The Quantum Electromagnetic Field on a Spatial Lattice

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Abstract This article introduces one of the simplest examples of a quantum model with a gauge field, treating D-dimensional space as a lattice so that the math is straightforward. The model is a special case of compact quantum electrodynamics (**compact QED**), namely the case with no electrically charged matter, so the quantum electromagnetic field is the only physical entity.

The adjective *compact* in the name refers to the fact that the model uses the compact group $U(1)$ as its gauged group, in contrast to traditional electrodynamics in which the gauged group is the noncompact group $\mathbb{R}$. The choice $U(1)$ is motivated by the fact that the electric charges of all known elementary particles appear to be precisely integer multiples of a single elementary unit of charge. The model constructed here does not include charged matter, but it uses $U(1)$ as the gauged group to prepare for models that do.

Contents

1 Introduction

Article 26542 sketches a model of the quantum electromagnetic field. That sketch isn't quite well-defined mathematically, because it tries to associate operators on a Hilbert space with individual points in continuous space. That sketch can be promoted to a mathematically well-defined construction by using only operators that are smeared in space,[1] but learning that consruction has a low value-to-cost ratio because it doesn't allow generalizing the model to include charged matter. This article uses a different approach to making the model well-defined, treating space as a large but finite number of closely spaced points.[2] This approach works just as well when charged matter is included.

As in article 26542, this article uses the hamiltonian formulation, in which time is continuous. The model also has a well-defined path-integral formulation in which time and space are both discrete. The path-integral formulation has several advantages, but the hamiltonian formulation makes the relationship to the general principles of quantum theory[3] more clear, so the hamiltonian formulation will be used here.

The equations of motion in this model are nonlinear,[4,5] even though interactions with charged matter are absent. The nonlinearity makes calculations challenging, but this article focuses on the easy part: constructing the model without any mathematical ambiguity, so that calculations have a solid place to start.

[1]Smearing in time works more generally than smearing only in space (Witten (2023)), but the smearing approach (in time and/or space) has not yet led to nonperturbative constructions of many models that are believed to have nontrivial continuum limits, much less of models that are not believed to have nontrivial continuum limits (like quantum electrodynamics with charged matter).

[2]Article 52890 uses this approach for scalar fields, for the same reason. The **holographic principle** (reviewed in Bousso (2002)) gives us a good reason to think that this lattice-like picture of space and the conventional picture of continuous space are *both* ultimately incorrect. With that in mind, the fact that we don't know how to define most quantum field models in continuous space is less disappointing, and the fact that we do know how to define so many quantum field models using a lattice-like approach is a welcome concession.

[3]Article 03431

[4]Section 20

[5]Section 21 will relate these nonlinear equations to Maxwell's equations, which are linear.

2 Outline

- Section 3 previews some notation.
- Section 4 introduces the gauged group.
- Sections 5-6 define the spatial lattice – actually two different versions of the spatial lattice, one periodic and one not, because they both have advantages when studying gauge theories in general.
- Sections 7-16 define a Hilbert space and the model's observables at time $t = 0$, represented as operators on that Hilbert space, and explain how this representation reproduces some of Maxwell's equations (the ones that don't involve time derivatives).
- Sections 17-18 introduce the hamiltonian and use it to define the model's observables at all times t in terms of those at $t = 0$.
- Sections 19-22 give some insight about the model's continuum limit. First, some simplistic calculations are used to help relate the lattice equations of motion to Maxwell's equations (the ones that involve time derivatives), and then some insights from more careful studies will be summarized.
- Section 23 will highlight a property of the model that might seem inconsistent with everyday experience, and section 24 will explain why it's not.

3 Preview of notation

For reference, here's a summary of some notation that will be introduced later in this article:

- $\mathbf{x}$ = lattice site (a point in the spatial lattice)
- ℓ = link (a pair of neighboring lattice sites)
- $\square$ = plaquette
- $u(\ell) = U(1)$-valued link variable
- $\theta(\ell)$ = angle-valued link variable defined (modulo 2π) by $e^{i\theta(\ell)} = u(\ell)$
- $E(\ell)$ = electric field operator associated with link ℓ
- $W(\square)$ = plaquette operator
- $W(C)$ = Wilson loop operator or Wilson line operator
- $B(C)$ = magnetic flux, defined (modulo $2\pi\hbar$) by $W(C) = e^{iB(C)/\hbar}$

This article uses the units conventions described in article 26542. That system of units uses a minimum electric charge that will be denoted q.

4 The gauged group

Article 70621 introduced the concept of a **principal G-bundle**, which is the mathematical foundation for the concept of a gauge field. In the physics literature, the group G is often called the **gauge group**, but that means something different in the math literature.[6] For clarity, this article calls G the **gauged group**. This name is not standard, but it is consistent with the important idea of *gauging* a symmetry group (using the word *gauge* as a verb).

In classical electrodynamics, the gauged group is usually taken to be $\mathbb{R}$, the additive group of real numbers. In quantum electrodynamics (QED), we have a good reason to take the gauged group to be the compact group $U(1)$ instead. This is called **compact QED**. Here's the reason: the magnitudes of the electric charges of all known elementary particles are precisely integer multiples of a single quantity, with no evidence of any deviations despite careful searches for exceptions.[7] This **charge quantization**[8] would be unexplained in models that use $G = \mathbb{R}$, but it is automatic in quantum models that use $G = U(1)$.[9] This is related to the fact that in a model with a charged entity, the term in the hamiltonian that implements its interaction with the electromagnetic field involves a link variable $e^{i\theta}$ (introduced in section 7) raised to the nth power, where n is the entity's electric charge expressed as a multiple of an elementary unit q of charge. When the gauged group is $U(1)$, the quantity θ is defined only modulo 2π, so n must be an integer for the nth power of $e^{i\theta}$ to make sense. If the gauged group were $\mathbb{R}$ instead, then n could be any real number, so the empirical quantization of charge would be unexplained.

The model constructed in this article doesn't include electrically charged objects, but it uses $G = U(1)$ anyway as practice for models that do.

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[6] In the math literature, the name *gauge group* is used for the group of gauge transformations, which is much larger than G (article 76708). G is often called the **structure group**, but that can also be ambiguous (article 70621).

[7] Dylla and King (1973), Marinelli and Morpurgo (1984), Baumann *et al* (1988)

[8] Here, *quantization* means limited to a discrete (not continuous) set of values.

[9] Harlow and Ooguri (2021), section 3.4, page 76: "...it would be crazy to ignore the observational fact that the charges of the electron and proton are exact opposites to within one part in 10^{21} [ref]. By far the most plausible explanation of this remarkable agreement is that the gauge group of electrodynamics is indeed $U(1)$..."

5 The short-distance cutoff

Each element of the Hilbert space will be represented by a function of an enormous number of variables, nominally D variables for each point in D-dimensional space. To keep the number of variables finite, so that the model's construction is straightforward, we will need both a **short-distance (UV) cutoff** and a **long-distance (IR) cutoff**. This section describes the short-distance cutoff, and section 6 will modify this picture to implement a long-distance cutoff.

Start with D-dimensional euclidean space, and choose a set of D mutually orthogonal basis vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_D$, all with the same magnitude ϵ. Choose any one point p in the D-dimensional space. The lattice consists of all points that may reached from p by adding integer multiples of the basis vectors $\mathbf{e}_k$. Each point in this infinite lattice has coordinates $(n_1, n_2, ..., n_D)$ in the chosen basis, where each coordinate n_k is an integer. The point p has coordinates $(0, 0, ..., 0)$.

Two points $\mathbf{x}$ and $\mathbf{y}$ in the lattice are called **nearest neighbors** if they have the same coordinates except for one coordinate in which they differ by ± 1, so the distance between them is ϵ. An ordered pair $(\mathbf{x}, \mathbf{y})$ of nearest neighbors will be called a **directed link**, and an unordered pair $\{\mathbf{x}, \mathbf{y}\}$ of nearest neighbors will be called an **undirected link**. The two directed links $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{y}, \mathbf{x})$ will be called **oppositely directed** compared to each other.

Instead of associating variables with each point in continuous space, variables will be associated only with the (directed) links in this lattice.[10] These variables will be called **link variables**. This is a type of short-distance cutoff, because the number of variables per unit volume is finite.

[10] Section 7

6 The long-distance cutoff

The Hilbert space inner product will be defined by integrating over all of the link variables. To ensure that this makes sense, a long-distance cutoff will be used so that the total number of integration variables is finite. Two different long-distance cutoffs will be considered. Both of them start with the lattice that was defined in section 5 and modify it to make the volume of space finite.

One long-distance cutoff uses a **truncated lattice**.[11,12] To define this, choose a convex open set O of D-dimensional euclidean space that contains a very large but finite number of the original lattice points. Choose O so that its boundary doesn't pass through any lattice point. Truncate the set of links by retaining only those links in the original lattice that have at least one endpoint inside O, and truncate the set of points by retaining only those that belong to at least one of the retained links. Points that are in O will be called **interior points**, points that are retained but are not inside O will be called **boundary points**, and links that include one boundary point will be called **boundary links**. This is illustrated in figure 1.

The other long-distance cutoff will be called a **wrapped lattice**,[11] because space wraps back on itself like a torus. To define this, choose an integer $K \ggg 1$. Start with the same infinite lattice as before, but declare two points $\mathbf{x}$ and $\mathbf{y}$ to be equivalent (the same point) if each coordinate of $\mathbf{x}$ is equal to the corresponding coordinate of $\mathbf{y}$ modulo K. With this equivalence relation, each point still has $2D$ nearest neighbors, the lattice still has (discrete) translation symmetry, and it still doesn't have any boundary points or boundary links, but now the total number of points in the lattice is finite (equal to K^D). This is illustrated in figure 2.

The model's construction will be described in a way that works equally well with either of these two long-distance cutoffs. When using a wrapped lattice, statements that apply only to boundary points and boundary links may simply be ignored, because the wrapped lattice doesn't have any.

[11] This name is not standard.

[12] A truncated lattice is technically no longer a *lattice* in the usual mathematical sense of the word, but this article still calls it a lattice. This is common in the literature about "lattice" quantum field theory.

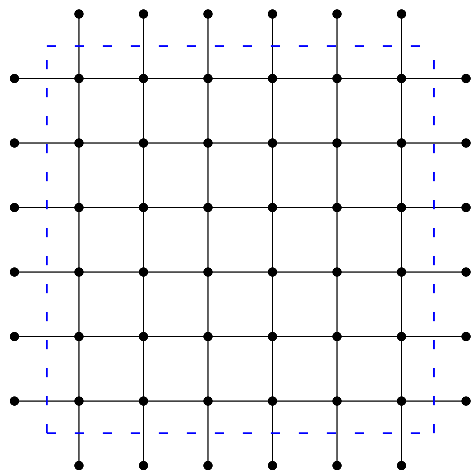


Figure 1 – Example of a two-dimensional truncated lattice ($D = 2$). Dots represent points in the lattice, and solid lines represent links. The dashed outline is the boundary of the region that was denoted O in the text. The points inside the dashed outline are interior points. The points outside the dashed outline are boundary points. Links that cross the dashed outline are boundary links. When $D = 3$, the one-dimensional dashed outline is replaced by a two-dimensional surface.

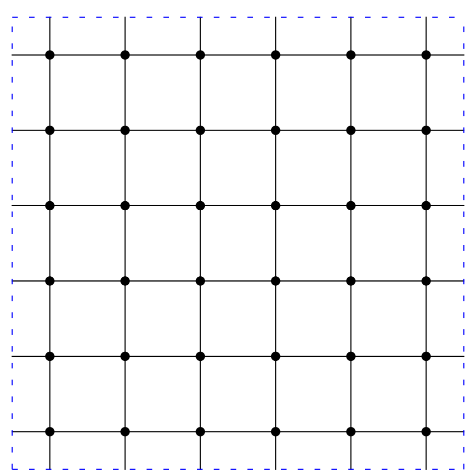


Figure 2 – Example of a two-dimensional wrapped lattice ($D = 2$). Opposite sides of the dashed outline are identified with each other, so space is topologically a torus. This lattice does not have any boundary points or links. Every point is an interior point with $2D$ nearest neighbors.

7 Link variables

Section 10 will construct the Hilbert space. Each element of the Hilbert space is a function of variables called *link variables*. This section introduces the link variables.

Gauge fields in smooth space are defined using a mathematical structure called a **principal G-bundle**.[13] The Lie group G is often called the **gauge group** in the physics literature, but the math literature uses that name for a much larger group (the group of all gauge transformations). To avoid confusion, this article will call G the **gauged group**, because the group of all gauge transformations is obtained by "gauging" the group G.[14]

In the standard hamiltonian formulation of lattice gauge theory, every directed link $(\mathbf{x},\mathbf{y})$ in the lattice has an associated **link variable** $u(\mathbf{x},\mathbf{y})$, which takes values in the gauged group G. Link variables associated with oppositely-directed links are related to each other by the condition

$$u(\mathbf{x},\mathbf{y})u(\mathbf{y},\mathbf{x}) = 1. \tag{1}$$

In this article, the gauged group G is $U(1)$, the multiplicative group of complex numbers with magnitude 1, so each link variable may be written in terms of a real-valued **angle variable** $\theta(\mathbf{x},\mathbf{y})$ like this:

$$u(\mathbf{x},\mathbf{y}) = e^{i\theta(\mathbf{x},\mathbf{y})}. \tag{2}$$

The angle variable $\theta(\mathbf{x},\mathbf{y})$ is only defined modulo 2π. Equation (1) implies

$$\theta(\mathbf{x},\mathbf{y}) = -\theta(\mathbf{y},\mathbf{x}) \qquad\qquad (\text{modulo } 2\pi).$$

The collection of link variables represents the **gauge field**, and any assignment of specific values (specific elements of G) to all of the link variables will be called a **configuration** of the gauge field. The name *compact QED* refers to the fact that the gauged group $U(1)$ is compact as a smooth manifold.

[13] Article 76708

[14] Harlow and Ooguri (2021), section 3.1

8 Gauge transformations

Section 10 will define the Hilbert space using functions that are invariant under the *gauge transformations.* This section explains what that means.

Let h be a map from the set of points in the lattice to the gauged group $U(1)$, so $h(\mathbf{x}) \in U(1)$ for each point $\mathbf{x}$, subject to the constraint that $h(\mathbf{x}) = 1$ for every boundary point $\mathbf{x}$.[15] A transformation that replaces the original value of every link variable with the new value

$$u^h(\mathbf{x}, \mathbf{y}) \equiv h(\mathbf{x})u(\mathbf{x}, \mathbf{y})h^{-1}(\mathbf{y}) \tag{3}$$

will be called a **gauge transformation**. A function $\Psi[u]$ of the link variables will be called **gauge invariant** if it is invariant under all of these gauge transformations:[16]

$$\Psi[u^h] = \Psi[u]. \tag{4}$$

This is the only part of the construction that treats boundary points differently than interior points, so let's consider what would happen if we didn't require $h(\mathbf{x}) = 1$ for boundary points. Section 10 will define the Hilbert space to consist of gauge invariant functions of the link variables. If we enlarged the group of all gauge transformations by allowing $h(\mathbf{x}) \neq 1$ for all points $\mathbf{x}$, including boundary points, then the space of gauge invariant functions would be smaller, so the set of linear operators that can act on the Hilbert space would also be smaller. Requiring $h(\mathbf{x}) = 1$ for boundary points $\mathbf{x}$ accommodates a slightly larger set of operators on the Hilbert space.[17,18]

[15] This constraint is empty on a wrapped lattice (section 6).

[16] This article uses the **temporal gauge**, in which $A_0 = 0$. Only time-independent gauge transformations are considered here.

[17] The text between equations (3.21) and (3.22) in Harlow and Ooguri (2021) mentions a context in which this can be important.

[18] The set would be even larger if we didn't require gauge invariance at all, but then the model wouldn't be consistent with electrodynamics.

9 Examples of gauge invariant functions

Let $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N$ be any sequence of points for which each pair $(\mathbf{x}_n, \mathbf{x}_{n+1})$ of consecutive points is a link. Let C denote this set of directed links, and define

$$u(C) \equiv \prod_{\ell \in C} u(\ell) = u(\mathbf{x}_1, \mathbf{x}_2)u(\mathbf{x}_2, \mathbf{x}_3) \cdots u(\mathbf{x}_{N-1}, \mathbf{x}_N). \tag{5}$$

This is the product of link variables along a path C made of links. The effect of a gauge transformation on this product is

$$u(C) \to h(\mathbf{x}_1)u(C)h^{-1}(\mathbf{x}_N).$$

The product is gauge invariant for either of these two types of path (figures 3-6):

- If the path's endpoints are both boundary points, then the product is gauge invariant because $h(\mathbf{x}) = 1$ for boundary points.[19]

- If the path is closed ($\mathbf{x}_N = \mathbf{x}_1$), then the product is gauge invariant. This works because the gauged group is abelian (commutative), so

$$h(\mathbf{x}_1)u(C)h^{-1}(\mathbf{x}_1) = u(C)h(\mathbf{x}_1)h^{-1}(\mathbf{x}_1) = u(C).$$

An important special case of the second type of path is the sequence of directed links that traces out the perimeter of a **plaquette**, the smallest possible loop in the lattice:

$$u(\square) \equiv \prod_{\ell \in \square} u(\ell) = u(\mathbf{x}_1, \mathbf{x}_2)u(\mathbf{x}_2, \mathbf{x}_3)u(\mathbf{x}_3, \mathbf{x}_4)u(\mathbf{x}_4, \mathbf{x}_1). \tag{6}$$

The product $u(\square)$ is called a **plaquette variable**. The plaquette $\square$ can have either of two possible **orientations**, corresponding to the two possible directions in which we can trace around the perimeter.

Any function of these gauge invariant products is still gauge invariant. This provides a rich supply of gauge invariant functions.

[19] Section 8

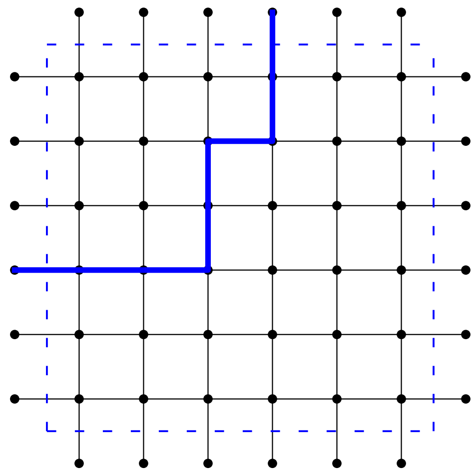


Figure 3 – Example of a path whose endpoints are both boundary points.

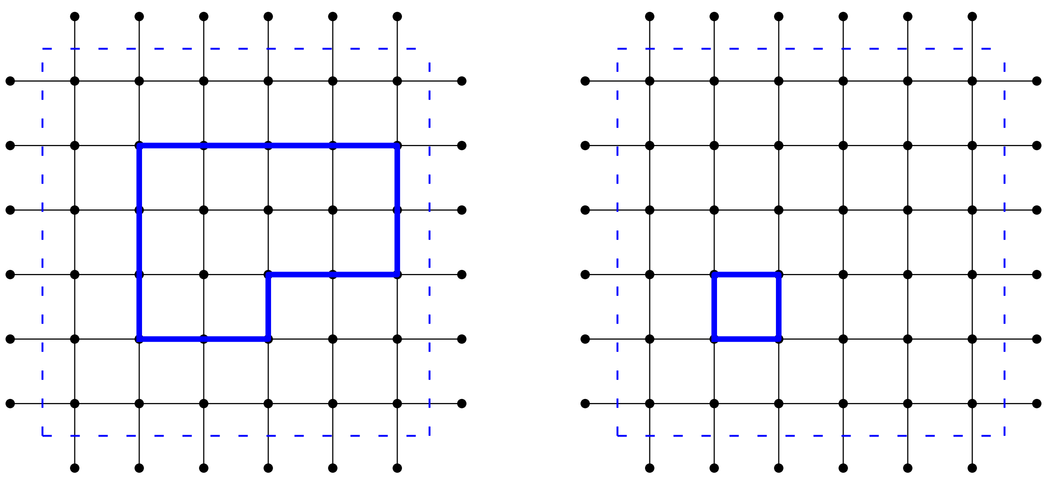


Figure 4 – Examples of closed paths. The example on the right is the boundary of a plaquette. The lattice in these pictures is two-dimensional ($D = 2$). On a D-dimensional lattice with $D \geq 3$, most paths do not lie in a single plane, but (the boundary of) a plaquette necessarily lies in a single plane no matter how many dimensions the lattice has.

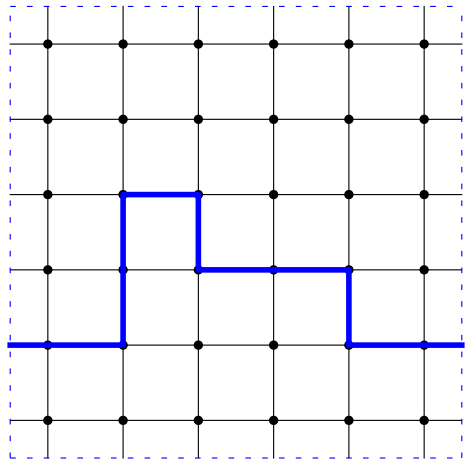


Figure 5 – Example of a closed path that is not the (whole) boundary of any surface made from plaquettes. Such paths exist on a wrapped lattice. If the space were continuous (but still topologically a torus), this loop would still not be contractible (article 61813).

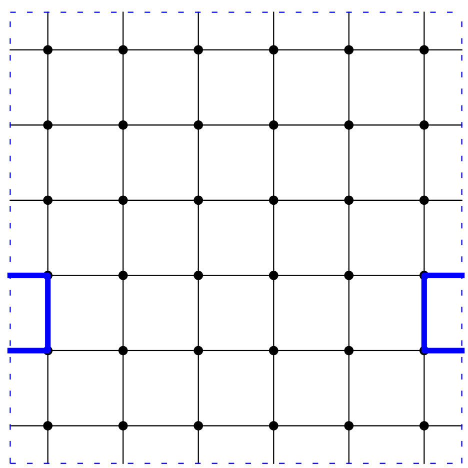


Figure 6 – Example of a closed path forming the boundary of a single plaquette, on a wrapped lattice. If space were continuous, this loop would be contractible.

10 The Hilbert space

Observables are represented by linear operators on a Hilbert space. This section constructs the Hilbert space that will be used for the rest of this article.

An element of the Hilbert space will be called a **state-vector**.[20] Each state-vector is represented by a gauge invariant[21] complex-valued function $\Psi[u]$ of the link variables. Given two states $\Psi_1[u]$ and $\Psi_2[u]$, their inner product is

$$\begin{aligned}\langle\Psi_1|\Psi_2\rangle &\equiv \int [du]\;\Psi_1^*[u]\Psi_2[u] \\ &\equiv \int \left(\prod_\ell du(\ell)\right)\;\Psi_1^*[u]\Psi_2[u]\end{aligned} \tag{7}$$

where the product is over all links and where $du(\ell)$ is defined for each link ℓ by

$$\int du(\ell)\cdots \equiv \int_0^{2\pi} d\theta(\ell)\cdots$$

with $u(\ell) = e^{i\theta(\ell)}$. The inner product is well-defined because the number of integration variables is finite (sections 5-7) and because the domain of each integration variable is finite.[22] This Hilbert space will be denoted $\mathcal{H}$.[23]

[20] Article 03431

[21] Section 8

[22] This is a technical advantage of working with the compact group $U(1)$ instead of the noncompact group $\mathbb{R}$. If the gauged group were noncompact, then defining the inner product would require *gauge fixing*.

[23] $\mathcal{H}$ is pronounced "curly H."

11 The electric field operators

This section defines some of the model's basic observables, namely the components of the electric field at a given locaion in space.

For each link ℓ, define the angle-valued link variable $\theta(\ell)$ by writing each $U(1)$-valued link variable $u(\ell)$ in the form (2), and define an operator $E(\ell)$ on $\mathcal{H}$ by

$$E(\ell)\Psi[u] \equiv i\kappa \frac{\partial}{\partial\theta(\ell)}\Psi[u], \tag{8}$$

where κ is a constant with units of $1/\epsilon^2$ that will be specified below. The factor of i makes this operator self-adjoint. When $\ell = (\mathbf{x}, \mathbf{x} + \mathbf{e}_j)$, the operator $E(\ell)$ will also be denoted $E_j(\mathbf{x})$:

$$E_j(\mathbf{x}) \equiv E(\ell) \qquad\qquad \text{when } \ell = (\mathbf{x}, \mathbf{x} + \mathbf{e}_j). \tag{9}$$

These are the **electric field operators** representing the components of the electric field. The list of components $E_1(\mathbf{x}), E_2(\mathbf{x}), ..., E_D(\mathbf{x})$ will be abbreviated $\mathbf{E}(\mathbf{x})$.

The value of the coefficient κ in equation (8) is[24]

$$\kappa \equiv \frac{q^2}{\epsilon^{D-1}}, \tag{10}$$

where q is the magnitude of the smallest electric charge that we would want the model to include when the model is extended to include charged matter. In the context of the full standard model of particle physics, the appropriate value would be 1/3 the charge of a proton.[25]

[24] The variable denoted θ here is related to the variable that was denoted a in article 26542 by equation (17) in section 14 (which writes A instead of a). That's why (10) doesn't include a factor of $\hbar$. To relate the factor ϵ^{D-1} in the denominator to article 26542, note that article 26542 implicitly uses $\partial a_j(\mathbf{x})/\partial a_k(\mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$, whose lattice version has ϵ^D in the denominator, and the integral in (17) cancels one of those factors of ϵ. The sign is consistent with article 26542, because $E^j = -E_j$ when the mostly-minus convention is used for the Minkowski metric.

[25] Section 23

12 Gauge invariance and the electric field

Gauss's law (one of Maxwell's equations) is implicit in the fact that the Hilbert space uses only gauge invariant functions. Given a gauge transformation (3), we can define a set of angle variables $\phi(\mathbf{x})$, one for each non-boundary point $\mathbf{x}$, by $h(\mathbf{x}) = \exp(i\phi(\mathbf{x}))$. If $\Psi[u]$ is any smooth complex-valued function of the link variables, not necessarily gauge invariant, then equation (3) implies

$$\frac{\partial}{\partial\phi(\mathbf{x})}\Psi[u^h] \propto \nabla\cdot\mathbf{E}(\mathbf{x})\,\Psi[u^h] \tag{11}$$

with $\nabla\cdot\mathbf{E} \equiv \sum_j \nabla_j E_j$, where ∇ is this lattice version of the gradient:

$$\nabla_j f(\mathbf{x}) \equiv \frac{f(\mathbf{x}) - f(\mathbf{x}-\mathbf{e}_j)}{\epsilon}.$$

If the function $\Psi[u]$ is gauge invariant, then (11) implies

$$\nabla\cdot\mathbf{E}(\mathbf{x})\Psi[u] = 0. \tag{12}$$

This is the quantum version of Gauss's law in a model where the quantum electromagnetic field is the only physical entity (no charged matter). Equation (12) is another way to write equation (4).

13 Wilson loop and Wilson line operators

This section defines more of the model's basic observables. Section 14 will explain how these observables relate to the magnetic field.

Every gauge invariant function $\Psi[u]$ represents an element of the Hilbert space $\mathcal{H}$ that was constructed in section 10. Any gauge invariant function $\omega[u]$ may also be used to define a linear operator W on $\mathcal{H}$, like this:

$$W\Psi[u] \equiv \omega[u]\Psi[u] \qquad\qquad \text{for all } \Psi \in \mathcal{H}. \tag{13}$$

Section 9 described examples of gauge invariant functions. One example is the product $u(C)$ of link variables around a closed path C. The corresponding operator $W(C)$, defined by

$$W(C)\Psi[u] \equiv u(C)\Psi[u], \tag{14}$$

is called a **Wilson loop operator** or just **Wilson loop**.[26] An important special case of a Wilson loop is the **plaquette operator**

$$W(\Box)\Psi[u] = u(\Box)\Psi[u] \tag{15}$$

with $u(\Box)$ defined by (6). Another example is the product $u(C)$ of link variables along a path C whose endpoints are boundary points. In this case, the operator defined by (14) is called a **Wilson line**.[27]

If $\omega[u]$ is not a gauge invariant function, then (13) does not define an operator on the Hilbert space, because the product $\omega[u]\Psi[u]$ is not gauge invariant and so does not belong to the Hilbert space. In particular, multiplication by a single link variable does not define an operator on this Hilbert space.

[26] Some authors use the name *Wilson loop* for the expectation value of this operator, as in Montvay and Münster (1997), section 3.2.4. The way I'm using the name here is consistent with Peskin and Schroeder (1995), section 15.3.

[27] Wilson loop operators and Wilson line operators are both called **line operators** to emphasize that they are localized along a one-dimensional curve, as opposed to being localized at a point (Aharony *et al* (2013), Gaiotto *et al* (2015)).

14 Magnetic flux and the magnetic field

If a closed loop C is the boundary of a surface made from plaquettes,[28] then the observable $B(C)$ corresponding to the magnetic flux through C is defined by

$$W(C) = e^{iB(C)/\hbar}, \tag{16}$$

where $W(C)$ is the Wilson loop observable that was defined in section 13. The next two paragraphs help motivate the relationship (16).

Article 76708 introduces the concept of a **connection** on a principal G-bundle, which is the mathematical foundation for the general concept of a classical gauge field. A principal G-bundle associates a copy of the **fiber**, a smooth manifold that is almost the Lie group G but without the full structure of a group, to each point of the **base space**. In this article, the base space is ordinary three-dimensional physical space, generalized to D dimensions for better insight, and the group G is $U(1)$. A *connection* defines a way of lifting each path in the base space to a path through the collection of fibers. When the path in the base space is a closed loop C, the lifted path starts and ends in the same (copy of the) fiber, but not necessarily at the same point in the fiber. The transformation from one point to the other is reproduced by an element of G called the **holonomy** associated with the loop C.[29]

The quantity $u(C) \in G$ defined in equation (5) is a lattice version of the holonomy associated with C. To relate this to magnetic flux, consider classical electromagnetism. Article 76708 explains that when G is abelian, the holonomy has the form $\exp\left(i\int_C A\right)$, where A is the gauge field one-form and the integral is around the loop C. This is related to (5) through

$$\theta(\mathbf{y}, \mathbf{y}') = \frac{1}{\hbar}\int_{\mathbf{y}}^{\mathbf{y}'} A_k(\mathbf{x})\, dx^k, \tag{17}$$

where θ is the angle-valued link variable defined in (2). In classical electrodynamics with gauged group $\mathbb{R}$, if S is a two-dimensional surface whose boundary is C, then

[28] Section 15 will define this more carefully.

[29] If the gauged group were nonabelian, then this element of G would also depend on the starting point in the fiber.

Stokes's theorem[30] gives

$$\int_C A = \int_S B, \tag{18}$$

where B is the magnetic field two-form. The integral $\int_S B$ defines the magnetic flux through the surface S in classical electromagnetism. This motivates the definition (16) of the magnetic flux operator in the quantum $U(1)$ model.

The definition (16) of the magnetic flux operator is unchanged when $B(C)$ is replaced by $B(C) + 2\pi\hbar n$ for any integer n, so in this model, $B(C)$ itself is only defined modulo $2\pi\hbar$. This is a consequence of defining magnetic flux in terms of holonomy when the gauged group is the compact group $U(1)$. Section 23 will acknowledge that this **periodicity** of the flux might seem to be inconsistent with real-world experience, but section 24 will explain why it actually isn't.

In continuous space, the components of the magnetic field may be expressed in terms of the magnetic flux like this:

$$B_{jk}(\mathbf{x}) = \lim_{\alpha(C)\to 0} \frac{B(C)}{\alpha(C)} \qquad \text{(in continuous space)}, \tag{19}$$

where C is the boundary of a surface element[31] in the j-k plane with area $\alpha(C)$ containing the point $\mathbf{x}$. On a lattice, the minimum possible area is ϵ^2, where ϵ is the distance between neighboring lattice sites. By analogy with (19), we can define

$$B_{jk}(\mathbf{x}) \equiv \frac{B(\Box)}{\epsilon^2} \qquad \text{(on a lattice)}, \tag{20}$$

where $\Box$ is the plaquette whose links trace through this sequence of points:

$$\mathbf{x} \;\to\; \mathbf{x}+\mathbf{e}_j \;\to\; \mathbf{x}+\mathbf{e}_j+\mathbf{e}_k \;\to\; \mathbf{x}+\mathbf{e}_k \;\to\; \mathbf{x}. \tag{21}$$

This is consistent with the continuous-space relationship $B_{jk} = \nabla_j A_k - \nabla_k A_j$ when θ is related to A by equation (17).

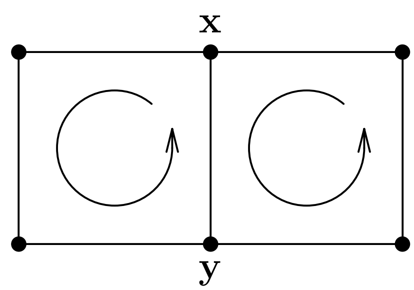
[30] https://www.englishrules.com/writing/2005/possessive-form-of-singular-nouns-ending-with-s/

[31] To make this definition unambiguous, an orientation (one of the two directions around the boundary C) would need to be specified. The sequence (21) does that for the lattice version.

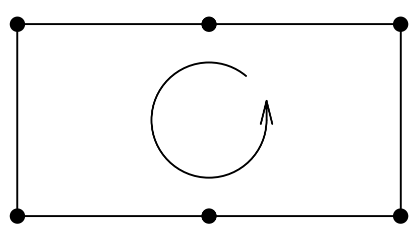
15 The magnetic flux through a closed surface

One of Maxwell's equations involves only the magnetic field. In continuous space, this equation may be written $dB = 0$, where B is the magnetic field two-form and d is the exterior derivative. Equivalently, by Stokes's theorem, the integral of the magnetic field two-form over any contractible closed surface is zero. This section shows that the magnetic flux defined in (16) also has this property, modulo $2\pi\hbar$.

Consider two adjacent plaquettes, directed so that their shared link occurs with opposite directions, as shown here:



Call these two plaquettes $\Box_1$ and $\Box_2$, and define $u(\Box_1)$ and $u(\Box_2)$ as in equation (6). One plaquette includes the link $(\mathbf{y}, \mathbf{x})$, and the other includes the link $(\mathbf{x}, \mathbf{y})$. Equation (1) says that those two link variables cancel each other in the product $u(\Box_1)u(\Box_2)$, leaving the product of the six link variables around the perimeter of the pair, as illustrated here:



This is still true if the two adjacent plaquettes are not coplanar. This generalizes to any number of plaquettes whose shared links all occur in oppositely-directed pairs. Such a collection of plaquettes will be called **consistently directed**.

Consider a surface S formed by consistently directed plaquettes, and suppose for simplicity that its boundary ∂S is a single closed loop. Then the product of all of those plaquette variables satisfies

$$\prod_{\square\in S} u(\square) = \prod_{\ell\in\partial S} u(\ell) \tag{22}$$

because equation (1) says that the contributions of the other link variables (the ones that are not in ∂S) cancel each other in pairs, as illustrated above. In terms of the angle-valued link variables $u(\ell)$, equation (22) is

$$\sum_{\square\in S} \theta(\square) = \sum_{\ell\in\partial S} \theta(\ell). \tag{23}$$

This is a lattice version of equation (18).

According to equation (16), the operator defined by (14) with $C = \partial S$ corresponds to the magnetic flux through the surface S. If the surface S is closed, so that the boundary ∂S is empty, then all of the link variables cancel in pairs, so the right-hand side of (22) is equal to 1 in this case. Equivalently, the right-hand side of (23) is equal to 0 (modulo 2π). This implies that the operator representing the magnetic flux through a closed surface S is zero (modulo $2\pi\hbar$). This is the integral version of one of Maxwell's equations, as described at the beginning of this section.

16 A commutation relation

Define $E(\ell)$ as in section 11, and let $W(C)$ be any Wilson loop or Wilson line as defined in section 13. Let ℓ^{rev} denote the link obtained by reversing the direction of ℓ, so if $\ell = (\mathbf{x}, \mathbf{y})$, then $\ell^{\text{rev}} = (\mathbf{y}, \mathbf{x})$. If the loop C does not intersect itself, then the definitions of $E(\ell)$ and $W(C)$ imply

$$\big[E(\ell),\, W(C)\big] = \begin{cases} -\kappa W(C) & \text{if } \ell \in C, \\ \kappa W(C) & \text{if } \ell^{\text{rev}} \in C, \\ 0 & \text{otherwise,} \end{cases} \tag{24}$$

using the standard notation $[A, B] \equiv AB - BA$. In particular, the electric field operator $E(\ell)$ commutes with a Wilson loop or Wilson line $W(C)$ if the loop C does not include the link ℓ or its oppositely-directed version ℓ^{rev}. If it does, then $E(\ell)$ doesn't commute with $W(C)$.

The commutation relation (24) is a lattice version of the commutation relation shown in article 26542. To infer this, use the fact that the representations of the electric and magnetic field operators in sections 11 and 14 are lattice versions of the representations of the electric and magnetic field operators in article 26542. The commutation relations are consequences of those representations, both in smooth space and on the lattice.

17 Time dependent observables

In quantum field theory, part of the task of defining a model is to associate observables with regions of spacetime.[32] Sections 11 and 13 defined the model's basic observables at time $t = 0$. These are the electric field operators $E(\ell)$ and the Wilson loop and Wilson line operators $W(C)$. This section defines observables at arbitrary times t in terms of those at $t = 0$, using a hamiltonian that will be specified in section 18.

If R is any region of space, then **observables** localized in R at time t are represented by operators of the form

$$\mathcal{O}(t) \equiv U(-t)\mathcal{O}U(t) \tag{25}$$

with

$$U(t) \equiv e^{-iHt/\hbar}, \tag{26}$$

where $\mathcal{O}$ is any linear operator on the Hilbert space – like an electric field operator $E(\ell)$ or a Wilson loop $W(C)$ – that can be expressed using only links in R, and the hamiltonian is defined by (28). The hamiltonian is (unbounded but) self-adjoint, so the operators (26) are unitary, and $\mathcal{O}(t)$ is self-adjoint for each t if $\mathcal{O}$ is self-adjoint.

The Hilbert space consists only of gauge invariant functions, so if $\mathcal{O}$ is a linear operator on the Hilbert space, then applying $\mathcal{O}$ to any gauge invariant function gives another gauge invariant function. In this sense, observables are gauge invariant.

Equations (24) and (25) imply

$$\big[E(\ell,t),\, W(\square,t)\big] = U^{-1}(t)\big[E(\ell),\, W(\square)\big]U(t) = \big[E(\ell),\, W(\square)\big] \tag{27}$$

for all t. This is the **equal-time commutation relation**. It is a lattice version of the commutation relation between the electric and magnetic field operators that was described in article 26542.

[32] Article 21916

18 The hamiltonian

This section introduces the hamiltonian, the operator that section 17 used to define the time dependence of the model's observables. The hamiltonian is[33,34]

$$
\begin{aligned}
H &= \frac{1}{q^2}\left(\epsilon^D\sum_\ell \frac{E^2(\ell)}{4} + \epsilon^D\sum_\Box \frac{1-W(\Box)}{2\epsilon^4}\hbar^2\right) + \text{constant} \\
&= \frac{1}{q^2}\left(\epsilon^D\sum_\ell \frac{E^2(\ell)}{4} + \epsilon^D\sum_\Box \frac{2-W(\Box)-W^\dagger(\Box)}{4\epsilon^4}\hbar^2\right) + \text{constant}.
\end{aligned}
\tag{28}
$$

The operators $E(\ell)$ and $W(\Box)$ are defined by equations (8) and (15). The sums over ℓ and $\Box$ are over all directed links and all oriented plaquettes, respectively.[35]

For any t, equation (25) and the obvious identity $U^{-1}(t)HU(t) = H$ imply that the hamiltonian (28) may also be written[36]

$$
H = \frac{1}{q^2}\left(\epsilon^D\sum_\ell \frac{E^2(\ell,t)}{4} + \epsilon^D\sum_\Box \frac{1-W(\Box,t)}{2\epsilon^4}\hbar^2\right) + \text{constant}. \tag{29}
$$

Section 19 will show that this is a lattice version of the more familiar hamiltonian for electrodynamics in continuous space.

[33]This is the **Kogut-Susskind hamiltonian** specialized to the gauged group $G = U(1)$.

[34]The two expressions for H are equal because reversing the direction of a plaquette is the same as replacing $W(\Box) \to W^\dagger(\Box)$.

[35]Equation (28) has an extra factor of 2 in the denominator compared to equation (3.66) in Montvay and Münster (1997), because the sum in their equation includes only one orientation of each unoriented plaquette.

[36]Footnote 34 in section 18

19 Formal continuum limit of the hamiltonian

This section shows that the hamiltonian (28) is a lattice version of the more familiar hamiltonian for electrodynamics in continuous space.

Use the relationships (16) and (20) to get

$$\begin{aligned}
\sum_{\square}\frac{2-W(\square)-W^\dagger(\square)}{4\epsilon^4}\hbar^2 &= \sum_{\square}\frac{2-e^{iB(\square)/\hbar}-e^{-iB(\square)/\hbar}}{4\epsilon^4}\hbar^2 \\
&= \sum_{\mathbf{x},j,k}\frac{2-e^{i\epsilon^2 B_{jk}(\mathbf{x})/\hbar}-e^{-i\epsilon^2 B_{jk}(\mathbf{x})/\hbar}}{4\epsilon^4}\hbar^2 \\
&= \sum_{\mathbf{x},j,k}\frac{\big(B_{jk}(\mathbf{x})\big)^2}{4}+O(\epsilon^2).
\end{aligned}$$

Use this and (9) in (28) to get

$$\begin{aligned}
H &\approx \frac{1}{q^2}\left(\epsilon^D\sum_{\ell}\frac{E^2(\ell)}{4}+\epsilon^D\sum_{\mathbf{x},j,k}\frac{\big(B_{jk}(\mathbf{x})\big)^2}{4}\right)+\text{constant} \\
&= \frac{1}{2q^2}\,\epsilon^D\sum_{\mathbf{x}}\left(\sum_j E_j^2(\mathbf{x})+\sum_{j<k}\big(B_{jk}(\mathbf{x})\big)^2\right)+\text{constant},
\end{aligned}$$

where "≈" means up to terms that are negligible when the resolution is low compared to the lattice scale ϵ, ignoring the fact that the magnetic flux is only defined modulo $2\pi\hbar$.[37] This matches the form of the hamiltonian that was used in article 26542.

[37] Section 14 explained why it's only defined modulo $2\pi\hbar$.

20 Equations of motion

This section derives expressions for the time derivatives of $E(\ell,t)$ and $W(C,t)$, and section 21 will show that the resulting equations are a lattice version of Maxwell's equations.[38]

Use equation (29) to get[39]

$$\begin{aligned} i\hbar\dot{E}(\ell,t) &= \big[E(\ell,t),H\big] = \frac{-\hbar^2}{2q^2}\epsilon^{D-4}\sum_{\square}\big[E(\ell,t),\,W(\square,t)\big] \\ i\hbar\dot{W}(\square,t) &= \big[W(\square,t),H\big] = \frac{1}{4q^2}\epsilon^{D}\sum_{\ell}\Big(E(\ell,t)\big[W(\square,t),\,E(\ell,t)\big] \\ &\qquad + \big[W(\square,t),\,E(\ell,t)\big]E(\ell,t)\Big), \end{aligned}$$

and then use the commutation relations (24) and (27) to get

$$i\dot{E}(\ell,t) = \frac{\kappa\hbar}{2q^2}\epsilon^{D-4}\left(\sum_{\square\ni\ell}W(\square,t) - \sum_{\square\ni\ell^{\text{rev}}}W(\square,t)\right) \tag{30}$$

$$\begin{aligned} i\hbar\dot{W}(\square,t) &= \frac{\kappa}{4q^2}\left(\epsilon^D\sum_{\ell\in\square}\Big(E(\ell,t)W(\square,t)+W(\square,t)E(\ell,t)\Big)\right. \\ &\qquad \left. - \epsilon^D\sum_{\ell^{\text{rev}}\in\square}\Big(E(\ell,t)W(\square,t)+W(\square,t)E(\ell,t)\Big)\right) \\ &= \frac{\kappa}{2q^2}\epsilon^D\sum_{\ell\in\square}\Big(E(\ell,t)W(\square,t)+W(\square,t)E(\ell,t)\Big). \end{aligned} \tag{31}$$

The sum $\sum_{\square\ni\ell}$ is over all directed plaquettes that include the directed link ℓ, and the sum $\sum_{\ell\in\square}$ is over all directed links that occur in the directed plaquette $\square$.

[38] Some of Maxwell's equations don't involve time derivatives. Sections 12 and 14 already showed how those are reproduced in this model.

[39] $\dot{X}$ denotes the derivative of X with respect to t.

21 Relationship to Maxwell's equations

To show that equation (30) is a lattice version of one of Maxwell's equations, use equation (16) and ignore the modulo-$2\pi\hbar$ ambiguity in the magnetic flux to get

$$\dot{E}(\ell,t) = \frac{\kappa}{q^2}\epsilon^{D-4}\sum_{\Box\ni\ell} B(\Box,t) + O(B^2).$$

Then use (9) and (20)-(21) to get

$$\begin{aligned}\dot{E}_j(\mathbf{x},t) &= \frac{\kappa}{q^2}\epsilon^{D-2}\left(\sum_k \big(B_{jk}(\mathbf{x},t) - B_{jk}(\mathbf{x}-\mathbf{e}_k,t)\big) + O(\epsilon^2)\right)\\ &= \frac{\kappa}{q^2}\epsilon^{D-1}\left(\sum_k \nabla_k B_{jk}(\mathbf{x},t) + O(\epsilon^2)\right)\end{aligned} \tag{32}$$

where ∇ is a lattice version of the gradient. To show that equation (31) is a lattice version of one of Maxwell's equations, use equations (16) and (20) on the right-hand side to get

$$i\hbar\dot{W}(\Box,t) = \frac{\kappa}{q^2}\epsilon^{D}\left(\sum_{\ell\in\Box} E(\ell,t) + O(\epsilon^2)\right).$$

Then use (16) and (20)-(21) again on the left-hand side to get

$$\begin{aligned}-\dot{B}_{jk}(\mathbf{x},t) &= \frac{\kappa}{q^2}\epsilon^{D-2}\big(E_j(\mathbf{x}) + E_k(\mathbf{x}+\mathbf{e}_j) - E_j(\mathbf{x}+\mathbf{e}_k) - E_k(\mathbf{x})\big)\\ &= \frac{\kappa}{q^2}\epsilon^{D-1}\big(\nabla_j E_k(\mathbf{x},t) - \nabla_k E_j(\mathbf{x},t) + O(\epsilon^2)\big)\end{aligned} \tag{33}$$

where ∇ is another lattice version of the gradient. When κ is given by (10), equations (32) and (33) agree with the zero-current case of Maxwell's equations as presented in article 31738, up to terms that are negligible when the resolution is low compared to ϵ.

22 Notes about the continuum limit

Equations (30)-(31) look more complicated than Maxwell's equations in smooth space, for two reasons. One reason is that space is being treated here as a lattice instead of as a continuum, so we have sums $\epsilon^D \sum_{\mathbf{x}} \cdots$ in place of integrals $\int d^D x \cdots$. Another reason is that these equations are nonlinear in the basic observables E and W, both explicitly and implicitly. The products of E with W on the right-hand side make them explicitly nonlinear, and a further nonlinearity is implicit in the constraint $W^\dagger W = 1$.

The fact that the equations of motion are nonlinear makes extracting this model's predictions more difficult. Even something as basic as the existence of a Lorentz-symmetric continuum limit governed by Maxwell's equations is far from obvious. As a concession, section 21 showed that equations (30)-(31) are a lattice version of Maxwell's equations, in the sense that they reproduce Maxwell's equations when the parameter ϵ is formally sent to zero, ignoring the noncommutativity of the operators and the periodicity of the magnetic flux. The periodicity of the flux does need to be taken into account when studying the continuum limit of the model as a whole, though. The rest of this section summarizes some insights from such studies.

For $D \geq 3$, studies using the path-integral formulation have shown that compact QED has a **Coulomb phase** with massless photons when the overall coefficient of the action is large enough. For a given ϵ, the coefficient of the action is proportional to $1/q^2$, just like the coefficient of the hamiltonian, so the Coulomb phase with massless photons occurs when q^2 is small enough (for fixed ϵ). Masslessness implies infinite correlation length, so this is evidence that the expected continuum limit exists.[40] For larger values of q^2, compact QED is in a **confinement phase** with no massless particles. When $D = 3$, the phase transition between the Coulomb phase and the confinement phase appears to be weakly first order. That means that the correlation length doesn't diverge in units of the lattice spacing ϵ when

[40] These statements are based on Frölich and Spencer (1982), section 2.11 (for $D = 3$) and remark 1 at the end of section 2.12 (for $D > 3$). Their analytic results agree with numerical studies for $D = 3$, recent examples of which include Lewis and Woloshyn (2018), Loveridge *et al* (2021), and and Loveridge *et al* (2021b).

the transition is approached from the confinement phase, so a strict continuum limit probably doesn't exist for the confinement phase[41] even though it does for the Coulomb phase.

For $D = 2$, compact QED is in the confinement phase for all nonzero values of q^2, so if a continuum limit with massless photons exists at all, it can only be for $q^2 \to 0$ (if ϵ is fixed).[42]

[41] Majumdar *et al* (2004), Espriu and Tagliacozzo (2003)

[42] The second-to-last paragraph in section 2 of Banks *et al* (1977) says that such a limit does exist, and this was more recently emphasized in footnote 47 on page 59 of Harlow and Ooguri (2021). Göpfert and Mack (1982) says that $q^2 \to 0$ gives a free massive scalar field instead. This discrepancy may come from using different continuum limits (holding different quantities fixed), but I haven't checked this carefully.

23 The period of the magnetic flux

In the model that was constructed in this article (compact QED), the magnetic flux is defined only modulo the period $2\pi\hbar$. Instead of using the units convention that was used in this article,[43] we can remove a factor of q from the definitions of the electric and magnetic fields to recover the electrical-engineering convention. With that convention, the period is $2\pi\hbar/q$. As explained in section 4, the constant q should be the magnitude of the smallest nonzero electric charge that the model includes after charged matter is included.[44] The smallest nonzero electric charge in the standard model of particle physics is 1/3 the charge of a proton. If we use this as the value of q, then the period is[45]

$$\frac{2\pi\hbar}{q} \approx 10^{-14} \text{ weber} \tag{34}$$

in standard international units. The assertion that magnetic flux is defined only modulo $2\pi\hbar/q$ means that for any given surface S, values of the magnetic flux $B(S)$ that differ from each other by an integer multiple of the quantity (34) are physically equivalent to each other.

For comparison, a typical value for the flux of the earth's magnetic field through one square meter is more than 10^{-5} weber,[46] which is enormous compared to (34), and other values we encounter routinely are even larger. In compact QED, such everyday values of the magnetic flux are physically indistinguishable from values smaller than (34)! This might seem to make compact QED inconsistent with reality, but when we think through it carefully, no inconsistencies can be found. Section 24 will explain why.

[43] The units convention used in this article is described in detail in article 26542.

[44] If a nonzero electric charge could be arbitrarily small, then the flux period would be infinite – the gauged group would be $\mathbb{R}$ instead of $U(1)$.

[45] The **flux quantum** that is famous in the study of conventional (BCS) superconductivity has the form $2\pi\hbar/q$, but in that case q is two times the magnitude of an electron's charge. References are cited in Loder *et al* (2007), which points out that the flux quantum can have a different value in unconventional superconductors.

[46] https://climate.nasa.gov/news/3105/

24 Flux periodicity and strong magnets

A strong electromagnet can lift a heavy load of metal.[47] This requires a magnetic field that is unambiguously strong. In standard electrodynamics with gauged group $\mathbb{R}$, this would imply that the magnetic flux must also be unambiguously large. However, the important question is whether compact QED is consistent with the real world, not whether compact QED is consistent with standard electrodynamics.

In compact QED, an unambiguously-strong magnetic field is compatible with the fact that the magnetic flux is always physically indistinguishable from one whose magnitude is less than (34). To see why these are compatible with each other, recall the relationship (20), repeated here for convenience:

$$\text{magnetic field} = \frac{\text{magnetic flux through } \square}{\epsilon^2} \tag{35}$$

where ϵ is the distance between neighboring lattice sites. The lattice is artificial, so when we treat space as a lattice for the purpose of defining the model, we should take ϵ to be much smaller than any practical measurement can resolve. As an example, suppose we choose $\epsilon \sim 10^{-20}$ meter, so that the lattice is indistinguishable from a continuum for most practical purposes. Then

$$\frac{2\pi\hbar/q}{\epsilon^2} \sim 10^{25} \text{ tesla}, \tag{36}$$

which is much greater than any of the magnetic field strengths that been measured so far.[48]

If we could take the strict $\epsilon \to 0$ limit of compact QED, then we could still use (16) to define the magnetic flux, which would still be periodic, and the relationship (19) (instead of (20)) could be used to define the magnetic field. Then the range of distinguishable values of the magnetic field strength would be unbounded, even

[47] To find examples, search online using the keywords *electromagnet scrap metal*.

[48] According to the text below equation (1) in Kong *et al* (2022), the magnetic field of a particular neutron star, with a strength of $\sim 10^{13}$ Gauss $= 10^9$ Tesla, became the new record-holder in 2022.

though the magnetic flux would still be periodic with period (34). A nontrivial strict limit $\epsilon \to 0$ might be obstructed when charged matter is included[49] but that's not a problem for physical applications, because (36) is plenty big enough already.

The argument in the preceding paragraph assumes that the thing that needs to be strong is the quantity (35), not the magnetic flux by itself. To understand why this is valid, remember that everything in standard electrodynamics is derived from a small set of fundamental equations (like Maxwell's equations and the Lorentz force equation), and those equations are spatially local: they relate quantities at the same point in space to each other. Some of those quantities involve derivatives, so we can integrate them to derive non-local relationships (like relationships involving the magnetic flux through a large surface), but everything starts with strictly local relationships. The same thing is true in QED,[50] even when space is treated as a lattice. The relationships may involve a few nearest-neighbor points (or links), but equations (30)-(31) show that they are still essentially local, even though they are nonlinear. Those equations do involve the magnetic flux through a plaquette, but only in the combination (35), and – here's the key – without any factors of ϵ that would cancel the factors of ϵ in the denominator of (35). To make this explicit, use the value (10) of κ in equations (32) and (33) to get

$$\begin{aligned}\dot{E}_j(\mathbf{x},t) &= \sum_k \nabla_k B_{jk}(\mathbf{x},t) + O(\epsilon^2)\\ -\dot{B}_{jk}(\mathbf{x},t) &= \nabla_j E_k(\mathbf{x},t) - \nabla_k E_j(\mathbf{x},t) + O(\epsilon^2).\end{aligned}$$

This shows that the time derivatives of the electric and magnetic fields depend only on the gradients of the electric and magnetic fields without any factors of ϵ except the ones used to define the magnetic field B_{jk} in (35). The absence of other factors of ϵ in these equations illustrates the fact that the ratio (35), not the flux by itself, is the thing that matters. This analysis didn't include charged matter, but a similar conclusion holds in that case.

[49] Göckeler *et al* (1998a) and (1998b)

[50] In QED, matter is not made of pointlike particles, but quantum matter fields still interact only locally with the quantum electromagnetic field.

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