Tubular Neighborhoods

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Abstract This article reviews the concept of a *tubular neighborhood*. Intuitively, if S is a lower-dimensional smooth submanifold of an ambient m-dimensional smooth manifold M, then a tubular neighborhood of S is like a slightly thickened version of S, the union of sufficiently small m-dimensional neighborhoods of all the points in S. The precise definition uses the concept of a vector bundle over S, and the tubular neighborhood is equivalent to the total space of the bundle. A tubular neighborhood may be trivial or nontrivial. This article describes several examples.

Contents

1 Introduction and notation

Let M be a smooth manifold that may or may not have a boundary, and let S be a lower-dimensional submanifold of M. If we choose small enough neighborhoods in M of each point of S, then the union of these neighborhoods is a thickened version of S. That's an intuitive picture of a *tubular neighborhood*. This article explains how to make it precise.[1]

A tubular neighborhood of S is called *trivial* if it is topologically equivalent to $S \times D$ where D is a ball with $\dim(M) - \dim(S)$ dimensions. Some submanifolds have a trivial tubular neighborhood and some do not. Sections 9-13 will describe some examples.

Notation and conventions used in this article include:

- M is a smooth manifold with m dimensions that may or may not have a boundary.
- S is a smooth manifold with n dimensions that does not have a boundary.
- D^j is the j-dimensional open ball, and $\bar{D}^j$ is the closed ball.
- S^j is the j-dimensional sphere, the boundary of a $(j+1)$-dimensional closed ball. In particular, S^1 is a circle.
- $\mathbb{R}\mathrm{P}^j$ is the j-dimensional real projective space
- $O(k)$ is the **orthogonal group** of origin-preserving rotations and reflections in k-dimensional euclidean space. $SO(k) \subset O(k)$ is the **special orthogonal group**, which excludes orientation-reversing reflections.
- In this article, a vector space is always a **real vector space** – a vector space whose field of coefficients is the field $\mathbb{R}$ of real numbers.

[1] In this article, all manifolds are understood to be smooth. Section 5 will explain why this matters.

2 The normal bundle of a submanifold

Let S be a smooth manifold without boundary. A **(real) vector bundle** over S is a fiber bundle with S as the base space and a real vector space as the fiber.[2] The vector bundle is said to have **rank $\boldsymbol{k}$** if the vector space (the fiber) is k-dimensional.[3]

The **tangent bundle** of S, denoted TS, is defined by taking the fiber at each point $p \in S$ to be the space of tangent vectors at that point.

If M is an m-dimensional smooth manifold and S is a submanifold with fewer than m dimensions, then we can also define the *normal bundle* of S in M, which is complementary to the tangent bundle of S. The normal bundle can be defined in two ways:

- If a Riemannian metric is given on M,[4] then we can use that metric to define the angle between two vectors. In that case, the **normal bundle** of $S \subset M$ can be defined like this:[5] when the tangent bundle of M is restricted to S, we get a vector bundle of rank m over S, and discarding all vectors at $p \in S$ that are not orthogonal to all the tangent vectors at p leaves the normal bundle.

- If a Riemannian metric is not given, then we can use this definition instead:[6] the **normal bundle** is the quotient bundle T_SM/TS, where T_SM is the tangent bundle of M restricted to points in S. Intuitively, the quotient bundle consists of vectors tangent to M at $p \in S$ modulo vectors tangent to S at p.

We can think of the first definition as using additional structure (a Riemannian metric) to select one representative of each equivalence class of tangent-to-M-modulo-tangent-to-S. Without such additional structure, we have no natural way to select a single representative of each equivalence class, and that's okay. Instead of choosing an arbitrary individual vector from each equivalence class, we can use the equivalence classes themselves. That gives the second definition.

[2] Article 70621

[3] Tu (2017), definition 7.1

[4] Every smooth manifold with or without boundary admits a Riemannian metric (Lee (2013), proposition 13.3).

[5] Milnor and Stasheff (1965), corollary 3.4

[6] Kosinski (1993), chapter 3, definition 2.1

3 Definition and existence of tubular neighborhoods

Let M be a m-dimensional smooth manifold with or without boundary,[7] let S be an embedded n-dimensional submanifold of M without boundary, and use the abbreviation $k \equiv m - n$. An **(open) tubular neighborhood** of $S \subset M$ is an m-dimensional submanifold of M that is diffeomorphic to the total space of a rank k real vector bundle over S with the original n-dimensional submanifold $S \subset M$ as its zero section.[8] Every submanifold S has a tubular neighborhood diffeomorphic to (the total space of) the normal bundle[9] of S in M,[10] and every tubular neighborhood of S is equivalent to this one.[11,12]

Intuitively, a tubular neighborhood of S is a neighborhood of S that looks like $S \times D^k$ locally, where D^k is a k-dimensional ball, but may have a more complicated topology overall. At each point $p \in S$, we can think of the ball $p \times D^k$ as a space of vectors normal to S at p. The preceding paragraph roughly says[13] that a submanifold $S \subset M$ has an essentially unique neighborhood looking like a vector bundle over S.

A vector bundle over S is called **trivial** if it has the form $S \times V$, where V is the fiber (a vector space). Similarly, a tubular neighborhood that looks like $S \times D^k$ globally (not just locally) is called **trivial**, and one that doesn't is sometimes called **twisted**.[14]

[7] This is what *manifold* means in Kosinski (1993) (bottom of page 2).

[8] Kosinski (1993), chapter 3, definition 2.4; Bott and Tu (1982), text after equation (6.21); Hirsch (1976), chapter 4, theorem 5.2

[9] Section 2 defined *normal bundle*.

[10] Milnor and Stasheff (1965), theorem 11.1; Bott and Tu (1982), text after equation (6.22)

[11] Kosinski (1993), chapter 3, corollary 3.2 and the text after definition 2.4; Bott and Tu (1982), text after equation (6.22); Hirsch (1976), chapter 4, theorem 5.3

[12] This is sometimes called the **tubular neighborhood theorem** (Cohen (2023), theorem 8.1 or 8.2), but the name *tubular neighborhood theorem* is also used for a related-but-different result (Kosinski (1993), chapter 3, theorem 3.5).

[13] Hirsch (1976), page 5

[14] The name *twisted* can be motivated by an example: the real projective plane $\mathbb{RP}^2$ contains a a noncontractible closed curve whose tubular neighborhood is a Möbius band.

4 Example

For an easy example, take M to be 3-dimensional euclidean space with a coordinate system (x, y, z), and take S to be the circle defined by $x^2+y^2 = 1$ and $z = 0$. Then S has a trivial tubular neighborhood, diffeomorphic to a solid torus $S\times D^2$. The same circle also has non-tubular neighborhoods, like the ball defined by $x^2 + y^2 + z^2 \leq 2$, but a tubular neighborhood is special because it preserves the essence of the original circle's shape, just slightly thickened. More precisely, it is homotopy equivalent[15] to the original circle.

A manifold that can be embedded in a euclidean space with trivial normal bundle is said to admit a **framing**.[16]

[15] Article 61813 defines *homotopy equivalent*.

[16] Kosinski (1993), introduction, page xiv

5 Restrictions

In this article, all manifolds are understood to be smooth. This is important because if the manifolds were merely topological or merely piecewise-linear, then a submanifold would not necessarily have a tubular neighborhood.[17] Example: a piecewise-linear embedding of S^{19} in $S^{19} \times S^9$ exists that does not have a tubular neighborhood.[18]

In this article, the submanifold S does not have a boundary. Submanifolds without boundary are a special case of **neat submanifolds**.[19] Only neat submanifolds have tubular neighborhoods.[20]

In this article, the ambient manifold M may have a boundary, but the submanifold S does not intersect the boundary of M. The boundary of the ambient manifold does not have a tubular neighborhood, but it can have a **collar**, which is intuitively like half of a tubular neighborhood (the vectors in the would-be normal bundle can only point inward, not outward).[21]

[17] Ho (2024), text after proposition 1.2

[18] Hirsch (1968), page 65

[19] Article 44113 reviews what this means.

[20] Hirsch (1976), beginning of section 5

[21] Hirsch (1976), chapter 4, beginning of section 6

6 Orientability of vector bundles

A given vector bundle may or may not be orientable. Roughly,[22,23] a vector bundle E over S is called *orientable* if the fiber (a vector space) can be given an orientation that is constant over a sufficiently small neighborhood of each point in the base space S. An orientable vector bundle over a connected manifold has two possible orientations.[24] Orientability of a (smooth) manifold can be defined as orientability of its tangent bundle.[25,26]

The total space of a vector bundle is itself a manifold. Orientability of the total space (as a manifold) is not the same as orientability of the vector bundle.[27] Example: If E is the total space of a tangent bundle, then E is always orientable as a manifold,[28] but the tangent bundle may or may not be orientable as a vector bundle.[29]

Vector bundles over S^1 are classified by their rank (the number of dimensions of the vector space) and orientability. For any given rank, two vector bundles over S^1 exist: one that is nonorientable and nontrivial, and one that is orientable and trivial.[30]

[22] Zinger (2010), text before lemma 6.1

[23] A precise definition of *orientable* for vector bundles is given in Hirsch (1976), chapter 4, section 4, page 104.

[24] Hirsch (1976), chapter 4, section 4, pages 104-105

[25] Hirsch (1976), chapter 4, text after lemma 4.1

[26] Section 1.0.1 in Cohen (2023) gives a more general definition of *orientable* that can be applied to topological manifolds.

[27] https://math.stackexchange.com/questions/50809

[28] Tu (2017), problem 21.9; Bott and Tu (1982), chapter 1, example 6.3

[29] $\mathbb{R}\mathrm{P}^2$ has a nonorientable tangent bundle, which is another way of saying that the manifold $\mathbb{R}\mathrm{P}^2$ itself is nonorientable. The total space of the tangent bundle of $\mathbb{R}\mathrm{P}^2$ is itself a (4-dimensional) manifold, and this manifold is orientable. In more detail: the unit tangent bundle of $\mathbb{R}\mathrm{P}^2$ – defined by keeping only unit vectors in the tangent spaces – is a lens space (Konno (2002)), and lens spaces are orientable (Huisman and Mangolte (2005), text before lemma 2.3).

[30] Auyeung (2020), example 1.2

7 Orientability and the normal bundle

Given a submanifold $S \subset M$, the orientability of the normal bundle is related to the orientability of (the tangent bundles of) S and M. The relationship may be expressed like this: among the three vector bundles

- tangent bundle to M,
- tangent bundle to S,
- normal bundle to S,

the number of nonorientable ones must be even. This follows from the **Whitney sum formula**[31] together with the fact that a vector bundle is orientable if and only if its first Stiefel-Whitney class is zero.[32] The relationship may be stated more explicitly like this:

- If M is orientable, then S and its normal bundle are either both orientable or both nonorientable.[33]
- If M is nonorientable, then either S or its normal bundle is orientable and the other is not.[34]

If S is simply-connected,[35] then its normal bundle is orientable for any ambient manifold M. This follows from the fact that every vector bundle over a simply-connected manifold S is orientable.[36,37]

[31] Debray (2017), proposition 2.3

[32] Hatcher (2003), proposition 3.11

[33] Ebert (2014), text before definition 6.1.2: If M is oriented, then an orientation of $S \subset M$ determines an orientation of the normal bundle and vice versa.

[34] Example: the normal bundle of $S = \mathbb{R}\mathrm{P}^2$ in $M = \mathbb{R}\mathrm{P}^2 \times D^k$ is orientable even though S and M (which is also the total space of the normal bundle in this case) are not. Compare this to footnote 29 in section 6, which is about the tangent bundle instead of the normal bundle.

[35] Article 61813 reviews what this means.

[36] Hirsch (1976), chapter 4, section 4, page 104

[37] In particular, (the tangent bundle of) every simply-connected manifold is orientable (Hirsch (1976), chapter 4, text after lemma 4.1).

8 Sufficient conditions for triviality

This section uses basic properties of S and M to deduce two sufficient conditions for a tubular neighborhood of $S \subset M$ to be trivial without using any details of how S is embedded in M. Here are the results, using the abbreviation $k \equiv \dim M - \dim S$:

- If every principal $O(k)$-bundle over S is trivial, then a tubular neighborhood of S in M must also be trivial.
- If S and M are both orientable and every principal $SO(k)$-bundle over S is trivial, then a tubular neighborhood of S in M must also be trivial.

To deduce these results, use the fact that a tubular neighborhood of S is the total space of a particular vector bundle over S. The first result follows from these statements:

- Every rank k vector bundle E over S has an associated principal $O(k)$-bundle P over S.[38] If P is trivial, then E is also trivial.[39]

The second result follows from these statements:

- If E is orientable, then it has an associated principal $SO(k)$-bundle over S.[40] If S and M are both orientable, then the normal bundle of $S \subset M$ must also be orientable,[41] so the normal bundle E has an associated principal $SO(k)$-bundle P over S. If P is trivial, then E is also trivial.[39]

If S is 1-dimensional, then a tubular neighborhood of S is always trivial.[42,43]

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[38] Cohen (2023), definition 2.7, exercise before definition 2.9, and exercise 1 after definition 2.9; also the text leading to proposition 3.31 in section 3.3.6

[39] Article 70621

[40] Cohen (2023), definition 2.7 and theorem 2.7

[41] Section 6

[42] This follows from the fact that a principal $SO(k)$-bundle over a 1-dimensional manifold is necessarily trivial because $SO(k)$ is connected (article 33600).

[43] If $\dim S \geq 2$, then S may admit a nontrivial tubular neighborhood even if S is orientable (section 12).

9 Nontrivial tubular neighborhoods of closed curves

In this section, S is a one-dimensional manifold (a closed curve, denoted S^1).

The group $SO(k)$ is connected.[44] This implies that every principal $SO(k)$-bundle over S is trivial.[45] A closed curve is orientable, so if the ambient manifold M is also orientable, then the results reviewed in section 8 imply that a tubular neighborhood of S in M is necessarily trivial.

If the ambient manifold M is not orientable, then a tubular neighborhood of S in M is not necessarily trivial. The group $O(k)$ has two connected components, so for any given k, some principal $O(k)$-bundles over S are nontrivial. We can use that to construct an example of a manifold M and a one-dimensional submanifold S whose tubular neighborhood is nontrivial:[46] take M to be the total space of a nontrivial rank k vector bundle over S, and take $S \subset M$ to be the zero section of that bundle. In this example, a tubular neighborhood of S is homeomorphic to the whole ambient manifold M.

Section 10 will describe a few specific examples of nontrivial tubular neighborhoods of one-dimensional submanifolds.

[44] Article 92035

[45] Article 11617

[46] In any manifold M, some closed curves in M have trivial tubular neighborhoods, namely closed curves that bound a disk in M.

10 Examples

Examples of nontrivial tubular neighborhoods for $S = S^1$ with $\dim(M) = 2$:[47]

- If M is a Möbius band, then the tubular neighborhood of a closed curve $S \subset M$ may be trivial or nontrivial, depending on which closed curve we choose. If the curve forms the "center" of the Möbius band, then its tubular neighborhood is nontrivial, diffeomorphic to the interior of the Möbius band.
- If $M = \mathbb{R}\mathrm{P}^2$, then the tubular neighborhood of a closed curve $S \subset M$ may be trivial or nontrivial, depending on which closed curve we choose. If we choose a noncontractible closed curve,[48] then its tubular neighborhood is nontrivial (again diffeomorphic to the interior of a Möbius band).[49]

Examples of nontrivial tubular neighborhoods for $S = S^1$ with $\dim(M) = 3$:

- If M is a **solid Klein bottle**, then the tubular neighborhood of a closed curve $S \subset M$ may be trivial or nontrivial, depending on which closed curve we choose. If it bounds a disk, then its tubular neighborhood is trivial (diffeomorphic to a solid torus $S^1 \times D^2$). If it forms the "center" of the solid Klein bottle, then its tubular neighborhood is nontrivial, diffeomorphic to the interior of the solid Klein bottle itself.[50]
- If $M = S^1 \times \mathbb{R}\mathrm{P}^2$, then the tubular neighborhood of a closed curve $S \subset M$ may be trivial or nontrivial, depending on which closed curve we choose. If it's a noncontractible curve in one of the S^1's worth of copies of $\mathbb{R}\mathrm{P}^2$ (figure 1), then its tubular neighborhood is nontrivial, diffeomorphic to a solid Klein bottle.[51]

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[47]In this case, the difference $k \equiv \dim(M) - \dim(S)$ is 1, so the orthogonal group $O(k)$ is the finite group with two elements.

[48]All noncontractible closed curves in $\mathbb{R}\mathrm{P}^2$ are isotopic to each other because $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$ (article 61813).

[49]Hirsch (1976), chapter 9, section 1, exercises 2 and 3a

[50]The torus $S^1 \times S^1$ and the **Klein bottle** are the two fiber bundles whose base space and fiber are both circles. Replacing the fiber with a disk gives the solid torus and the solid Klein bottle.

[51]The solid Klein bottle and $S^1 \times \mathbb{R}\mathrm{P}^2$ are both examples of nonorientable 3-dimensional manifolds, but they are different: the solid Klein bottle has a boundary (namely a Kelin bottle), and $S^1 \times \mathbb{R}\mathrm{P}^2$ does not.

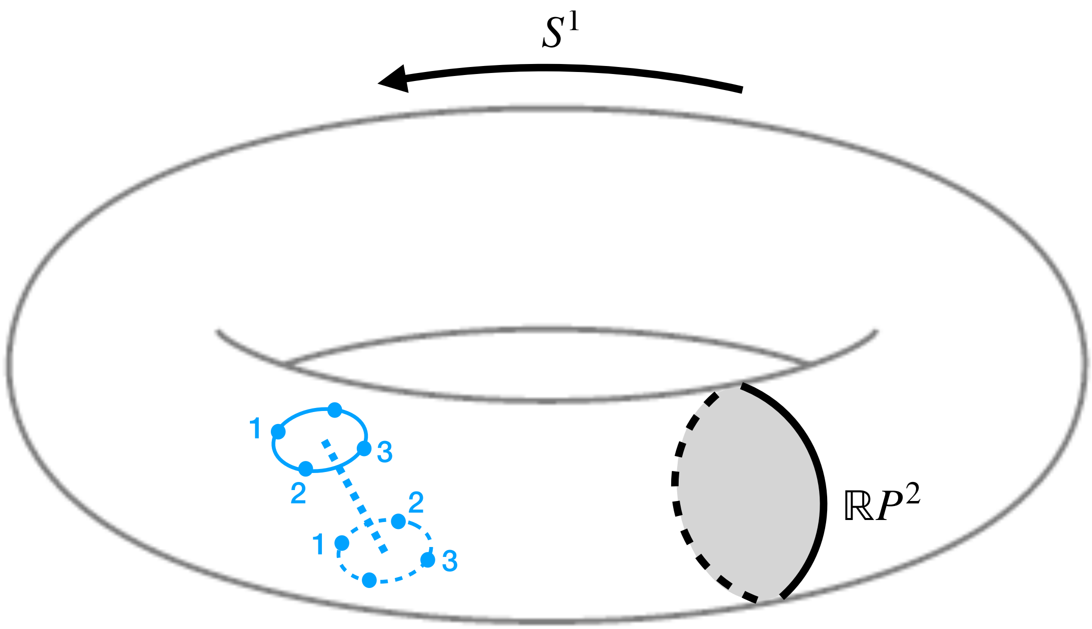

Figure 1 – The manifold $S^1 \times \mathbb{R}\mathrm{P}^2$, visualized as a solid torus $S^1 \times \bar{D}^2$ with antipodal points on each disk $\bar{D}^2$ identified with each other to convert the disk to a copy of $\mathbb{R}\mathrm{P}^2$. The thick dashed blue line is a noncontractible closed curve connecting antipodal points in one of the $\mathbb{R}\mathrm{P}^2$s. A tubular neighborhood of this curve may be visualized as a disk bundle over the curve. The blue circles indicate one of these disks: the solid blue circle (foreground) and the dashed blue circle (background) are identified with each other, with corresponding points labeled by corresponding integers. This shows that the disk bundle is nonorientable, so the tubular neighborhood is nontrivial.

11 Examples with trivial tubular neighborhoods

Here are a few examples with trivial tubular neighborhoods:

- If S is a compact, orientable n-dimensional submanifold of $(n+1)$- or $(n+2)$-dimensional euclidean space, then its normal bundle is trivial.[52]
- Let S be an n-dimensional smooth manifold that is homeomorphic to an n-sphere. Then the normal bundle to S is trivial for any embedding in $(n+3)$-dimensional euclidean space.[53]
- The normal bundle to an n-sphere imbedded in $(n+k)$-dimensional Euclidean space is trivial if $k > (n+1)/2$.[54]

[52] Massey (1959), section 1, fact (b)
[53] Massey (1959), section 1, corollary
[54] Massey (1959), section 1, fact (c)

12 Nontrivial tubular neighborhoods of surfaces

This section describes two examples of 2-dimensional surfaces S with nontrivial tubular neighborhoods, one with an orientable S and one with a nonorientable S.

Take M to be the total space of a nontrivial rank 3 vector bundle over S^2 associated with the nontrivial principal $SO(3)$-bundle over S^2 (which exists because $SO(3)$ is not 1-connected).[55] By construction, this gives a nontrivial tubular neighborhood of S^2 in a 5-dimensional manifold. The 5-dimensional manifold is the tubular neighborhood itself.

For an example of a (nonorientable) surface with a nontrivial tubular neighborhood in euclidean space, combine these two facts:

- An even-dimensional real projective space embedded in a euclidean space of any number of dimensions has a nontrivial normal bundle.[56] Proof: even-dimensional real projective spaces are not orientable, so the first Stiefel-Whitney class (of the tangent bundle) is nontrivial.[57] The Whitney sum formula mentioned in section 7 then implies that the normal bundle of an embedding in euclidean space also has a nontrivial Stiefel-Whitney class.[58]
- $\mathbb{R}\mathrm{P}^2$ can be embedded in $\mathbb{R}^4$.[59,60]

Altogether, the euclidean space $M = \mathbb{R}^4$ has a submanifold $S = \mathbb{R}\mathrm{P}^2$, and a tubular neighborhood of this submanifold is nontrivial.

[55] Article 33600

[56] https://math.stackexchange.com/questions/108467

[57] Theorem 4.5 in Milnor and Stasheff (1965) gives all the Stiefel-Whitney classes of $\mathbb{R}\mathrm{P}^n$. The coefficients in that expression should be understood modulo 2 (table on pages 55-56).

[58] Section 8; and section 13

[59] Article 44113

[60] https://math.stackexchange.com/questions/91981 describes such an embedding explicitly.

13 An orientable example in $\mathbb{R}^m$

This section mentions an example of a smooth closed oriented submanifold of $\mathbb{R}^m$ whose normal bundle is not trivial. The submanifold has codimension 3.

The complex projective space $\mathbb{C}\mathrm{P}^2$ is a smooth, closed, orientable 4-dimensional manifold.[61,62] It can be realized as a smooth submanifold of 7-dimensional euclidean space $\mathbb{R}^7$.[63,64] The tubular neighborhood of such a submanifold is nontrivial (not homeomorphic to $\mathbb{C}\mathrm{P}^2 \times D^3$). That can be deduced from these results:

- If m is even, then the second Stiefel-Whitney class of the tangent bundle of $\mathbb{C}\mathrm{P}^m$ is nontrivial.[65,66]
- If M is a submanifold of $\mathbb{R}^m$ with a nontrivial Stiefel-Whitney class (of its tangent bundle), then its normal bundle also has a nontrivial Stiefel-Whitney class.[67,68]
- The Stiefel-Whitney classes of a trivial bundle are trivial.[69]

[61] $\mathbb{C}\mathrm{P}^m$ is smooth (Nakahara (1990), example 8.4 combined with the end of section 8.1) and orientable (Nakahara (1990), theorem 11.21 combined with equation (11.140)).

[62] Every complex manifold is orientable (Eschrig (2011), text around equation 9.111).

[63] Every closed smooth orientable 4–manifold admits a smooth embedding in $\mathbb{R}^7$ (Ghanwat and Pancholi (2002), text above theorem 6.1). In contrast, $\mathbb{R}\mathrm{P}^4$ cannot be smoothly embedded in $\mathbb{R}^7$ (Milnor and Stasheff (1965), section 11.1, page 130).

[64] An embedding $\mathbb{C}\mathrm{P}^2 \subset \mathbb{R}^7$ is reviewed concisely at the beginning of Coffman (2002). Dunajski and Tod (2001) gives a more detailed analysis. In fact, $\mathbb{C}\mathrm{P}^2$ can be realized as a submanifold of $\mathbb{R}^7$ in two ways that are not isotopic to each other (Skopenkov (2010), section 1.2, footnote 8).

[65] If m is even, then the second Stiefel-Whitney class of of the tangent bundle of $\mathbb{C}\mathrm{P}^m$ is the generator of $H^2(\mathbb{C}\mathrm{P}^m;\mathbb{Z}_2)$ (Nakahara (1990), equation (11.140)), and $H^2(\mathbb{C}\mathrm{P}^m;\mathbb{Z}_2)$ is nontrivial (`https://topospaces.subwiki.org/wiki/Cohomology_of_complex_projective_space`).

[66] This implies that $\mathbb{C}\mathrm{P}^m$ does not admit a spin structure if m is even (Nakahara (1990), theorem 11.23).

[67] Milnor and Stasheff (1965), lemma 4.2

[68] Article 70621 shows that the unit sphere in $\mathbb{R}^3$ has a nontrivial tangent bundle, but its normal bundle is clearly trivial. This doesn't contradict the statement in the text, because the Stiefel-Whitney classes (of the tangent bundle) of a sphere are all zero (Milnor and Stasheff (1965), section 4.1, example 6).

[69] Milnor and Stasheff (1965), section 4.1, proposition 2

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