still becomes unitary in the continuous-time limit, and that's good enough.

[13] Article 58590

[14] **Square-integrable** means that $\int_\infty^\infty dr\ |f(r)|^2$ is finite.

Equations (8) and (10) give

$$\Omega[\phi', \phi''] = \mathcal{N}^2 \exp\left( i s_0[\phi''] - i s_0[\phi'] + \frac{i\,\epsilon^D}{2\,dt} \sum_{\mathbf{x}} \Big( \big(\phi''(\mathbf{x})\big)^2 - \big(\phi'(\mathbf{x})\big)^2 \Big) \right)$$
$$\times \int [d\phi] \;\; \exp\left( i\epsilon^D \sum_{\mathbf{x}} \phi(\mathbf{x}) \frac{\phi'(\mathbf{x}) - \phi''(\mathbf{x})}{dt} \right)$$

and then the identity (13) gives

$$\Omega[\phi', \phi''] = \prod_{\mathbf{x}} \delta\Big( \phi'(\mathbf{x}) - \phi''(\mathbf{x}) \Big)$$

if we choose

$$\mathcal{N} = \left( \frac{2\pi\, dt}{\epsilon^D} \right)^{K^D/2}.$$

Use this in (14) to get

$$\langle \Psi_1 | T^\dagger T | \Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle. \tag{15}$$

The result $\langle \Psi_1 | T T^\dagger | \Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle$ may be derived the same way, so this shows that the transfer matrix $T$ is a unitary operator on the Hilbert space, at least in this family of models.

# 7 Field operators and time-ordered products

Using the representation defined in section 3, a state $|\Psi\rangle$ can be represented by a function $\Psi[\phi]$. For each of the field variables $\phi(\mathbf{x})$ on which the function $\Psi[\phi]$ depends, we can define a corresponding operator $\phi_{\text{op}}(\mathbf{x})$ on the Hilbert space, called a **field operator**. The field operator $\phi_{\text{op}}(\mathbf{x})$ is defined by this condition:

$$\text{The state } \phi_{\text{op}}(\mathbf{x})|\Psi\rangle \text{ is represented by the function } \phi(\mathbf{x})\Psi[\phi]. \tag{16}$$

The time-dependent field operators are then defined by

$$\phi_{\text{op}}(t,\mathbf{x}) = U(-t)\phi_{\text{op}}(\mathbf{x})U(t) \tag{17}$$

where $U(t)$ are the unitary time-translation operators. The operators $\phi_{\text{op}}(\mathbf{x})$ all commute with each other, but the operators $\phi_{\text{op}}(t,\mathbf{x})$ don't. In particular, $\phi_{\text{op}}(t,\mathbf{x})$ does not commute with its time derivative $\dot{\phi}_{\text{op}}(t,\mathbf{x})$.[15]

In the path integral formulation, instead of using (17) directly, time-dependent field operators usually enter through quantities like

$$\langle\tilde{\Psi}|U(t_f - t_N)\phi_{\text{op}}(\mathbf{x}_N)\cdots\phi_{\text{op}}(\mathbf{x}_2)U(t_2 - t_1)\phi_{\text{op}}(\mathbf{x}_1)U(t_1 - t_i)|\Psi\rangle \tag{18}$$

with

$$t_f > t_N > \cdots t_2 > t_1 > t_i. \tag{19}$$

Thanks to the identity $U(t+s) = U(t)U(s)$, the quantity (18) may also be written

$$\langle\tilde{\Psi}|U(t_f)\phi_{\text{op}}(t_N,\mathbf{x}_N)\cdots\phi_{\text{op}}(t_2,\mathbf{x}_2)\phi_{\text{op}}(t_1,\mathbf{x}_1)U(-t_i)|\Psi\rangle. \tag{20}$$

This product of field operators is called a **time-ordered product** because of the condition (19). To express (18) in the path integral formulation, we just need to use equations (3), (12), and (16). In words: start with the state $|\Psi\rangle$ at the initial time $t_i$, use (12) to evolve it forward by the amount $t_1 - t_i$, then use (16) to apply the

[15] Section 12 uses the path integral formulation to derive the commutation relation.

operator $\phi_{\rm op}(\mathbf{x}_1)$, then use (12) to evolve it farther forward by the amount $t_2 - t_1$, then use (16) to apply the operator $\phi_{\rm op}(\mathbf{x}_2)$, and so on until reaching the final time $t_f$, and then use equation (3) to form the inner product of that final state with $|\tilde{\Psi}\rangle$. The result is that (18) is proportional to[16]

$$\int [d\phi]\ \tilde{\Psi}^*\big[\phi(t_f)\big] \exp\Big(iS[\phi]\Big) F[\phi] \Psi\big[\phi(t_i)\big]. \tag{21}$$

with

$$F[\phi] = \prod_{n=1}^{N} \phi(t_n, \mathbf{x}_n). \tag{22}$$

In this context, the time interval $\mathcal{I}$ in equation (7) is understood to be[17]

$$\mathcal{I} = [t_i, t_f - dt]$$

so that $S$ involves only field variables $\phi(x)$ in the time interval $[t_i, t_f]$ (equation (7)). The integral $\int [d\phi] \cdots$ is over all of these field variables.

More generally, applying an operator $\mathcal{O}(t, \mathbf{x})$ at time $t$ is represented by inserting an appropriate function of the field variables into the integrand of the path integral, so operators are often called **insertions**. In this context, operators are also sometimes called **defects**. Which word a given author uses may depend on the intertion's form or just on the author's preference.[18]

The order of the factors in the product (22) doesn't matter, because the $\phi(t_n, \mathbf{x}_n)$s are ordinary real variables, so they all commute with each other. This is consistent with the fact that the corresponding field *operators* don't all commute with each other, because the operators are automatically time-ordered by the path integral (21) as explained in the text above (21).

[16]The proportionality factor (not written here) comes from the one in equation (10). It depends only on the time-difference $t_f - t_i$, not on the states or the operators.

[17]The standard notation $[x, y]$ means the interval that starts with $x$, ends with $y$, and includes both $x$ and $y$.

[18]Page 10 in Moradi *et al* (2022), and page 21 in McGreevy (2022)

# 8 The equation of motion

The equation of motion for a classical scalar field $\phi(x)$ can be expressed in terms of the action $S[\phi]$, like this:[19]

$$\frac{\partial S[\phi]}{\partial \phi(x)} = 0. \tag{23}$$

In the path integral formulation, the corresponding equation of motion for the time dependent field operators $\phi_{\text{op}}(x) = \phi_{\text{op}}(t, \mathbf{x})$ is an easy consequence of the time-evolution equation (21). To derive it, use basic identity

$$\int_{-\infty}^{\infty} ds \, \frac{dg}{ds} = 0,$$

which holds for all square-integrable functions $g$, to get

$$\int [d\phi] \, \frac{\partial}{\partial \phi(x)} \tilde{\Psi}^*\big[\phi(t_f)\big] \exp\Big(iS[\phi]\Big) F[\phi] \Psi\big[\phi(t_i)\big] = 0. \tag{24}$$

This holds for any spacetime point $x$ in the interval $[t_i, t_f]$. Now suppose that $x$ is such that the functions $\Psi[\phi(t_i)]$, $\tilde{\Psi}[\phi(t_f)]$, and $F[\phi]$ don't involve the variable $\phi(x)$. Then $S[\phi]$ is the only factor in the integrand that depends on $\phi(x)$, so equation (24) implies

$$\int [d\phi] \, \tilde{\Psi}^*\big[\phi(t_f)\big] \exp\Big(iS[\phi]\Big) \frac{\partial S[\phi]}{\partial \phi(x)} F[\phi] \Psi\big[\phi(t_i)\big] = 0. \tag{25}$$

This is the quantum-field analog of the classical equation of motion (23), and it agrees with the equation of motion highlighted in article 52890.

[19] Article 49705

# 9 The Schwinger-Dyson equation

The derivation of the field's equation of motion in section 8 assumed that the factor $F[\phi]$ did not involve $\phi(x)$ at the point $x$ of interest. More generally, if $F$ does involve $\phi(x)$ but $\Psi$ and $\tilde{\Psi}$ still don't, then equation (24) implies

$$\int [d\phi]\ \tilde{\Psi}^*\big[\phi(t_f)\big] \exp\Big(iS[\phi]\Big) \left(i\frac{\partial S[\phi]}{\partial \phi(x)}F[\phi] + \frac{\partial F[\phi]}{\partial \phi(x)}\right) \Psi\big[\phi(t_i)\big] = 0. \tag{26}$$

This is called the **Schwinger-Dyson equation**. It reduces to the previous result (25) when $\partial F/\partial\phi(x) = 0$.

# 10 Derivation of the hamiltonian, part 1

In the continuous-time limit, the path integral formulation introduced in the preceding sections reproduces the hamiltonian formulation that was used in article 52890. This section derives the hamiltonian, and section 12 derives the equal-time commutation relations for the time dependent field operators.

According to equations (8), (9), and (10), evolving a state $\Psi_t$ through a single time increment of duration $dt$ gives the state

$$\Psi_{t+dt}[\phi] = \mathcal{N} \int [d\phi'] \; \exp\left( i\epsilon^D \sum_{\mathbf{x}} \frac{\big(\phi'(\mathbf{x}) - \phi(\mathbf{x})\big)^2}{2\,dt} \right) \exp\big(i s_0[\phi']\big) \Psi_t[\phi']. \tag{27}$$

Now use the identity[20]

$$\frac{e^{-i\pi/4}}{\sqrt{2\pi a}} \int d\phi' \; \exp\left( i \frac{(\phi' - \phi)^2}{2\,a} \right) f(\phi') = \exp\left[ i \frac{a}{2} \left( \frac{\partial}{\partial \phi} \right)^2 \right] f(\phi). \tag{28}$$

Using this identity for each of the integration variables $\phi'(\mathbf{x})$ in equation (27) gives

$$\Psi_{t+dt}[\phi] = e^{-i\pi K^D/4} \exp\left( i\,dt\,\epsilon^D \sum_{\mathbf{x}} \frac{1}{2} \left[ \frac{1}{\epsilon^D} \frac{\partial}{\partial \phi(\mathbf{x})} \right]^2 \right) \exp\big(i s_0[\phi]\big)\; \Psi_t[\phi].$$

In the continuous-time limit $dt \to 0$, we may expand the exponentials to first order in $dt$ to get

$$\Psi_{t+dt}[\phi] = \big(1 - i\,dt\,H\big) \Psi_t[\phi] + O\big((dt)^2\big) \tag{29}$$

with

$$H = \epsilon^D \sum_{\mathbf{x}} \left( -\frac{1}{2} \left( \frac{1}{\epsilon^D} \frac{\partial}{\partial \phi(\mathbf{x})} \right)^2 + \frac{\big(\nabla \phi(\mathbf{x})\big)^2}{2} + V\big(\phi(\mathbf{x})\big) \right) + \text{constant}. \tag{30}$$

This agrees with the hamiltonian that was used in article 52890.

[20] To prove this, take the the Fourier transform of both sides. The resulting **Fresnel integral** cancels the complex normalization factor on the left-hand side (Leonard (2007), question 8). The normalization factor shown here assumes that $a$ is real and positive.

# 11 Derivation of the hamiltonian, part 2

The previous section considered only a single time-step. In the continuous-time limit, the number of time-steps in any finite time interval becomes infinite. To finish the derivation that section 10 started, we need to take the number $M$ of time-steps to infinity with $M\,dt$ held fixed.

According to equations (9) and (29), the transfer matrix is

$$T = 1 - i\,dt\,H + O\big((dt)^2\big).$$

The transfer matrix implements evolution through a single time-step, so the operator that implements evolution through $M$ time-steps is

$$\begin{aligned} T^M &= \big(1 - i\,dt\,H + O\big((dt)^2\big)\big)^M \\ &= \left(1 - i\,\frac{\Delta t}{M}\,H + O\big((\Delta t/M)^2\big)\right)^M \end{aligned}$$

with $\Delta t \equiv M\,dt$. In the limit $M \to \infty$ with $\Delta t$ fixed, the right-hand side becomes

$$\lim_{M\to\infty} \left(1 - i\,\frac{\Delta t}{M}\,H + O\big((\Delta t/M)^2\big)\right)^M = e^{-iH\,\Delta t}. \tag{31}$$

To prove this, take the derivative of both sides with respect to $\Delta t$ and notice that the term of order $(\Delta t/M)^2$ in large parentheses does not contribute to the derivative in the limit $M \to \infty$. This shows that the operator (30) really is the generator of time-translations in the continuous-time limit. In other words, it really is the hamiltonian.

# 12 Derivation of the commutation relations

This section uses the path integral formulation to evaluate the commutators of the field operators $\phi_{\text{op}}(t, \mathbf{x})$ with their time derivatives $\dot{\phi}_{\text{op}}(t, \mathbf{x}')$, denoted

$$[\phi_{\text{op}}(t, \mathbf{x}),\, \dot{\phi}_{\text{op}}(t, \mathbf{x}')] \tag{32}$$

using the standard notation[21] $[A, B] \equiv AB - BA$. The path integral (21) automatically puts the field operators in time-order if the times are all distinct, so we can calculate (32) by writing it as

$$[\phi_{\text{op}}(t, \mathbf{x}),\, \dot{\phi}_{\text{op}}(t, \mathbf{x}')] = \lim_{\delta t \to 0} \big(\phi_{\text{op}}(t + \delta t, \mathbf{x})\dot{\phi}_{\text{op}}(t, \mathbf{x}') - \dot{\phi}_{\text{op}}(t, \mathbf{x}')\phi_{\text{op}}(t - \delta t, \mathbf{x})\big).$$

We're using a formulation in which time is discrete, so this really means

$$[\phi_{\text{op}}(t, \mathbf{x}),\, \dot{\phi}_{\text{op}}(t, \mathbf{x}')] \equiv \phi_{\text{op}}(t, \mathbf{x}) \left(\frac{\phi_{\text{op}}(t, \mathbf{x}') - \phi_{\text{op}}(t - dt, \mathbf{x}')}{dt}\right) - \left(\frac{\phi_{\text{op}}(t + dt, \mathbf{x}') - \phi_{\text{op}}(t, \mathbf{x}')}{dt}\right) \phi_{\text{op}}(t, \mathbf{x}). \tag{33}$$

The two quantities in large parentheses are two different ways of discretizing the time derivative $\dot{\phi}_{\text{op}}(t, \mathbf{x}')$, chosen so that time-ordering enforces the desired order of multiplication.

According to section 7, the result of sandwiching (33) between two states $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ at times $t - dt$ and $t + dt$, respectively, is

$$\big\langle \tilde{\Psi} \big| [\phi_{\text{op}}(t, \mathbf{x}),\, \dot{\phi}_{\text{op}}(t, \mathbf{x}')] \big| \Psi \big\rangle \propto \int [d\phi]\ \tilde{\Psi}^*\big[\phi(t + dt)\big] \exp\big(iS[\phi]\big) F[\phi] \Psi\big[\phi(t - dt)\big] \tag{34}$$

with

$$F[\phi] = \phi(t, \mathbf{x}) \left(\frac{\phi(t, \mathbf{x}') - \phi(t - dt, \mathbf{x}')}{dt}\right) - \left(\frac{\phi(t + dt, \mathbf{x}') - \phi(t, \mathbf{x}')}{dt}\right) \phi(t, \mathbf{x}).$$

[21] This should not be confused with the same-looking notation defined in section 7, footnote 17.

In equation (34), the interval $\mathcal{I}$ (equation (7)) consists of the two times $t$ and $t-dt$, and the path integral is over all of the field variables in the integrand. To keep the equations compact, use the abbreviations

$$\phi_\pm \equiv \phi(t \pm dt) \qquad\qquad \phi_0 \equiv \phi(t)$$

for the sets of field variables associated with the indicated times. According to equations (7)-(8), the factor $\exp\big(iS[\phi]\big)F[\phi]$ in the integrand may be written

$$\begin{aligned}\exp\big(iS[\phi]\big)F[\phi] &= \exp\big(is_1[\phi_+,\phi_0] + is_0[\phi_0] + is_1[\phi_0,\phi_-] + is_0[\phi_-]\big)F[\phi] \\ &= \exp\big(is_0[\phi_0] + is_0[\phi_-]\big)\phi(t,\mathbf{x}) \\ &\quad\times \frac{1}{i\epsilon^D}\frac{\partial}{\partial\phi(t,\mathbf{x}')}\exp\big(is_1[\phi_0,\phi_-] + is_1[\phi_+,\phi_0]\big).\end{aligned}$$

Since this is in the integrand of the path integral, we can integrate-by-parts to move the derivative to the other $\phi_0$-dependent factors, which gives

$$\begin{aligned}\exp\big(iS[\phi]\big)F[\phi] &\to -\exp\big(is_1[\phi_0,\phi_-] + is_1[\phi_+,\phi_0]\big) \\ &\quad\times \frac{1}{i\epsilon^D}\frac{\partial}{\partial\phi(t,\mathbf{x}')}\Big(\exp\big(is_0[\phi_0] + is_0[\phi_-]\big)\phi(t,\mathbf{x})\Big) \\ &= \exp\big(iS[\phi]\big) \times \frac{1}{i\epsilon^D}\left(\delta_{\mathbf{x},\mathbf{x}'} + \frac{\partial}{\partial\phi(t,\mathbf{x}')}is_0[\phi_0]\right).\end{aligned}$$

with

$$\delta_{\mathbf{x},\mathbf{x}'} \equiv \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{x}', \\ 0 & \text{otherwise.}\end{cases}$$

The quantity $s_0$ includes an overall factor of $dt$, which goes to zero in the continuous-time limit. Altogether, this gives

$$\Big\langle\tilde{\Psi}\Big|\big[\phi_{\rm op}(t,\mathbf{x}),\,\dot{\phi}_{\rm op}(t,\mathbf{x}')\big]\Big|\Psi\Big\rangle = \left(\frac{i}{\epsilon^D}\delta_{\mathbf{x},\mathbf{x}'} + O(dt)\right)\langle\tilde{\Psi}|\Psi\rangle, \tag{35}$$

which agrees with the equal-time commutation relation in article 52890.

# 13 Are time-ordered products sufficient?

The order of the factors in a product of field variables doesn't matter in the integrand of the path integral (equations (21)-(22)), but order of the factors in a product of field *operators* does matter (equations (18)-(19)). When the initial and final states are the vacuum state, so that $U(t)|0\rangle = |0\rangle$, the relationship can be expressed concisely like this:

$$\int [d\phi]\ \Psi^*_{|0\rangle}\big[\phi(t_f)\big] \exp\big(iS[\phi]\big) \left(\prod_{n=1}^{N} \phi(t_n, \mathbf{x}_n)\right) \Psi_{|0\rangle}\big[\phi(t_i)\big]$$

$$\propto \langle 0|\mathcal{T} \prod_{n=1}^{N} \phi_{\text{op}}(t_n, \mathbf{x}_n)|0\rangle \qquad (36)$$

The symbol $\mathcal{T}$ means "arrange the factors in chronological order from right to left," and $\phi_{\text{op}}$ is the field operator corresponding to the field variable $\phi$. The quantity (36) is called a **time-ordered correlation function**.

In the path integral formulation, representing correlation functions involving products that are not time-ordered is more awkward, but the time-slice principle[22] ensures that it can be done. Section 12 showed that commutation relations among operators at the "same time" are encoded in such time-ordered products, and article 52890 showed that the field operators at all times can be expressed entirely in terms of those at a single time (say $t = 0$), so we can rest assured that the path integral formulation is not missing anything.

Another general principle says that if the order of two events is observer-dependent (which is possible only if the events are separated by a spacelike interval), then observables associated with those two events must commute with each other[22] – so the order doesn't matter in that case.[23] This ensures that time-ordering is

[22] Article 21916

[23] Field operators aren't necessarily observables, but the time-slice principle implies that time-ordered products of field operators are sufficient for constructing observables.

consistent with the fact that time is observer-dependent.[24]

Time-ordering is also important when using projection operators to describe what we know about a system's history, which we must do (at least implicitly) in order to choose an appropriate state for making predictions about subsequence measurement outcomes.[25]

[24] "Time is observer-dependent" is a loose way of saying that proper duration is a property of a specific worldline connecting two events and not a property of just the pair of events (article 48968).

[25] Article 03431

# 14 Wick rotation and the vacuum state

One of the inputs to the path integral is the time step-size $dt$. The path integral remains well-defined if the real number $dt$ is replaced by a complex number:

$$dt = e^{-i\theta} \times (\text{real number}).$$

This is called **Wick rotation**. A quantity like (21) – regarded as a function of $dt$ – can be analytically continued from the real $dt$ axis to the complex $dt$ plane by rotating from $\theta = 0$ to $\theta > 0$.

Wick rotation is possible because the path integral's dependence on the value of $dt$ comes from only two places: in the definition of the "integral" over time (equation (5)), and in the definition of the "derivative" with respect to time (equation (6)). The $t$ in the arguments of the field variables $\phi(t, \mathbf{x})$ is really just an index: $t$ is an integer multiple of $dt$, and the integer itself would serve just as well as the index. For that reason, the value of $dt$ doesn't affect $\phi(t, \mathbf{x})$.

To see why Wick rotation can be useful, consider what happens to equation (29) when $\theta = \pi/2$:

$$dt = -i\, dt_E \qquad \text{with } dt_E \text{ real and positive.} \tag{37}$$

After making that substitution, the time-evolution operator (31) becomes

$$e^{-H\,\Delta t_E} \tag{38}$$

with $\Delta t_E$ defined by $\Delta t = -i\,\Delta t_E$. (Section 15 will explain the subscript $E$.) If $\Delta t_E$ is real and positive, then applying (38) to a typical[26] state suppresses the contribution of all higher-energy states relative to the contribution of the vacuum state. If we use (38) as the time-evolution operator in the ratio[27]

$$\frac{\langle\tilde{\Psi}|U(t_f - t_N)\phi_{\text{op}}(\mathbf{x}_N)\cdots\phi_{\text{op}}(\mathbf{x}_2)U(t_2 - t_1)\phi_{\text{op}}(\mathbf{x}_1)U(t_1 - t_i)|\Psi\rangle}{\langle\tilde{\Psi}|U(t_f - t_i)|\Psi\rangle} \tag{39}$$

[26]Here, *typical* means not exactly orthogonal to the vacuum state. State-vectors that are exactly orthogonal to the vacuum state-vector (or to any other given state-vector) are exceptional.

[27]The numerator is the time-ordered function (18), and the denominator makes the ratio independent of the normalizations of the initial and final states.

with typical[26] states $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$, then taking the limits

$$t_f \to \infty \qquad\qquad t_i \to -\infty \tag{40}$$

(with the other times $t_n$ held fixed) converts the ratio to

$$\frac{\langle 0|U(-t_N)\phi_{\rm op}(\mathbf{x}_N)\cdots\phi_{\rm op}(\mathbf{x}_2)U(t_2-t_1)\phi_{\rm op}(\mathbf{x}_1)U(t_1)|0\rangle}{\langle 0|0\rangle}, \tag{41}$$

where $|0\rangle$ is the vacuum state.[28] If we hadn't taken the limits (40), then making $\Delta t$ real-valued again would restore the original ratio (39). After taking the limits, though, only the vacuum state remains, and making $\Delta t$ real-valued again cannot undo that projection.

This works for any pair of states $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ that are not orthogonal to the vacuum state. Even better, we can eliminate the reference to any specific initial/final states by considering the ratio

$$\frac{\text{trace}\big(U(t_f-t_N)\phi_{\rm op}(\mathbf{x}_N)\cdots\phi_{\rm op}(\mathbf{x}_2)U(t_2-t_1)\phi_{\rm op}(\mathbf{x}_1)U(t_1-t_i)\big)}{\text{trace}\big(U(t_f-t_i)\big)} \tag{42}$$

insterad of (39). with

$$\text{trace}(\cdots) \equiv \sum_n \langle n|\cdots|n\rangle, \tag{43}$$

where the sum is over any orthonormal basis for the Hilbert space. The initial/final factors of (38) ensure that the trace is well-defined. As before, taking the limits (40) leaves (41). This gives us a way to construct vacuum expectation values of time-ordered products – quantities like (41) – without needing to know in advance which state-vector represents the vacuum state (the state with lowest energy). The next section explains how to implement this explicitly in the path integral formulation.

[28] For simplicity, this section assumes that the lowest-energy state is unique. If a symmetry is spontaneously broken, making the vacuum state non-unique, then we can add a small symmetry-breaking term to the hamiltonian before taking the limits (40) and then remove it after the limits have been taken. The result is (41) with whichever state $|0\rangle$ had the lowest energy when the symmetry-breaking term was present.

# 15 Wick rotation and the euclidean action

According to equations (8), the effect of the substitution (37) on the factor $e^{iS}$ in the path integral is

$$e^{iS}\big|_{dt=-i\,dt_E} = e^{-S_E} \tag{44}$$

with

$$S_E[\phi] = dt_E\,\epsilon^D \sum_x \left( \frac{\dot\phi_E^2(x) + \big(\nabla\phi(x)\big)^2}{2} + V\big(\phi(x)\big) \right), \tag{45}$$

where $\dot\phi_E$ is obtained from $\dot\phi$ by replacing the $dt$ in the denominator of (6) with $dt_E$. The functional $S_E[\phi]$ is called the **euclidean action**, because it is just like the original action (section 4) except that the signature of the metric[29] is euclidean instead of lorentzian. The subscript $E$ stands for *euclidean*.

The reasoning in section 14 referred to the hamiltonian. In particular, the substitution (37) was used only after expressing the time-ordered function (39) in terms of the hamiltonian. We can get the same result by using the substitution (37) first to get (44) and then repeating the steps that were used in section 10 to derive the hamiltonian, but now with $a = -i\,dt_E$ in the identity (28).[30] Even though $S_E \neq S$, the resulting hamiltonian is the same either way because the signs of $1/a$ and $a$ are equal when $a$ is real but are opposite when $a$ is imaginary.

To implement the trace (43) in the path integral formulation, use the identity

$$\sum_n \langle n|a\rangle\,\langle b|n\rangle = \langle b|a\rangle$$

where the sum is over any orthonormal basis for the Hilbert space. In the path integral formulation, if we write $a[\phi]$, $b[\phi]$, and $n[\phi]$ for the functions that represent

[29] The signature of the metric is implicit in the pattern of signs of the derivative terms: the derivative terms may be written $\sum_{a,b} \eta^{ab}(\partial_a\phi)(\partial_b\phi)$, where $\eta^{ab}$ are the components of the metric tensor (article 49705).

[30] The prefactor on the left-hand side of the identity (28) is ambiguous when $a$ is complex, but it can be defined so that the identity still holds.

the states $|a\rangle$, $|b\rangle$, and $|n\rangle$, then the preceding identity becomes

$$\sum_n \int [d\phi]\,[d\phi']\; n^*[\phi]a[\phi]b^*[\phi']n[\phi'] = \int [d\phi]\; b^*[\phi]a[\phi].$$

This holds for all normalizable functions $a$ and $b$, so it implies

$$\sum_n \int [d\phi]\,[d\phi']\; n^*[\phi]f[\phi,\phi']n[\phi'] = \int [d\phi]\; f[\phi,\phi]$$

for all functions $f$ for which the right-hand side is well-defined. In particular, this implies

$$\sum_n \int [d\phi]\; n^*\big[\phi(t_f)\big] \exp\Big(-S_E[\phi]\Big) F[\phi] n\big[\phi(t_i)\big] = \int [d\phi] \exp\Big(-S_E[\phi]^{\text{periodic}}\Big) F[\phi] \tag{46}$$

where the superscript *periodic* means that the field variables $\phi(t_f)$ and $\phi(t_i)$ have been identified with each other, as though time were periodic (wrapped back on itself). This shows that the trace in quantities like (42) can be evaluated by treating the time dimension as as periodic, just like the spatial dimensions are often treated.

This euclidean path-integral formulation makes quantum field theory in $d$-dimensional spacetime look like classical statistical mechanics in $d$-dimensional euclidean space, because the right-hand side of (46) can be viewed as the expectation value of a function $F$ of random variables $\phi(x)$ with distribution $\exp(-S_E[\phi]^{\text{periodic}})$, which is analogous to the Boltzmann distribution. Quantum field theory and classical statistical mechanics are not equivalent, of course, because a theory consists of more than just a mathematical formulation. It also consists of a relationship between ingredients of that formulation and features of the real world, and quantum field theory and classical statistical mechanics still differ from each other in that respect.

# 16 The generating functional

The euclidean version of a time-ordered correlation function (36) is the quantity

$$\int [d\phi] \ \exp\big(-S_E[\phi]^{\text{periodic}}\big) \left(\prod_{n=1}^{N} \phi(x_n)\right). \tag{47}$$

As explained in section 14, treating time as periodic on the right-hand side effectively selects the vacuum state. From now on, the superscript "periodic" will be omitted. Whenever a path integral is written without any initial/final states, time is understood to be periodic. More generally, (euclidean) spacetime is understood to become a closed manifold in the continuum limit.

Conceptually, Wick rotation converts the signature of the spacetime metric from lorentzian to euclidean (equation (45)), so Lorentz symmetry in $d$-dimensional spacetime becomes ordinary rotation symmetry in $d$-dimensional euclidean space. In this context, we might as well set $dt = \epsilon$, so that the lattice spacings in the "time" and "space" directions are equal to each other.

The **generating functional**

$$Z[J] = \int [d\phi] \ e^{-S_E[\phi,J]} \tag{48}$$

with

$$S_E[\phi, J] \equiv S_E[\phi] - \epsilon^d \sum_x \phi(x) J(x) \tag{49}$$

is a concise way to encode all of the quantities (47). The variable function $J(x)$ is called an **external source** or just a **source**, and correlation functions (47) are recovered from $Z[J]$ by taking derivatives with respect to the source:

$$\int [d\phi] \ \exp\big(-S_E[\phi]\big) \left(\prod_{n=1}^{N} \phi(x_n)\right) = \left(\prod_{n=1}^{N} \frac{1}{\epsilon^d} \frac{\partial}{\partial J(x_n)}\right) Z[J]\bigg|_{J=0}. \tag{50}$$

# 17 The Schwinger-Dyson equation again

The (euclidean version of the) Schwinger-Dyson equation derived in section 9 can be expressed (and derived) more concisely using the generating functional $Z[J]$. Start with the identity

$$\int [d\phi]\ \frac{1}{\epsilon^d}\,\frac{\partial}{\partial\phi(x)}\,e^{-S_E[\phi,J]} = 0,$$

and use (49) to write this as

$$\int [d\phi]\ e^{-S_E[\phi,J]}\left(\frac{1}{\epsilon^d}\,\frac{\partial S_E[\phi]}{\partial\phi(x)} - J(x)\right) = 0. \tag{51}$$

This one equation encodes the Schwinger-Dyson equations (26) for all polynomials $F[\phi]$ when the initial and final states are the vacuum state.

# 18 Another derivation of the commutation relations

Section 12 showed how the equal-time commutation relations can be derived from the path integral formulation. This section uses the generating functional $Z[J]$ to re-derive the same result.

This section uses the lorentzian version instead of the euclidean version, but to help reduce clutter, the initial and final states won't be written explicitly. For the same reason, integrals and derivatives with respect to spacetime coordinates will be written as though spacetime were continuous, with the understanding that these are really just abbreviations for the corresponding sums and finite differences. As before, a point in spacetime will be denoted $x = (t, \mathbf{x})$, where $t$ the time coordinate and $\mathbf{x}$ is the spatial coordinates.

With those abbreviations, the lorentzian version of the Schwinger-Dyson equation (51) is[31]

$$\int [d\phi]\; e^{iS[\phi,J]} \left( \frac{1}{\epsilon^d} \frac{\partial S[\phi]}{\partial \phi(x)} + J(x) \right) = 0 \tag{52}$$

with

$$S[\phi, J] = S[\phi] + \int d^d x\; \phi(x) J(x).$$

Write the action as

$$S[\phi] = \int dt\; L(t),$$

where $L(t)$ is constructed from the field variables $\phi$ and $\dot{\phi}$ at time $t$, and it includes an integral over the spatial coordinates $\mathbf{x}$. Then[32]

$$\frac{1}{\epsilon^d} \frac{\partial S[\phi]}{\partial \phi(t, \mathbf{x})} = \frac{1}{\epsilon^{d-1}} \frac{\partial L(t)}{\partial \phi(t, \mathbf{x})} - \dot{\pi}(t, \mathbf{x}) \tag{53}$$

[31] The different signs of the $J$ terms in equations (51) and (52) is a consequence of Wick rotation.

[32] Article 46044

with $\dot{\pi} \equiv d\pi/dt$ and

$$\pi(t, \mathbf{x}) \equiv \frac{1}{\epsilon^{d-1}} \frac{\partial L(t)}{\partial \dot{\phi}(t, \mathbf{x})}.$$

Use (53) in (52) to get

$$\int [d\phi]\ e^{iS[\phi,J]} \dot{\pi}(t, \mathbf{x}) = \int [d\phi]\ e^{iS[\phi,J]} \left( \frac{1}{\epsilon^{d-1}} \frac{\partial L(t)}{\partial \phi(t, \mathbf{x})} + J(t, \mathbf{x}) \right),$$

which implies

$$\int [d\phi]\ e^{iS[\phi,J]} \big( \pi(t, \mathbf{x}) - \pi(t_0, \mathbf{x}) \big)$$
$$= \int [d\phi]\ e^{iS[\phi,J]} \int_{t_0}^{t} dt\ \left( \frac{1}{\epsilon^{d-1}} \frac{\partial L(t)}{\partial \phi(t, \mathbf{x})} + J(t, \mathbf{x}) \right) \tag{54}$$

for any $t_0 < t$. As in equation (50), applying

$$\frac{-i}{\epsilon^d} \frac{\partial}{\partial J(t', \mathbf{x}')} \tag{55}$$

to either side of (54) is equivalent to inserting a factor of $\phi(t', \mathbf{x}')$ into the integrand, except for an extra term proportional to $\delta_{\mathbf{x},\mathbf{x}'}$ on the right-hand side in the case $t_0 \le t' \le t$. This extra term comes from the derivative (55) hitting the $J(t, \mathbf{x})$ in large parentheses on the right-hand side. This occurs only if $t'$ is within the interval from $t_0$ to $t$. It doesn't occur if $t' > t$. To exploit this, define $t^{\pm} \equiv t \pm \epsilon$, apply $\partial/\partial J(t^-, \mathbf{x}')$ to both sides of (54), and subtract that from the result of applying $\partial/\partial J(t^+, \mathbf{x}')$ to both sides of (54). That gives

$$\int [d\phi]\ e^{iS[\phi,J]} \pi(t, \mathbf{x}) \big( \phi(t^+, \mathbf{x}') - \phi(t^-, \mathbf{x}') \big) = \frac{i}{\epsilon^{d-1}} \delta_{\mathbf{x},\mathbf{x}'} \int [d\phi]\ e^{iS[\phi,J]} + O(\epsilon),$$

where $O(\epsilon)$ represents terms that vanish in the continuous-time limit. The path integral automatically enforces time-ordering of the operators that correspond to the factors in the integrand, so this implies the same equal-time commutation relation that was derived in section 12, namely equation (35).

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