Lifting Elements from a Lie Group to its Universal Covering Group

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Abstract A connected Lie group G has a universal covering group $\tilde{G}$. The covering homomorphism $\sigma : \tilde{G} \to G$ is typically not one-to-one. A map from G back into $\tilde{G}$ is called a **lift** if its composition with σ is the identity map on G. If σ is not one-to-one, then a lift cannot be continuous everywhere, but any continuous path γ in G can be lifted to a continuous path $\tilde{\gamma}$ in $\tilde{G}$ with $\sigma(\tilde{\gamma}) = \gamma$. This article introduces a special lift and applies it to the idea of lifting a practically continuous path (sequence of closely-spaced points) in G to a practically continuous path in $\tilde{G}$. Article 40191 uses this to define Wilson operators associated with representations of the covering group $\tilde{G}$ of the gauged group G.

Contents

1 Introduction

Let G be a compact connected Lie group $\tilde{G}$ its universal covering group,[1] and let K be the kernel of the covering map $\sigma : \tilde{G} \to G$. Then K is a discrete subgroup of the center $Z(\tilde{G})$ of the covering group $\tilde{G}$.[2] Examples:[3]

- If $G = U(1)$, then $\tilde{G} \simeq \mathbb{R}$ and $K \simeq \mathbb{Z}$, where $\mathbb{Z}$ is the additive group of integers.
- If $G = SU(n)/Z(SU(n))$, then $\tilde{G} = SU(n)$ and $K \simeq \mathbb{Z}_n$, the additive group of integers modulo n.

A map from G back into $\tilde{G}$ is called a **lift** if its composition with σ is the identity map on G. If K has more than one element, then a lift cannot be continuous everywhere in G, but any continuous path γ in G can be lifted to a continuous path $\tilde{\gamma}$ in $\tilde{G}$.

Quantum models with gauge fields have gauge-invariant operators called *Wilson operators* that are constructed using representations r of the gauged group G or of its covering group $\tilde{G}$.[4] The only known mathematically legitimate constructions of many of these models treat spacetime as discrete, and when r is a representation of $\tilde{G}$ but not of G, the construction of the corresponding Wilson operator involves lifting a finely-discretized path in G to a finely-discretized path in $\tilde{G}$. This article explains how to define such lifts and clarifies how closely spaced the points along the path must be.

[1] Every connected Lie group has a universal covering group (article 92035).

[2] The kernel of a covering map is discrete (article 61813), and for Lie groups the kernel is a subgroup of the center (Mimura and Toda (1991), theorem 4.8 on page 71).

[3] Article 92035 reviews examples and properties compact Lie groups and their universal covering groups.

[4] Article 40191

2 Notation and conventions

- If G and H are Lie groups, then $G \simeq H$ means they're **isomorphic** to each other as Lie groups.[5]
- $Z(G)$ is the **center** of a Lie group G.[5]
- 1_G is the identity element of G.
- $[g]$ is the **conjugacy class** of an element $g \in sg$, defined as the set of all elements of the form $h^{-1}gh$ with $h \in G$.
- $\text{dist}(g, g')$ is the *distance* between two elements $g, g' \in G$ (section 4).
- $\text{trace}_r(g)$ is the trace of the matrix $r(g)$ that represents an element $g \in G$ is the representation r.
- $\tilde{G}$ is the **universal covering group**[5] of a Lie group G.
- $\sigma : \tilde{G} \to G$ is the covering homomorphism, abbreviated **covering map**.[6]
- K is the **kernel** of the covering map $\tilde{G} \to G$. This is the subgroup of $\tilde{G}$ that is mapped to the identity element of G.
- G_1 is a special open subset of G (section 8).
- This article uses multiplicative notation and terminology: if $g, h \in G$, then their composition is denoted $gh \in G$ and called the *product*.[7]

For the rest of this article, G is a compact connected Lie group.

[5] Article 92035 explains what this means.

[6] Article 61813 defines *covering map* more generally, and article 92035 defines **homomorphism** for Lie groups.

[7] When $G = U(1)$, elements of the universal covering group $\tilde{G}$ are traditionally treated as real numbers composed using addition (instead of multiplication), but that's just a difference in notation and terminology. The definitions and results in this article still apply to that case except where stated otherwise.

3 Bi-invariant metrics on Lie groups

A smooth manifold is not automatically equipped with any concept of geometry (lengths and angles), but we can define lengths and angles by specifing a **riemannian metric** (abbreviated **metric** from now on)[8] on the manifold, then we can use it to define length of any curve connecting two points in the manifold. A Lie group is both a group and a smooth manifold, and points in this smooth manifold are elements of the Lie group. The metric is said to be **invariant under left** (respectively **right**) **translations** if, for each curve γ and each element g of the group, the length of γ does not change when each of its points is multiplied on the left (respectively right) by g. The metric is called **bi-invariant** if it is invariant under both left and right translations.[9]

A connected Lie group admits a bi-invariant metric if and only if it is isomorphic to the cartesian product of a compact group and a commutative group.[10,11] If G is a compact connected Lie group, then its universal covering group $\tilde{G}$ is isomorphic to the cartesian product of a compact group and a commutative group,[12] so $\tilde{G}$ admits a bi-invariant metric. If $\tilde{G}$ happens to be a simple Lie group,[13] then the bi-invariant metric is unique up to normalization (up to an overall positive constant factor).[14,15]

[8] Article 21808 introduces the concept of a riemannian metric in a context where the smooth manifold is a spacelike submanifold of spacetime. Here, the smooth manifold will be an abstract Lie group, not related to spacetime.

[9] Milnor (1976), definition at beginning of section 7; Gallier and Quaintance (2025), definition 21.1

[10] Milnor (1976), lemma after corollary 1.3, repeated in lemma 7.5; Gallier and Quaintance (2025), theorem 21.9

[11] In particular, every compact Lie group admits a bi-invariant metric (Milnor (1976), corollary 1.4; Gallier and Quaintance (2025), proposition 21.6)

[12] This can be inferred from information in article 92035.

[13] A Lie group is called **simple** whenever its Lie algebra is simple (Salamon (2022), text above theorem 11.1). Article 92035 restates the definition without referring to the Lie algebra.

[14] Milnor (1976), lemma 7.6; Gallier and Quaintance (2025), proposition 21.22

[15] This metric has constant Ricci curvature (Gallier and Quaintance (2025), proposition 21.22).

4 A distance function on the covering group

Let $\tilde{G}$ be the universal covering group of a compact connected Lie group G, and suppose that a bi-invariant metric on $\tilde{G}$ is given.[16] Using that metric, define the **distance** $\text{dist}(a, b)$ between any two elements $a, b \in \tilde{G}$ to be the minimum length among all curves from a to b.[17] We can build some intuition about this distance function using these facts:

- Every minimum-length curve between two points is a **geodesic**.
- When the metric is bi-invariant, the geodesics are (segments of) the 1-parameter subgroups.[18] A **1-parameter subgroup** is the image of a smooth homomorphism from $\mathbb{R}$ into $\tilde{G}$, so it is isomorphic to $U(1)$ or $\mathbb{R}$.[19,20]

Write $1_{\tilde{G}}$ for the identity element of $\tilde{G}$. Properties of the distance function include:[21]

- $\text{dist}(1_{\tilde{G}},\, a) = \text{dist}(1_{\tilde{G}},\, b^{-1}ab)$ for all $a, b \in \tilde{G}$. In words: distance from the identity element is invariant under conjugation.
- $\text{dist}(a,\, b) = \text{dist}(1_{\tilde{G}},\, a^{-1}b)$ for all $a, b \in \tilde{G}$. In words: the distance between a and b is same as distance from the identity element to $a^{-1}b$.
- $\text{dist}(a,\, b) = \text{dist}(a^{-1},\, b^{-1})$ for all $a, b \in \tilde{G}$. In words: the distance is invariant under inversion.

[16] The concepts in this article don't depend on which bi-invariant metric we use. If $\tilde{G}$ is simple, then the metric is unique up to normalization (section 3). More generally, $\tilde{G}$ is a cartesian product of simple groups and 1-dimensional groups (section 3), so we could (if desired) specify a bi-invariant metric up to normalization by requiring its restriction to each simple or 1-dimensional factor to give the same the minimum nonzero distance between elements of K, where K is the kernel of the covering map $\tilde{G} \to G$.

[17] Remember that these are curves in the smooth manifold $\tilde{G}$, not in spacetime.

[18] Milnor (1963), lemma 21.2; Gallier and Quaintance (2025), proposition 21.21, and the text near the beginning of chapter 21

[19] Milnor (1963), text before lemma 21.2

[20] Even a compact group may have a noncompact 1-parameter subgroup, but it's not a closed subgroup (article 92035). Example: think of an of the compact group $U(1) \times U(1)$ as a pair $(e^{i\theta_1}, e^{i\theta_2})$ of complex numbers and consider the subgroup $(e^{ic_1\theta}, e^{ic_2\theta})$ where c_1/c_2 is irrational.

[21] These are easy consequences of bi-invariance.

5 The concept of a lift

Define G, $\tilde{G}$, σ, and K as in section 1. The Wilson operators introduced article 40191 are defined by inserting a complex-valued function of the G-valued link variables into the integrand of the path integral, but the function is defined using a representation of $\tilde{G}$. For this to make sense, we need a way to convert a function $\tilde{f}$ of a $\tilde{G}$-valued variable to a function f of a G-valued variable so it makes sense in the integrand of a path integral with gauged group G.

We can do that by choosing a map $\tilde{\sigma} : G \to \tilde{G}$ that satisfies

$$\sigma(\tilde{\sigma}(g)) = g \qquad \text{for all } g \in G. \tag{1}$$

Such a map $\tilde{\sigma}$ will be called a **lift** from G to $\tilde{G}$. For any lift $\tilde{\sigma}$ and any pair of elements $a, b \in G$, the condition (1) implies

$$\tilde{\sigma}(ab) = \kappa\tilde{\sigma}(a)\tilde{\sigma}(b) \tag{2}$$

for some $\kappa \in K$ that may depend on a and b. This says that a lift is almost a homomorphism: it's a homomorphism modulo elements of the kernel K of the covering map. Given a a lift $\tilde{\sigma} : G \to \tilde{G}$ and a function $\tilde{f} : \tilde{G} \to \mathbb{C}$, we can compose them to get a function $f : G \to \mathbb{C}$:

$$f(g) \equiv \tilde{f}\big(\tilde{\sigma}(g)\big) \qquad \text{for all } g \in G. \tag{3}$$

If the function $\tilde{f}$ happens to satisfy the condition

$$\tilde{f}(\tilde{g}\kappa) = \tilde{f}(\tilde{g}) \qquad \text{for all } \kappa \in K, \tag{4}$$

then we can define a corresponding function f of a G-valued variable without the help of a lift:

$$f(g) \equiv \tilde{f}(\tilde{g}) \qquad \text{whenever } g \equiv \sigma(\tilde{g}). \tag{5}$$

The shift-invariance condition (4) makes this definition unambiguous, because $\sigma(\tilde{g}\kappa) = \sigma(\tilde{g})$. In general, though, then the definition (5) is ambiguous: the value of $\tilde{f}(\tilde{g}\kappa)$ may depend on $\kappa \in K$ even though $\sigma(\tilde{g})$ does not. In that situation, we must choose a lift and use the definition (3) instead.

6 Example: $G = SO(3)$

This example uses a specific function $\tilde{f}$ to show that the function f defined by (3) depends on which lift $\tilde{\sigma}$ we use.

Suppose $G = SO(3)$, which implies $\tilde{G} \simeq SU(2)$ and $K \simeq \mathbb{Z}_2$. Define the trace of $\tilde{g} \in SU(2)$ by treating $SU(2)$ as a unitary matrix group, so $\tilde{g}$ is a 2×2 unitary matrix. The covering map $\sigma : SU(2) \to SO(3)$ is defined by treating each matrix in $SU(2)$ as equivalent to its negative. If the function $\tilde{f}$ is

$$\tilde{f}(\tilde{g}) = \text{trace}(\tilde{g}) \text{ for } \tilde{g} \in \tilde{G},$$

then the definition (3) gives

$$f(g) = \text{trace}\big(\tilde{\sigma}(g)\big) \text{ for } g \in G. \tag{6}$$

Given any map $\tilde{\sigma} : SO(3) \to SU(2)$ that satisfies (1), we can get another one by multiplying $\tilde{\sigma}$ by the non-identity element of K, but this changes the overall sign of (6). This shows that the function f defined by (3) depends on $\tilde{\sigma}$.

7 Example: $G = U(1)$

This section illustrates the fact that if $\tilde{G} \neq G$, then a lift from G to the covering group $\tilde{G}$ cannot be a homomorphism and cannot be continuous.

Suppose G is the group $U(1)$ whose elements have the form $e^{i\theta}$ with $\theta \in \mathbb{R} \simeq \tilde{G}$. (In this section, the group operation in $\tilde{G}$ will be written as addition.) The covering map is $\sigma(\theta) = e^{i\theta}$. A lift is any map $\tilde{\sigma} : U(1) \to \mathbb{R}$ for which[22] $\sigma\big(\tilde{\sigma}(e^{i\theta})\big) = e^{i\theta}$. Any lift $\tilde{\sigma}$ has two undesirable features:

- It cannot be a homomorphism. To deduce this, suppose it's a homomorphism in a neighborhood of the identity element of $U(1)$. Then its image generates the whole group $\mathbb{R}$ but does not include the whole group $\mathbb{R}$, so it cannot be a homomorphism everywhere.
- It cannot be continuous. To deduce this, use the fact that a continuous function from a circle to the real line cannot be injective, but the condition $\sigma\big(\tilde{\sigma}(e^{i\theta})\big) = e^{i\theta}$ implies that $\tilde{\sigma}$ is injective.

Given a a lift $\tilde{\sigma} : U(1) \to \mathbb{R}$ and a function $\tilde{f} : \mathbb{R} \to \mathbb{C}$, we can compose them to get a function $f : U(1) \to \mathbb{C}$:

$$f(e^{i\theta}) \equiv \tilde{f}\big(\tilde{\sigma}(\theta)\big). \tag{7}$$

If the function $\tilde{f}$ happens to have period 2π, which means[23] $\tilde{f}(\theta + 2\pi) = \tilde{f}(\theta)$, then equation (7) reduces to $f(e^{i\theta}) \equiv \tilde{f}(\theta)$, which doesn't depend on the lift. If $\tilde{f}$ doesn't have period 2π, then the definition (7) makes f discontinuous somewhere even if $\tilde{f}$ is continuous everywhere, because the lift $\tilde{\sigma}$ is discontinuous somewhere.

[22] This is equation (1).
[23] This is equation (4).

8 The canonical lift

Section 4 defined a distance function on the covering group $\tilde{G}$. Refer to an element $g \in G$ as being **on the fence** if $g = \sigma(\tilde{g})$ for some $\tilde{g}$ that is the same distance from two (or more) distinct elements of K. The subset of G consisting of elements that are *not* on the fence will be denoted G_1.[24] Sections 9-12 will use examples to help build some intuition about the subset $G_1 \subset G$, but its most important properties are already evident:[25]

- G_1 includes a neighborhood of the identity element.
- If $g \in G_1$, then $g^{-1} \in G_1$.
- If $g \in G_1$, then $h^{-1}gh \in G_1$ for all $h \in G$.

In words, the last two properties say that G_1 is self-contained under inverses and under conjugation.

For each $g \in G_1$, define its **canonical lift** $\tilde{\sigma}_1(g)$ to be the unique element of $\tilde{G}$ that satisfies the condition (1) and whose distance from the identity element $1_{\tilde{G}}$ is less than its distance from any other element of K. We could use arbitrary choices to extend the definition of $\tilde{\sigma}_1$ to elements on the fence, but that won't be necessary because $\tilde{\sigma}_1$ is only meant to be applied to elements that are close to the identity element. Sections 16-17 will explain why this is enough to be useful.

[24]Mnemonic: the subscript 1 alludes to the identity element. (G_1 is defined using proximity to the identity element.)

[25]The first property is obvious because K is discrete. The second and third properties follow from the corresponding properties of the distance function (section 4).

9 Examples of elements on the fence

Here are a few examples of elements on the fence:

- Represent elements of the group $G = U(1)$ as complex numbers $e^{i\theta}$ with $\theta \in \mathbb{R} = \tilde{G}$. The kernel K consists of integer multiples of 2π in $\mathbb{R}$, and the element of $U(1)$ with $\theta = \pi$ is on the fence: the distance from π to $1_{\tilde{G}} = 1_{\mathbb{R}} = 0$ is the same as the distance from π to 2π.

- Let $G = SO(n)$ be the group of rotations about the origin in n-dimensional euclidean space. Its universal covering group is $\tilde{G} = \text{Spin}(n)$. Any element $g \in SO(n)$ whose rotation angle is π is on the fence. To deduce this, use $g^2 = 1_G$. The covering map $\sigma : \tilde{G} \to G$ is 2-to-1, so the preimage of the two-element subgroup $\{1_G, g\}$ of G is a 4-element cyclic group $\{1_{\tilde{G}}, \tilde{g}, \tilde{g}^2, \tilde{g}^3\}$ in $\tilde{G}$ with $\sigma(\tilde{g}) = g$. This shows that $\tilde{g}^2$ is the non-identity element of the kernel of σ. The left- or right-invariance of the distance function implies $\text{dist}(1_{\tilde{G}}, \tilde{g}) = \text{dist}(\tilde{g}, \tilde{g}^2)$, so g is on the fence.

- Take $\tilde{G} = SU(3)$, represented as the group of 3×3 unitary matrices with determinant equal to 1, and let G be the quotient of $SU(3)$ by its center. Let $z \neq 1$ be a complex number with $z^3 = 1$, and define $\tilde{g} \equiv \text{diag}(1, z, z^2)$. This is an element of $SU(3)$, and its image under the covering map is on the fence. To deduce this, define $\kappa \equiv \text{diag}(z, z, z) \in K$ and use the properties of the distance function that were listed at the end of section 4 combined with the fact that $\tilde{g}$ and $\kappa\tilde{g} = \text{diag}(z, z^2, 1)$ are conjugate to each other to get

$$\text{dist}(1_{\tilde{G}}, \tilde{g}) = \text{dist}(\kappa, \kappa\tilde{g}) = \text{dist}(\kappa, \tilde{g}).$$

10 The identity sheet

Let $\tilde{\sigma}_1$ be the canonical lift defined in section 8. The image $\tilde{\sigma}_1(G_1) \subset \tilde{G}$ will be called the **identity sheet**.[26] Example: if $G = U(1)$ is parameterized as in section 9, then the identity sheet in $\tilde{G} = \mathbb{R}$ is the open interval $-\pi < \theta < \pi$.

For any G, the identity sheet is connected. To deduce this, use the fact that each $g \in G_1$ belongs to some 1-parameter subgroup, and the shortest route from 1_G to g within that subgroup is all in G_1, so the lifted route is all in $\tilde{\sigma}_1(G_1)$. This shows that both G_1 and $\tilde{\sigma}_1(G_1)$ are connected.

The covering space $\tilde{G}$ may be described as the union of $|K|$ copies of (the manifold isometric to) the identity sheet, one centered on each element of the kernel K, separated from each other by lower-dimensional "fences." This is clear in the 1-dimensional case $G = U(1)$, and sections 11-12 will describe other examples. The fact that the fences are lower-dimensional (isolated points in the 1-dimensional case) follows from the fact that within any given 1-parameter subgroup of G, any on-the-fence element of that subgroup is disconnected from all other on-the-fence elements of that subgroup. This, in turn, follows from the definition of the distance function and the fact that 1-parameter subgroups of $\tilde{G}$ are geodesics.

[26] This is consistent with the usual meaning of *sheet* in the context of covering spaces (article 61813).

11 Example of an identity sheet: $G = SO(4)$

Think of $G = SO(4)$ as the group of rotations about the origin in 4-dimensional euclidean space. Elements on the fence are not all the same distance from 1_G in G. To deduce this, consider two 1-parameter subgroups, both parameterized by an angle θ: one consists of rotations through θ in a single plane, and one consists of the composition of rotations through θ in two planes that are totally orthogonal to each other (they intersect only at the origin). In both cases, elements with $\theta = \pi$ are on the fence. Both return to 1_G when the angle reaches 2π, but the second path is longer.

The covering group is $\tilde{G} = \text{Spin}(4)$. To describe elements of $\tilde{G}$, let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be an orthonormal basis for a Clifford algebra, so $(\gamma_j\gamma_k)^2 = -1$ whenever $j \neq k$.[27,28] Spin(4) is generated by elements of the form

$$e^{\gamma_j\gamma_k\phi} = \cos\phi + \gamma_j\gamma_k \sin\phi \qquad\qquad \text{with } j \neq k.$$

A vector $\mathbf{v}$ in 4-dimensional euclidean space may be represented as a linear combination of the basis vectors γ_j, and the linear transformation

$$\mathbf{v} \to e^{\gamma_j\gamma_k\phi}\mathbf{v}e^{-\gamma_j\gamma_k\phi} \tag{8}$$

is a rotation about the origin, which is an element of $SO(4)$. The rotation angle is 2ϕ,[29] so the covering map $\tilde{G} \to G$ is 2-to-1. The kernel of the covering map is $K = \{1, -1\}$.[30]

To build some intuition about the identity sheet in this case, consider any maximal torus in $\tilde{G}$, say the one consisting of elements of the form

$$\tilde{g}(\phi_{12}, \phi_{34}) \equiv \exp(\gamma_1\gamma_2\phi_{12}) \exp(\gamma_3\gamma_4\phi_{34}).$$

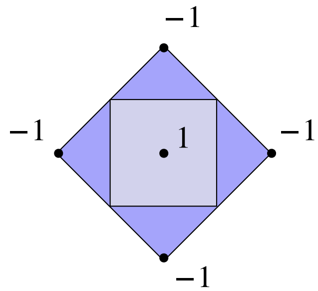
[27] Article 08264

[28] This section uses the symbol γ differently than the rest of this article.

[29] To deduce this, suppose $\mathbf{v}$ is in the j-k plane so it anticommutes with $\gamma_j\gamma_k$.

[30] Replacing $\phi \to \phi + \pi$ is equivalent to replacing $e^{\gamma_j\gamma_k\phi} \to -e^{\gamma_j\gamma_k\phi}$. The corresponding element of $SO(4)$ is unaffected by this replacement because the signs cancel in (8).

This is a 2-dimensional torus parameterized by the two angles ϕ_{12} and ϕ_{34}. The torus is depicted here as a diamond whose opposite sides are identified with each other:



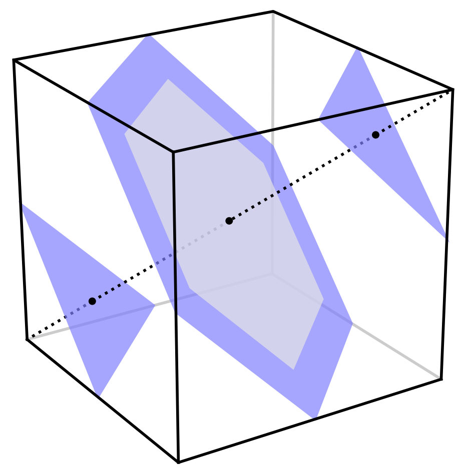
The dot at the center is the element $1_{\tilde{G}} = 1$, and the dot at the outer corners (all four of which are identified with each other) is the element -1. Elements on the fence form the boundary of the inscribed square. The interior of the inscribed square (lighter shade) is the intersection of the maximal torus with the identity sheet. The torus is covered by the interiors of two squares, one centered on 1 (lighter shade) and one centered on -1 (darker shade), together with the fence between them.

The 1-parameter subgroups $\tilde{g}(0,\phi)$ and $\tilde{g}(\phi,0)$ correspond to horizontal and vertical lines (not shown) through the center of the picture. In both cases, setting $\phi = \pi$ gives the element $-1 \in K$. The elements $\tilde{g}(0,\pi/2)$ and $\tilde{g}(\pi/2,0)$ are both on the fence: they are halfway between 1 and -1.

The 1-parameter subgroups $\tilde{g}(\phi,\phi)$ and $\tilde{g}(\phi,-\phi)$ correspond to diagonal lines (not shown) through the center of the picture. These subgroups don't include $-1 \in K$. Each of them returns to 1 at $\phi = \pi$, and its maximum distance from 1 occurs at $\phi = \pi/2$. Bi-invariance implies that $-\tilde{g}(\phi,\phi)$ is also a geodesic, even though it's not a subgroup. The geodesics $\tilde{g}(\phi,-\phi)$ and $-\tilde{g}(\phi,\phi)$ intersect each other at $\tilde{g}(\pi/2,-\pi/2) = -\tilde{g}(\pi/2,\pi/2)$, which shows that this point is equidistant from both 1 and -1, so this point is on the fence. In the picture, it's the upper-left corner of the inscribed square, which is identified with the lower-right corner because opposite sides of the diamond are identified with each other.

12 Example of an identity sheet: $G = SU(3)/\mathbb{Z}_3$

Consider the case $\tilde{G} = SU(3)$ and $G = \tilde{G}/Z(\tilde{G})$. The group $SU(3)$ is a subgroup of $U(3)$. Think of an element of $U(3)$ as a unitary 3×3 matrix. Such a matrix belongs to $SU(3)$ if and only if its determinant is 1. To build some intuition about the identity sheet, consider any maximal torus in $U(3)$, which is a 3-torus. The intersection of that 3-torus with $SU(3)$ is a maximal torus for $SU(3)$, which is a 2-torus. The relationship between these two tori is depicted here:



The cube contains all the points $\text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$, with $-\pi \leq \phi_k \leq \pi$ for each k. Each of those points represents an element of the diagonal subgroup of $U(3)$, which is a maximal torus for $U(3)$. Opposite faces of the cube are identified with each other, giving a 3-torus. The diagonal dashed line is $\phi_1 = \phi_2 = \phi_3$. After identifying opposite faces of the cube with each other, the shaded areas fit together to form the 2-torus $\phi_1 + \phi_2 + \phi_3 = 0$. Each point on that 2-torus is an element of the diagonal subgroup of $SU(3)$, which is a maximal torus for $SU(3)$. The three dots are the three elements of the center of $SU(3)$. The 2-torus is covered by the interiors of three hexagons, each one centered on a different element of the center, together with the fences between them. The central hexagon (lighter shade) is the intersection of the maximal torus with the identity sheet $\tilde{\sigma}_1(G_1)$.

13 A conjugation property

Section 4 mentioned that for any $\tilde{g} \in \tilde{G}$, all conjugates $\tilde{h}^{-1}\tilde{g}\tilde{h}$ of $\tilde{g}$ are the same distance from the identity element $1_{\tilde{G}}$. This section uses that property of the canonical lift $\tilde{\sigma}_1$ to deduce

$$\tilde{\sigma}_1(h^{-1}gh) = \tilde{\sigma}_1(h^{-1})\tilde{\sigma}_1(g)\tilde{\sigma}_1(h) \qquad \text{for all } h \in G \text{ and } g \in G_1. \tag{9}$$

To derive (9), start by using equation (2) to get

$$\tilde{\sigma}_1(h^{-1}gh) = \kappa(h)\tilde{\sigma}_1(h^{-1})\tilde{\sigma}_1(g)\tilde{\sigma}_1(h) \tag{10}$$

for some $\kappa(h) \in K$. The goal is to show that $\kappa(h)$ must be the identity element. Let r be any faithful representation of $\tilde{G}$, and consider the function

$$f(h) \equiv \text{trace}_r\big(\tilde{\sigma}_1(h^{-1}gh)\big) \tag{11}$$

for any given $g \in G_1$. Equation (10) gives

$$f(h) = \text{trace}_r\big(\kappa(h)\tilde{\sigma}_1(h^{-1})\tilde{\sigma}_1(g)\tilde{\sigma}_1(h)\big),$$

and the fact that $\kappa(h) \in K$ commutes with everything gives

$$f(h) = \text{trace}_r\big(\kappa(h)\tilde{\sigma}_1(g)\big). \tag{12}$$

The set of allowed values of $\kappa(h)$ is discrete, but the right side of (11) is a continuous function of h, so (12) implies that $f(h)$ must be independent of h. The premise that the representation r is faithful then shows that $\kappa(h)$ must be the identity element. Using this in (10) gives the result (9).

14 Lifting paths and homotopies of paths

A continuous path in G may be described as a continuous function γ from the closed interval $[0,1] \subset \mathbb{R}$ into G. The covering map $\sigma : \tilde{G} \to G$ has this property:[31] for any such path γ in G and any point $\tilde{\gamma}(0) \in \tilde{G}$ with $\sigma\big(\tilde{\gamma}(0)\big) = \gamma(0)$, a unique continuous path $\tilde{\gamma}$ exists in $\tilde{G}$ such that $\sigma\big(\tilde{\gamma}(t)\big) = \gamma(t)$ for all $t \in [0,1]$. This is called the **(unique) path lifting property** of the covering map.

A lift from G to $\tilde{G}$ can't be continuous everywhere on G, but a continuous path in G can always be lifted to a continuous path in $\tilde{G}$. That's not a contradiction, because the result of lifting a path depends on more than just the path's endpoint: it depends on the whole path. If a continuous path γ in G starts at 1_G and ends at an arbitrary element $g \in G$, then it has a unique lift to a continuous path $\tilde{\gamma}$ in $\tilde{G}$ that starts at $1_{\tilde{G}}$. Where it ends in $\tilde{G}$ depends on the path in G. If the path in G were not specified, then the endpoint in $\tilde{G}$ would be defined only modulo elements of the kernel K of the covering map, but specifying the path resolves this ambiguity.

Now let γ and γ' be two continuous paths $[0,1] \to G$. A **homotopy** from γ to γ' is a continuous map $H : [0,1] \times [0,1] \to G$ with $H(t,0) = \gamma(t)$ and $H(t,1) = \gamma'(t)$. Intuitively, the homotopy continuously morphs one path into the other path. A covering map $\sigma : \tilde{G} \to G$ has the **homotopy lifting property**:[32] given the homotopy H and a lift $\tilde{\gamma}$ of the path γ, a unique homotopy $\tilde{H} : [0,1] \times [0,1] \to \tilde{G}$ exists with $\tilde{H}(t,0) = \tilde{\gamma}(t)$. Intuitively, a continuous morph from one path to another in G lifts to a continuous morph from one path to another in $\tilde{G}$, all uniquely determined by the lift of the first point in the initial path.

[31] May (2007), chapter 3, section 2; Hatcher (2001), text after proposition 1.30

[32] Hatcher (2001), proposition 1.30

15 Small increments

Choose a small positive real number $0 < \varepsilon \ll 1$ and let $G_\varepsilon \subset G$ be the set of elements whose distance from 1_G is less than ε. The radius ε should be small enough to ensure that $G_\varepsilon \subset G_1$, with G_1 defined as in section 8. Like G_1, the neighborhood G_ε has these properties:

- If $g \in G_\varepsilon$, then $g^{-1} \in G_\varepsilon$.
- If $g \in G_\varepsilon$, then $h^{-1}gh \in G_\varepsilon$ for all $h \in G$.

Elements of G_ε are close to 1_G, so for any $h' \in G$ and $g \in G_\varepsilon$, we can reasonably write $h'g \approx h'$. The identity

$$h'gh = (h'h)(h^{-1}gh)$$

implies that we should also write

$$h'gh \approx h'h \qquad\qquad \text{whenever } g \in G_\varepsilon, \tag{13}$$

because $h^{-1}gh \in G_\varepsilon$. The relationship "$\approx$" is not transitive because many small increments can accumulate to a large overall change, but a small enough number of small enough increments is still small overall, so we can still write

$$h''g'h'gh \approx h''h'h \qquad\qquad \text{whenever } g, g' \in G_\varepsilon.$$

16 Lifting discrete paths

The usual path lifting theorem[33] refers to strictly continuous paths, but it also holds for any sequence of sufficiently closely-spaced points in G. Define $\tilde{\sigma}_1$ as in section 8 and G_ε and as in section 15. Let $\gamma_0, \gamma_1, \gamma_2, ..., \gamma_N$ be a sequence of elements of G with

$$\gamma_0 \equiv 1_G \qquad \gamma_n = \gamma_{n-1} g_n \qquad g_n \in G_\varepsilon$$

for all $n \in \{1, 2, ..., N\}$. Such a sequence of points will be called a **practically continuous** path. Given this practically continuous path in G, we can define a corresponding sequence in $\tilde{G}$ like this:[34]

$$\tilde{\gamma}_0 \equiv 1_{\tilde{G}} \qquad \tilde{\gamma}_n \equiv \tilde{\gamma}_{n-1} \tilde{\sigma}_1(g_n).$$

Even if the endpoint γ_N is not in G_1, this recipe defines a unique lift of γ_N using the canonical lift $\tilde{\sigma}_1$ and a practically continuous path from 1_G to γ_N. This is a discrete version of the path lifting property.

[33] Section 14

[34] This is determined by the path $\gamma_0, \gamma_1, \gamma_2, ..., \gamma_N$ because $g_n \equiv \gamma_{n-1}^{-1} \gamma_n$.

17 Lifting discrete homotopies

Even though the usual homotopy lifting theorem[35] refers to continuous morphs of continuous paths, it also holds for any sequence of sufficiently small changes to practically continuous paths. To deduce this, let γ and γ' be two practically continuous paths in G that both start at 1_G and differ from each other only slightly. Let γ_n be the sequence of points that constitutes γ, and γ'_n likewise for γ'. Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be their lifts into $\tilde{G}$, both starting at $1_{\tilde{G}}$. The premise that γ is practically continuous means that the increment $g_n \equiv \gamma_{n-1}^{-1}\gamma_n$ is in G_ε for each n, and similarly for γ'. The premise that γ_n and γ'_n differ from each other only slightly means $\gamma_n^{-1}\gamma'_n \approx 1_G$ for all n. Use

$$\begin{aligned}
\tilde{\gamma}'_n\tilde{\gamma}_n^{-1} &= \tilde{\sigma}_1(g'_1)\tilde{\sigma}_1(g'_2)\cdots\tilde{\sigma}_1(g'_n)\big(\tilde{\sigma}_1(g_1)\tilde{\sigma}_1(g_2)\cdots\tilde{\sigma}_1(g_n)\big)^{-1} \\
&= \kappa_n\tilde{\sigma}_1(\gamma'_n\gamma_n^{-1}) && \text{(equation (2))} \\
&\approx \kappa_n && (\gamma'_n\gamma_n^{-1} \approx 1_G)
\end{aligned}$$

to conclude that the points $\tilde{\gamma}_n$ and $\tilde{\gamma}'_n$ must be separated from each other by something close to an element of the kernel K. The kernel is discrete, with no elements arbitrarily close to $1_{\tilde{G}}$ other than $1_{\tilde{G}}$ itself, so the fact that the two paths in $\tilde{G}$ are initially identical and are both practically continuous[36] implies that the points $\tilde{\gamma}_n$ and $\tilde{\gamma}'_n$ must be separated from each other by something close to $1_{\tilde{G}}$. In other words, the paths must remain close to each other in the covering group $\tilde{G}$.

In particular, if γ and γ' end at the same point in G, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ must end at the same point in $\tilde{G}$ because they are equal modulo elements of the kernel K, and two nearly-equal endpoints that are equal modulo the discrete kernel K must be strictly equal.

[35] Section 14

[36] Section 16

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