Charged Particles in an Electromagnetic Field: the Lorentz Force Equation in Flat Spacetime

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Abstract This article introduces the Lorentz force equation, which governs the behavior of a charged particle in a prescribed electromagnetic field in a spacetime with any number of dimensions. The case of a constant electromagnetic field is used as an example.

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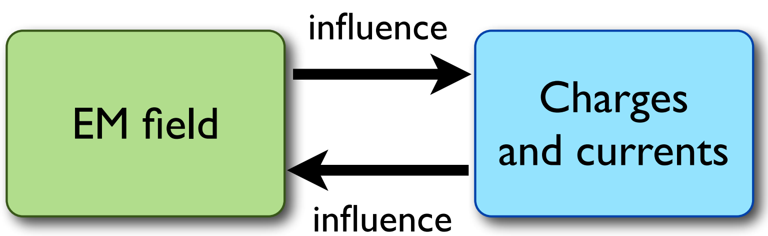
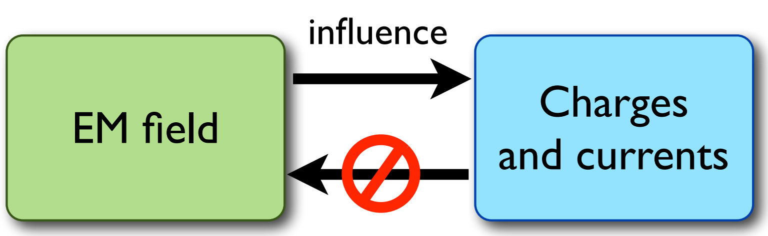
1 Context

The electromagnetic interaction is mediated by the electromagnetic (EM) field. The action principle says that all physical influences must go both ways. In particular, influences between the EM field and charges/currents go both ways:



For pedagogical purposes (and even for some practical purposes), we can consider models in which the influence goes only one way. This article treats the EM field as a prescribed background field, exempt from the action principle. In such a model, the influence goes only one way:



Specifically, this article considers a classical[1] model of pointlike charged particles under the influence of a prescribed EM field. Since the influence between field and particle goes only one way in this model, the charged particles do not interact with each other at all: their behavior is influenced by the EM field, but the EM field is prescribed independently of the particles' behavior, so it cannot mediate any interaction from one particle to another. This article considers only one particle, because the particles don't interact with each other anyway.

[1]Classical = not quantum.

2 Notation and conventions

The EM field is introduced in article 31738. This article uses the same notation and conventions, which are briefly reviewed here.

Let $F_{ab}(x)$ denote the components of the Faraday tensor, which represents the electromagnetic field. The argument x is an abbreviation for the list of spacetime coordinates $(x^0, x^1, ..., x^D)$ Work in a coordinate system where the proper time τ along any timelike worldline satisfies

$$d\tau^2 = (dx^0)^2 - \big[(dx^1)^2 + \cdots + (dx^D)^2\big] = \eta_{ab}\, dx^a\, dx^b \tag{1}$$

in natural units, where the metric is

$$\eta_{ab} = \begin{cases} 1 & \text{if } a = b = 0, \\ -1 & \text{if } a = b > 0, \\ 0 & \text{if } a \neq b. \end{cases} \tag{2}$$

This is the **Minkowski metric**, which defines **flat** spacetime. The index 0 corresponds to the time dimension, and D is the number of spatial dimensions. Indices from the beginning of the alphabet $(a, b, c, ...)$ are space-time indices: they take the values $0, 1, 2, ..., D$. Indices from the middle of the alphabet $(j, k, ...)$ are spatial indices: they take the values $1, 2, ..., D$.

The electric and magnetic components of the Faraday tensor F_{ab} are

$$E_k \equiv F_{k0} \qquad\qquad B_{jk} \equiv F_{jk}. \tag{3}$$

This article uses **natural units** (article 00669) to clarify the underlying symmetry of the model and to avoid extraneous coefficients.[2]

[2]The coefficient q in equation (10) could be considered extraneous in a model with only one particle, but it's important in a model with multiple particles having different charges (article 98002), so I'll retain it.

3 The Lorentz force equation

This section introduces the **Lorentz force equation**, which governs the behavior of a structureless charged particle[3] under the influence of a given EM field.

Consider a particle of mass m and charge q. The particle's worldline can be described by specifying its coordinates as functions of its proper time: $x^a(\tau)$. The particle's behavior is governed by the **Lorentz force equation**[4]

$$\boxed{\frac{dp^c}{d\tau} = \frac{q}{m}\, p^a\, F_{ab}\big(x(\tau)\big)\eta^{bc}} \tag{4}$$

with

$$p^a \equiv m\frac{dx^a}{d\tau}. \tag{5}$$

The Lorentz force equation is the **equation of motion** governing the behavior of charged particles in this model.

The component p^0 is the particle's **energy**, and $\mathbf{p} \equiv (p^1, ..., p^D)$ is the particle's **momentum** vector.[5] The more familiar form of the Lorentz force equation shown later (section 6) comes from specializing equation (4) to $D = 3$ and separating F_{ab} into its the electric and magnetic parts.

[3] The particle is assumed to have negligible size and no intrinsic angular momentum (no **spin**).

[4] Sums over a and b are implied.

[5] The words "energy" and "momentum" are in boldface here because these can be regarded as *definitions*. Articles 98002 and 78463 relate these definitions to the action principle and conservation laws.

4 Example: uniform EM field

We can use a matrix notation in which the Lorentz force equation (4) is

$$\frac{dp}{d\tau} = -\frac{q}{m}\eta F p,$$

where p is the column matrix with components p^a, and η and F are the square matrices with components η^{ab} and F_{ab}, respectively. For simple examples of solutions, suppose that F is constant in space and time. In this case, the Lorentz force equation is solved by[6]

$$p(\tau) = \exp\left(-\frac{q}{m}\eta F\,\tau\right)p(0). \tag{6}$$

This says that $p(\tau)$ is related to $p(0)$ by a Lorentz transformation[7] whose "angle" is proportional to the proper time τ.

For a specific example, set $D = 3$ and suppose that the only nonzero components of F are $F_{12} = -F_{21} = B_{12}$. Then

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_{12} & 0 \\ 0 & -B_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad -\eta F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_{12} & 0 \\ 0 & -B_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, the τ-dependent Lorentz transformation in equation (6) is

$$\exp\left(-\frac{q}{m}\eta F\,\tau\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(eB_{12}\tau/m) & \sin(eB_{12}\tau/m) & 0 \\ 0 & -\sin(eB_{12}\tau/m) & \cos(eB_{12}\tau/m) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an ordinary rotation with angle proportional to τ, so a charged particle in a constant magnetic field travels in a circle with constant centripetal acceleration.

[6] The exponential of a matrix is defined in article 18505.

[7] Articles 49705 and 18505 together explain why this qualifies as a Lorentz transformation.

Now suppose that the only nonzero components of F are $F_{01} = -F_{10} = -E_1$. Then

$$F = \begin{bmatrix} 0 & -E_1 & 0 & 0 \\ E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad -\eta F = \begin{bmatrix} 0 & E_1 & 0 & 0 \\ E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, the τ-dependent Lorentz transformation in equation (6) is

$$\exp\left(-\frac{q}{m}\eta F\,\tau\right) = \begin{bmatrix} \cosh(eE_1\tau/m) & \sinh(eE_1\tau/m) & 0 & 0 \\ \sinh(eE_1\tau/m) & \cosh(eE_1\tau/m) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is a Lorentz boost with "angle" (rapidity)[8] proportional to τ, so a charged particle in a constant electic field undergoes a constant linear acceleration. The acceleration is not constant in the relative sense, but it is constant in the *absolute* sense: the particle has constant weight from its own perspective.[9]

Realistic EM field configurations are not constant everywhere in space and time, but they are approximately constant within a sufficiently small region of space and time, so the examples shown above can be useful approximations within such a region.

[8] Article 77597

[9] To prove this, use the fact that this specific configuration of the electric field is invariant under this specific Lorentz transformation, so the field always the same strength from the particle's perspective.

5 Expression in terms of electric and magnetic fields

In this section, the timelike coordinate x^0 will be denoted t, so the spacetime coordinates are $x = (t, \mathbf{x}) = (t, x^1, x^2, ..., x^D)$. The ordinary velocity vector $\mathbf{v}$ is defined by

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt}. \tag{7}$$

To write the Lorentz force equation (4) in terms of the electric and magnetic fields (3), separate the timelike and spacelike components of equation (4) like this:

$$\begin{aligned} \frac{dp^0}{d\tau} &= \frac{q}{m}\sum_k p^k\, F_{k0} \\ -\frac{dp^k}{d\tau} &= \frac{q}{m}\left(p^0 F_{0k} + \sum_j p^j F_{jk}\right). \end{aligned}$$

Use the definitions (3) to get

$$\begin{aligned} \frac{d}{d\tau}p^0 &= \frac{q}{m}\sum_k E_k\, p^k = \frac{q}{m}\mathbf{E}\cdot\mathbf{p} \\ \frac{d}{d\tau}p^k &= \frac{q}{m}\left(p^0\, E_k - \sum_j p^j B_{jk}\right). \end{aligned} \tag{8}$$

Equation (5) implies

$$\mathbf{v} = \frac{\mathbf{p}}{p^0} \qquad\qquad \frac{d}{d\tau} = \frac{p^0}{m}\,\frac{d}{dt}. \tag{9}$$

Use (9) and exploit the antisymmetry of B_{jk} to rewrite equations (8) as

$$\boxed{\frac{dp^0}{dt} = q\,\mathbf{v}\cdot\mathbf{E} \qquad\qquad \frac{dp^k}{dt} = q\left(E_k - \sum_j v^j B_{jk}\right).} \tag{10}$$

Equations (10) are equivalent to the original equation (4), even though the original symmetry is obscured by the new notation and by the use of one coordinate as "time."

The first equation in (10) can be deduced from the second one. To see this, use equations (1) and (5) to get

$$p^a p^b \eta_{ab} = m^2.$$

Take the derivative of both sides with respect to t to get

$$p^a \frac{dp^b}{dt} \eta_{ab} = 0,$$

which can be re-arranged to get

$$p^0 \frac{dp^0}{dt} = \sum_k p^k \frac{dp^k}{dt}.$$

Use use $\mathbf{v} = \mathbf{p}/p^0$ and the second of equations (10) on the right-hand side to get

$$\frac{dp^0}{dt} = q\mathbf{v} \cdot \mathbf{E},$$

which is the first equation in (10). For this reason, the second equation in (10) can be called the Lorentz force equation by itself.

6 Specialization to 3-dimensional space

In 3-dimensional space, the electric field vector is

$$\mathbf{E} = (E_1,\, E_2,\, E_3),$$

and we can pretend that the magnetic field is also a "vector" given by

$$\mathbf{B} = (B_{23},\, B_{31},\, B_{12}).$$

Then the second of equations (10) can be written in the traditional form[10]

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{11}$$

[10]Griffiths (1989), section 7.4.4; Jackson (1975), equation (11.124)

7 The non-relativistic approximation

Equation (11) is valid for arbitrary speeds, even though its Lorentz symmetry is obscured by the notation. However, if we use the non-relativistic relationship

$$\mathbf{p} \approx m\mathbf{v},$$

then the resulting equation

$$\frac{d}{dt}\mathbf{v} \approx \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{12}$$

is valid only in the non-relativistic approximation.

8 References

Griffiths, 1989. *Introduction to Electrodynamics (Second Edition).* Prentice Hall

Jackson, 1975. *Classical Electrodynamics (Second Edition).* John Wiley and Sons

d'Inverno, 1995. *Introducing Einstein's Relativity.* Clarendon Press

9 References in this series

Article **00669** (https://cphysics.org/article/00669):
"Units in Electrodynamics" (version 2022-06-04)

Article **18505** (https://cphysics.org/article/18505):
"Matrix Math" (version 2023-02-12)

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