cphysics.org article **52890** 2023-11-12

Defining Scalar Quantum Fields on a Spatial Lattice

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Abstract This article constructs a family of models of a single scalar field with a not-necessarily-linear equation of motion, using a discrete lattice in place of continuous space so that the whole construction is mathematically unambiguous. These are toy models, not intended to have direct physical applications, but they illustrate some features of relativistic quantum field theory without some of the complications of more realistic models.

Contents

1 Introduction

In classical field theory, we normally treat spacetime as a smooth manifold. Treating spacetime as a smooth manifold would be nice in quantum field theory, too, except that in most cases we don't know how to do it.[1] Even when we do know how, it usually involves some heavy technical details.[2] This article uses a more straightforward appraoch. Instead of trying to treat spacetime as a smooth manifold, this article treats space as a discrete lattice. This is good enough as long as the step-size is much finer than any of the observables we care about. Time will still be treated as a continuous parameter.

The type of model constructed in this article involves only the simplest type of quantum field, namely a **scalar** quantum field. It consists of one self-adjoint operator $\phi(x)$ for each spacetime point x. In continuous spacetime, the equation of motion would be[3]

$$\eta^{ab}\partial_a\partial_b\phi(x) + V'\big(\phi(x)\big) = 0 \tag{1}$$

where η^{ab} are the components of the Minkowski metric (article 48968), ∂_a is the derivative with respect to the ath spacetime coordinate, and sums over the repeated indices a, b are implied. Conditions on the function $V'(\phi)$ will be specified later.

Equation (1) has Lorentz symmetry.[4] Discretizing space ruins exact Lorentz symmetry, but evidence from small-parameter expansions (not reviewed here) indicates that Lorentz symmetry can still be an excellent approximation at sufficiently low resolution.[5]

[1] Some models are believed to have nontrivial continuum limits even though we don't yet have watertight proofs. Some models are believe to *not* have nontrivial strict continuum limits, even though they can still have realistic applications at achievable resolutions. The approach illustrated in this article is appropriate in either case.

[2] Article 44563 reviews an approach that only works when the equation of motion is linear.

[3] This article uses natural units with $c = \hbar = 1$.

[4] Article 49705 studies the Lorentz symmetry of this equation in the context of classical field theory.

[5] This does not require the existence of a nontrivial strict continuum limit. Such a limit probably doesn't even exist for most models of the type considered in this article (footnote 1).

2 The equation of motion on a spatial lattice

A point in spacetime will be denoted $(\mathbf{x}, t)$, where $\mathbf{x}$ is the list of spatial coordinates and t is the time coordinate. Equation (1) may be written

$$\ddot{\phi}(\mathbf{x}, t) - \nabla^2\phi(\mathbf{x}, t) + V'\big(\phi(\mathbf{x}, t)\big) = 0, \tag{2}$$

where each overhead dot denotes a derivative with respect to t, and ∇ is the gradient with respect to $\mathbf{x}$. The function $V'(\phi)$ is a polynomial with real coefficients, such as $V'(\phi) = c_1\phi + c_2\phi^2 + c_3\phi^3$. The notation V' is used because it will be expressed later as the derivative $V' = dV/d\phi$ of another polynomial $V(\phi)$.

In this article, space is D-dimensional and is treated as a periodic lattice with finite size.[6] The lattice is defined by a set of D basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_D$, each with magnitude ϵ. Each lattice site $\mathbf{x}$ is a linear combination of the basis vectors with integer coefficients:

$$\mathbf{x} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + \cdots + n_D\mathbf{e}_D.$$

The time coordinate t is still continuous. The field $\phi(\mathbf{x}, t)$ is defined only at lattice sites $\mathbf{x}$, not between lattice sites.

The equation of motion may still be written as in (2), but now $\mathbf{x}$ is restricted to lattice sites, and the gradient term in (2) is defined by

$$\nabla^2\phi(\mathbf{x}, t) \equiv \sum_n \frac{\phi(\mathbf{x}+\mathbf{e}_n, t) + \phi(\mathbf{x}-\mathbf{e}_n, t) - 2\phi(\mathbf{x}, t)}{\epsilon^2}. \tag{3}$$

Taking the limit $\epsilon \to 0$ would give the usual gradient in continuous space, but we will keep ϵ fixed.[7] If ϵ is much smaller than the resolution of any observables that we care about, then space is still effectively smooth.

[6] Article 71852 reviews some tools for working with the lattice formulation.

[7] Defining the limit $\epsilon \to 0$ would require defining $\phi(\mathbf{x}, t)$ at all points in continuous space, which is problematic. The purpose of defining the model on a lattice is to avoid that problem.

3 Preview

Sections 4-5 construct a set of operators $\phi(\mathbf{x},t)$ called **field operators**. They will be constructed explicitly as operators on a Hilbert space. Operators representing observables are then constructed from the field operators (section 5).

Here's a preview of some consequences of the construction. Section 8 shows that the field operators satisfy the **equal-time commutation relations**[8,9]

$$\big[\phi(\mathbf{x},t),\phi(\mathbf{y},t)\big]=0 \qquad \big[\dot\phi(\mathbf{x},t),\dot\phi(\mathbf{y},t)\big]=0$$
$$\big[\phi(\mathbf{x},t),\dot\phi(\mathbf{y},t)\big]=i\delta(\mathbf{x}-\mathbf{y}), \tag{4}$$

with $\delta(\mathbf{x})$ defined in equation (8). Section 9 shows that the field operators satisfy the equation of motion (2), with $\nabla^2\phi(\mathbf{x},t)$ given by (3). Sections 7-8 show that the operator[10]

$$H=\int d^Dx\,\left(\frac{\dot\phi^2(\mathbf{x},t)+\big(\nabla\phi(\mathbf{x},t)\big)^2}{2}+V\big(\phi(\mathbf{x},t)\big)\right) \tag{5}$$

is independent of time, where $\int d^D\cdots$ is a lattice version of the integral and ∇ is a lattice version of the gradient. The operator H is the **hamiltonian** for this model. It generates translations in time (equation (12)), so it is the observable corresponding to the system's total energy. The fact that H is independent of time says that the total energy is conserved.

This was only a preview. The actual construction is in sections 4-5, and the consequences previewed here are derived in sections 7-9. The sequence is chosen carefully to avoid any circular logic.

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[8] $[A,B]\equiv AB-BA$

[9] This qualifies as a *quantum* model because its observables don't all commute with each other: measurements of these observables are not all compatible with each other (article 03431).

[10] The right-hand side of equation (5) has the same form as the expression for the total energy in the corresponding classical model (article 49705).

4 The Hilbert space and basic operators

The goal is to construct a model using operators that satisfy the equation of motion (2), with one operator $\phi(\mathbf{x},t)$ for each point $(\mathbf{x},t)$ in spacetime, with $\mathbf{x}$ restricted to lattice sites. These are operators on a Hilbert space. This section constructs a convenient representation of the Hilbert space.

Each element of the Hilbert space is represented by a complex-valued function $\Psi[s]$ of a collection $[s]$ of real variables, with one real variable $s(\mathbf{x})$ for each site $\mathbf{x}$ in the spatial lattice. The inner product is defined by

$$\begin{aligned}\langle\Psi_1|\Psi_2\rangle &\equiv \int [ds]\ \Psi_1^*[s]\Psi_2[s] \\ &\equiv \int \left(\prod_{\mathbf{x}} ds(\mathbf{x})\right)\ \Psi_1^*[s]\Psi_2[s].\end{aligned} \tag{6}$$

The integral is over the full range $-\infty < s(\mathbf{x}) < \infty$ of each of the real variables $s(\mathbf{x})$. For each lattice site $\mathbf{x}$, define a pair of operators $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ by the conditions[11]

$$\phi(\mathbf{x})\Psi[s] \equiv s(\mathbf{x})\Psi[s] \qquad\qquad \pi(\mathbf{x})\Psi[s] \equiv \frac{-i}{\epsilon^D}\frac{\partial}{\partial s(\mathbf{x})}\Psi[s].$$

These operators are self-adjoint, if the adjoint is defined with respect to the inner product (6). They clearly satisfy the commutation relations

$$\big[\phi(\mathbf{x}),\phi(\mathbf{y})\big] = 0 \qquad\qquad \big[\pi(\mathbf{x}),\pi(\mathbf{y})\big] = 0 \tag{7}$$

and

$$\big[\phi(\mathbf{x}),\pi(\mathbf{y})\big] = i\delta(\mathbf{x}-\mathbf{y}) \qquad \text{with } \delta(\mathbf{x}-\mathbf{y}) \equiv \begin{cases} 1/\epsilon^D & \text{if } \mathbf{x}=\mathbf{y}, \\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

[11] These operators are both **unbounded**, meaning that they are not defined on all elements of the Hilbert space, but every element of the Hilbert space can be arbitrarily well-approximated by elements on which they are defined.

5 Time evolution, field operators, and observables

Define the hamiltonian H by

$$H = \int d^Dx\; H(\mathbf{x}) \equiv \epsilon^D \sum_{\mathbf{x}} H(\mathbf{x}) \tag{9}$$

with

$$H(\mathbf{x}) \equiv \frac{\pi^2(\mathbf{x}) + \big(\nabla\phi(\mathbf{x})\big)^2}{2} + V\big(\phi(\mathbf{x})\big) \tag{10}$$

where V is a polynomial with a finite lower bound, and

$$\big(\nabla\phi(\mathbf{x})\big)^2 \equiv \sum_n \left(\frac{\phi(\mathbf{x}+\mathbf{e}_n) - \phi(\mathbf{x})}{\epsilon}\right)^2. \tag{11}$$

The hamiltonian is (unbounded but) self-adjoint, so the operators

$$U(t) \equiv \exp(-iHt)$$

are unitary. The field operators at time t are defined by[12]

$$\phi(\mathbf{x},t) \equiv U^{-1}(t)\phi(\mathbf{x})U(t). \tag{12}$$

In quantum field theory, observables are expressed in terms of field operators, but the field operators themselves are not necessarily observables. The present model is an exception: the field operators $\phi(\mathbf{x},t)$ are observables. A measurement of the observable $\phi(\mathbf{x},t)$ represents what we could call a measurement of the amplitude of the field at the location $\mathbf{x}$ at time t. All other observables associated with a given region R of spacetime are expressed in terms of the field operators $\phi(\mathbf{x},t)$ with $(\mathbf{x},t) \in R$.

This completes the construction of the model on a spatial lattice of finite size.

[12]This article uses the Heisenberg picture, where observables are time-dependent and states are not (article 22871).

6 The spectrum condition and infinite volume

The operators $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are self-adjoint, so if the function V in equation (10) has a finite lower bound, then the hamiltonian (9) satisfies the **spectrum condition** (article 22871). This provides the foundation for defining particles, as article 30983 illustrates using the special case $V(\phi) \propto \phi^2 + \text{constant}$.

The construction of the Hilbert space in section 4 assumed that the lattice has finite size (a finite number of points). In that case, the Hilbert space is automatically separable, as it should be in quantum theory. In the naïve limit of an infinite lattice (called the **infinite volume limit**), the Hilbert space would become nonseparable.[13] To get a separable Hilbert space in the infinite volume limit, we can use the completion of the space of states that can be reached from the lowest-energy state by applying sums and products of finite numbers of field operators.[14]

One quirk of the infinite volume limit in this family of models is that if we keep the coefficients in the hamiltonian (9) fixed while taking that limit, the hamiltonian's lower bound goes to $+\infty$ (article 00980). This isn't really a problem, because we can include a constant term in V that depends on the size of the lattice in such a way that the hamiltonian's lower bound remains finite, conventionally zero, in the infinite volume limit. This requires the constant term in V to become infinitely negative, and that's okay, because the important thing is that the hamiltonian *as a whole* remains well-defined in the limit. Ensuring that the hamiltonian as a whole remains well-defined is exactly the purpose of choosing the constant term in V this way.[15]

[13] This would still be true even if $s(\mathbf{x})$ were a ± 1-valued variable instead of a real variable, because the set of all binary digits with an infinite number of digits is uncountable.

[14] Witten (2021)

[15] This does have an interesting side-effect, though: even if we choose the volume-dependent constant so that the total energy H is positive, the energy density (10) can be negative in some places even if $V > 0$. This is reviewed in Fewster (2005a) and Fewster (2005b). Witten (2018), section 2.4, page 11, relates this phenomenon to the Reeh-Schlieder theorem.

7 Deriving the equation of motion, part 1

The next goal is to show that the field operators defined above satisfy the equation of motion (2). This section starts the derivation, and the next two sections finish it.

To begin, define

$$\pi(\mathbf{x},t) \equiv U^{-1}(t)\pi(\mathbf{x})U(t) \tag{13}$$

$$H(\mathbf{x},t) \equiv U^{-1}(t)H(\mathbf{x})U(t). \tag{14}$$

According to equations (12) and (13), $H(\mathbf{x},t)$ can be expressed by starting with equation (10) and replacing

$$\phi(\mathbf{x}) \to \phi(\mathbf{x},t) \qquad\qquad \pi(\mathbf{x}) \to \pi(\mathbf{x},t),$$

as previewed in equation (5). Even though $H(\mathbf{x},t)$ depends on time, its integral over space does not:

$$\begin{aligned}\int d^D x\ H(\mathbf{x},t) &= \int d^D x\ U^{-1}(t)H(\mathbf{x})U(t)\\ &= U^{-1}(t)\left(\int d^D x\ H(\mathbf{x})\right)U(t)\\ &= U^{-1}(t)HU(t)\\ &= H.\end{aligned} \tag{15}$$

The last step is true because H commutes with $U(t)$. This way of writing the hamiltonian will be used in the following sections to derive the equation of motion for the field operators.

8 Deriving the equation of motion, part 2

The time dependence of $\phi(\mathbf{x}, t)$ is defined by equation (12). This section uses that definition to derive a relationship between $\pi(\mathbf{x}, t)$ and the time derivative of $\phi(\mathbf{x}, t)$.

Use equations (7)-(8) and (12)-(13) to deduce the **equal-time commutation relations**

$$\begin{gathered} \big[\phi(\mathbf{x},t), \phi(\mathbf{y},t)\big] = 0 \qquad\qquad \big[\pi(\mathbf{x},t), \pi(\mathbf{y},t)\big] = 0 \\ \big[\phi(\mathbf{x},t), \pi(\mathbf{y},t)\big] = i\delta(\mathbf{x}-\mathbf{y}). \end{gathered} \tag{16}$$

Take the derivative of equation (12) with respect to t to get

$$\dot{\phi}(\mathbf{x},t) = -i\big[\phi(\mathbf{x},t), H\big]. \tag{17}$$

Use the results from the preceding section to see that this may also be written

$$\begin{aligned} \dot{\phi}(\mathbf{x},t) &= -i\int d\mathbf{y}\ \big[\phi(\mathbf{x},t), H(\mathbf{y},t)\big] \\ &= \frac{-i}{2}\int d\mathbf{y}\ \big[\phi(\mathbf{x},t), \pi^2(\mathbf{y},t)\big]. \end{aligned}$$

The other terms in $H(\mathbf{y}, t)$ do not contribute because $\phi(\mathbf{x}, t)$ commutes with $\phi(\mathbf{y}, t)$ for all $\mathbf{x}, \mathbf{y}$. Use the equal-time commutation relations (16) to evaluate the remaining commutator, which gives

$$\boxed{\dot{\phi}(\mathbf{x},t) = \pi(\mathbf{x},t).}$$

This shows that the operator $\pi(\mathbf{x}, t)$ is the time-derivative of the field operator $\phi(\mathbf{x}, t)$.

By combining this result with equation (15), the hamiltonian defined in section 5 may also be written as shown in section 3.

9 Deriving the equation of motion, part 3

Take the derivative of equation (17) with respect to t to get

$$\begin{aligned}\ddot{\phi}(\mathbf{x},t) &= -i\big[\dot{\phi}(\mathbf{x},t),H\big]\\ &= -i\big[\pi(\mathbf{x},t),H\big],\\ &= -i\int d\mathbf{y}\ \big[\pi(\mathbf{x},t),H(\mathbf{y},t)\big].\end{aligned}$$

The term $\pi^2(\mathbf{y},t)$ in $H(\mathbf{y},t)$ does not contribute because $\pi(\mathbf{x},t)$ commutes with $\pi(\mathbf{y},t)$ for all $\mathbf{x},\mathbf{y}$. To evaluate the remaining commutator, use[16]

$$\begin{aligned}\frac{-i}{2}\int d\mathbf{y}\ \Big[\pi(\mathbf{x},t),\big(\nabla\phi(\mathbf{y},t)\big)^2\Big] &= \nabla^2\phi(\mathbf{x},t)\\ -i\int d\mathbf{y}\ \big[\pi(\mathbf{x},t),V\big(\phi(\mathbf{y},t)\big)\big] &= -V'\big(\phi(\mathbf{x},t)\big)\end{aligned}$$

with $\nabla^2\phi$ defined by (3) and where V' is the derivative of V with respect to its argument. Altogether, this gives

$$\boxed{\ddot{\phi}(\mathbf{x},t)-\nabla^2\phi(\mathbf{x},t)+V'\big(\phi(\mathbf{x},t)\big)=0.} \tag{18}$$

This is equation (2).

[16] The first equation can be derived using an integration-by-parts identity described in article 71852, because the lattice is periodic. To second equation can be derived by writing the left-hand side as $-iU^{-1}(t)\int d\mathbf{y}\ \big[\pi(\mathbf{x}),V\big(\phi(\mathbf{y})\big)\big]\,U(t)$ and then using the definitions of ϕ and π in section 4.

10 Multiple scalar fields

The model introduced in the preceding sections involved a single scalar field. The generalization to multiple scalar fields is straightforward. For a system of N scalar fields, each element of the Hilbert space is represented by a complex-valued function $\Psi[s]$ of a collection of real variables, with N real variables $s_1(\mathbf{x}), s_2(\mathbf{x}), ..., s_N(\mathbf{x})$ for each site $\mathbf{x}$ in the spatial lattice. The inner product is defined as before, using an integral over all of these real variables. The hamiltonian is

$$H = \int d^D x \,\left(\sum_n \frac{\pi_n^2(\mathbf{x}) + \big(\nabla\phi_n(\mathbf{x})\big)^2}{2} + V\big(\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), ..., \phi_N(\mathbf{x})\big)\right), \tag{19}$$

where V is a polynomial in N real variables, and the operators $\phi_n(\mathbf{x})$ and $\pi_n(\mathbf{x})$ are defined by

$$\phi_n(\mathbf{x})\Psi[s] \equiv s_n(\mathbf{x})\Psi[s] \qquad\qquad \pi_n(\mathbf{x})\Psi[s] \equiv \frac{-i}{\epsilon^D}\frac{\partial}{\partial s_n(\mathbf{x})}\Psi[s].$$

The time-dependent field operators are defined by

$$\phi_n(\mathbf{x}, t) \equiv U^{-1}(t)\phi_n(\mathbf{x})U(t) \tag{20}$$

with $U(t) \equiv e^{-iHt}$. As before, this implies the equal-time commutation relations

$$\big[\phi_j(\mathbf{x}, t), \phi_k(\mathbf{y}, t)\big] = 0 \qquad\qquad \big[\dot\phi_j(\mathbf{x}, t), \dot\phi_k(\mathbf{y}, t)\big] = 0$$
$$\big[\phi_j(\mathbf{x}, t), \dot\phi_k(\mathbf{y}, t)\big] = i\delta(\mathbf{x}-\mathbf{y})\delta_{jk}$$

and the equations of motion (one for each n)

$$\ddot\phi_n(\mathbf{x}, t) - \nabla^2\phi_n(\mathbf{x}, t) + V_n\big(\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), ..., \phi_N(\mathbf{x})\big) = 0,$$

where V_n is the derivative of V with respect to its nth argument.

11 Lorentz symmetry and microcausality

The motive for using the lattice is to make the construction mathematically unambiguous, but it also has obvious drawbacks:

- It ruins exact Lorentz symmetry.
- It doesn't satisfy the **microcausality principle** (article 21916). The lattice version of the equation of motion wouldn't have a strict maximum speed even if the field operators were ordinary real variables, like they are in classical field theory. As a result, observables separated from each other by a spacelike interval do not necessarily commute with each other, even though they do commute at equal times (equations (16)).

The search for quantum field models that can be defined in continuous spacetime, with strict Lorentz symmetry and strict microcausality, is an important theme in the literature about quantum field theory.[17] In four-dimensional spacetime, models of the type constructed in this article probably don't have any such nontrivial continuum limit.[18] (In this context, **nontrivial** means distinct from anything that could be obtained with a linear equation of motion.) Another important theme, though, is that a model can *effectively* have those properties (Lorentz symmetry and microcausality) at sufficiently coarse resolution. Most of the articles in this series, including this one, are written with this empirically-oriented theme in mind. Evidence from small-parameter expansions (perturbation theory) indicates that nontrivial models of the type constructed in this article can be *effectively* consistent with Lorentz symmetry and microcausality at sufficiently coarse resolution, at least in four-dimensional spacetime.[19]

[17] Example: Heckman and Rudelius (2018)

[18] Smit (2002), end of section 3.8

[19] In quantum field theory, if the lattice spacing is changed relative to some physical scale of interest, then the coefficients in the hamiltonian must typically also be changed in order to keep the model's low-resolution predictions unchanged. This is called (nonperturbative) **renormalization**, and this is why the (non)existence of a nontrivial continuum limit depends on the number of dimensions of spacetime.

Even models that aren't expected to have emergent Lorentz symmetry at low resolution may still have a property analogous to effective microcausality, or at least an effective limit to the speed at which information can propagate. This is quantified by **Lieb-Robinson bounds**.[20] Lieb-Robinson bounds assume that the observables in question are represented by bounded operators, and they also assume that the hamiltonian is a sum of bounded local operators. The field operators used in this article are not bounded, and the local terms $H(\mathbf{x})$ in the hamiltonian (9) are not bounded, so Lieb-Robinson bounds cannot be directly applied to these models. On the other hand, these operators might as well be bounded when the model is restricted to low-energy states,[21] so Lieb-Robinson bounds might be applicable with this restriction. That might be a way to address the issue of effective microcausality in models that are expected to have emergent Lorentz symmetry at low energy, without relying on perturbation theory, but this hasn't been explored much yet as far as I know.

[20] Lieb-Robinson bounds quantify the group velocity in quantum models that use discrete space and continuous time (Lieb and Robinson (1972)). Lieb-Robinson bounds are reviewed in section 4 of Naaijkens (2013), in section III.B of Masanes (2009), and in Hastings (2010). Beware, though, that *group velocity* does not always represent the speed at which information propagates (Robinett (1978)).

[21] For a hamiltonian of the form described in section 5, with a finite lower bound, low energy implies low resolution: if a state is restricted to low energies (relative to the lower bound), then it's automatically also restricted to low values of the gradient term (11), because the gradient term is nonnegative.

12 References

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13 References in this series

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