Newton's Model of Gravity

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Abstract A good way to learn physics is to start with simple models and then to move on to better models that have more uses and fewer limitations. This article introduces a relatively simple model of gravity, Newton's model, which is good enough for many purposes. Students who are already familiar with Newton's model may still benefit from reading this article, because the perspectives used here are also useful when learning other subjects.

Contents

1 Notation

This article considers a system of several objects in D-dimensional space. The special case $D = 3$ corresponds to the real world. At any given time, the location of one object can be described using a list of D numbers:

$$\mathbf{x} = (x_1,\, x_2,\, ...,\, x_D).$$

These D numbers are called the **coordinates** of the object or the **components** of $\mathbf{x}$. The object can move, so the components of $\mathbf{x}$ can change with time. We can express this by writing $\mathbf{x}(t)$, but often I'll just write $\mathbf{x}$ even though it depends on time. The notation $\dot{\mathbf{x}}$, with an overhead dot, denotes the derivative of $\mathbf{x}$ with respect to time:

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \equiv \left(\frac{dx_1}{dt},\, \frac{dx_2}{dt},\, ...,\, \frac{dx_D}{dt}\right).$$

If $\mathbf{x}$ is an object's location, then $\dot{\mathbf{x}}$ is its velocity. Similarly, $\ddot{\mathbf{x}}$ denotes the object's accelration, the derivative of its velocity $\dot{\mathbf{x}}$ with respect to time.

Given two points $\mathbf{x}$ and $\mathbf{y}$, the quantity $\mathbf{x} + \mathbf{y}$ or $\mathbf{x} - \mathbf{y}$ is defined by the list of D sums or differences:

$$\mathbf{x} \pm \mathbf{y} = (x_1 \pm y_1,\, x_2 \pm y_2,\, ...,\, x_D \pm y_D).$$

This article uses a coordinate system in which the distance between two points $\mathbf{x}$ and $\mathbf{y}$ is

$$|\mathbf{x} - \mathbf{y}| \equiv \sqrt{\sum_n (x_n - y_n)^2}.$$

With this notation, we rarely need to write out the components explicitly. That's good, because we'll also use subscripts for a different purpose, namely to label the different objects. We'll write $\mathbf{x}_k$ for the location of the kth object. For each value of the index k (each object), $\mathbf{x}_k$ is a list of D components describing the location of that object in D-dimensional space. Remember: the subscript on the non-boldface symbol x_n refers to the nth component of $\mathbf{x}$, and the subscript on the boldface symbol $\mathbf{x}_k$ refers to the kth object.

2 Newton's model in D-dimensional space

Consider a system of several objects whose sizes are negligible compared to the distances between them. Let $\mathbf{x}_k$ be the location of the kth object. The objects can move, so their locations $\mathbf{x}_k$ can depend on time. Newton's model of gravity is a set of **equations of motion** that specify which motions are physically allowed.[1] In D-dimensional space for any $D \geq 3$, the equations of motion are

$$\boxed{\ddot{\mathbf{x}}_k = \sum_{j\neq k} M_j \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|^D}.} \tag{1}$$

The sum is over all objects except the kth one, so the denominator is never zero as long as the objects are all in different locations.

I'm using **natural units** (article 37431) in which Newton's gravitational constant G is equal to 1. In these units, the masses M_k have units of acceleration $\times$ distance^{D-1}. If desired, we could write $M_k = Gm_k$ where m_k has the conventional units of mass (kilograms), but for the purposes of this article, that would just clutter the equations without adding any new insight.

The jth term in the sum (1) has magnitude

$$M_j \frac{|\mathbf{x}_j - \mathbf{x}_k|}{|\mathbf{x}_j - \mathbf{x}_k|^D} = M_j \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^{D-1}}.$$

In the realistic case $D = 3$, this is inversely proportional to the square of the distance between the jth and kth objects. We call this an **inverse square law**. When $D = 4$, it's an inverse cube law instead, and so on. The reason for this D-dependence will be explained in section 7.

[1] Here, "physically allowed" means allowed in the simplified world that this simple model describes. For many purposes, the world that Newton's model describes is a good approximation to the real world. General relativity is a better model, but Newton's model is easier, so we might as well use Newton's model whenever we can.

3 Comments about the model

- The quantity $\mathbf{x}_j - \mathbf{x}_k$ is a vector directed from the kth object toward the jth object, so equation (1) says that gravity is attractive: the objects tend to accelerate toward each other.[2,3]

- The acceleration of the kth object depends only on the masses of the other objects, not on the mass of the kth object itself. This is what physicists mean when they say that all objects "fall at the same rate" in Newton's model of gravity.

- Equation (1) says that the acceleration $\ddot{\mathbf{x}}_k$ of the kth object at time t depends on the locations of all of the other objects at the same time t. This is what physicists mean when they say that Newton's model involves "action at a distance." In Newton's model, gravity propagates infinitely fast.[4]

- The model treats each object as a point. Real objects are not points, but the model can still be a good approximation when the distances between the objects are large compared to the sizes of the objects – in other words, when the sizes are negligible compared to the distances.

- The right-hand side of (1) is undefined whenever two objects have the same location. We don't need to worry about that, because the model isn't perfectly realistic anyway.

[2] This assumes that the masses are positive.

[3] The words "tend to" are in this sentence because in a system with multiple objects, the influences of different objects compete with each other. In a system with only two objects, the words "tend to" can be omitted.

[4] In general relativity, gravity propagates at a finite speed, which is more realistic.

4 Example of an exact solution

If we specify the locations and velocities of all of the objects at any one time, then equation (1) implicitly tells us what their locations will be at all other times. Making that implicit information explicit is called **solving** the equations of motion. Solving a system of coupled differential equations like (1) is usually too difficult for us. Much of the study of physics consists of finding special cases or special conditions under which we can solve the equations, or finding ways to extract the information we need from the equations without actually solving them. This section highlights one special case in which can write down at least one exact solution of the equations of motion.

In a system with only two objects, equations (1) reduce to the pair of equations

$$\ddot{\mathbf{x}}_1 = M_2 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^D} \qquad\qquad \ddot{\mathbf{x}}_2 = M_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^D}. \tag{2}$$

For one example of an exact solution, let R be a positive constant and define

$$\omega \equiv \sqrt{\frac{M_1 + M_2}{R^D}}.$$

Then, for any constant (time-independent) point $\mathbf{c}$, equations (2) are satisfied by[5]

$$\mathbf{x}_1(t) = \frac{M_2}{M_1 + M_2}\big(R\cos\omega t,\ R\sin\omega t,\ 0,\ 0,\ ...,\ 0\big) + \mathbf{c}$$
$$\mathbf{x}_2(t) = \frac{-M_1}{M_1 + M_2}\big(R\cos\omega t,\ R\sin\omega t,\ 0,\ 0,\ ...,\ 0\big) + \mathbf{c}.$$

This describes two objects in circular orbits about the point

$$\mathbf{c} = \frac{M_1\mathbf{x}_1 + M_2\mathbf{x}_2}{M_1 + M_2},$$

which is called their **center of mass**.[6]

[5] Notice that $|\mathbf{x}_1 - \mathbf{x}_2| = R$.

[6] The orbits are *stable* only if $D = 3$. This is one implication of **Bertrand's theorem**, which is reviewed in Goldstein (1980), *Classical Mechanics (second edition)*, Addison-Wesley, section 3-6, page 93.

5 Example of an approximate model

Consider a universe with N objects, and focus on objects 1 and 2. Suppose that:

- For $k > 1$, the kth objects are all either so far away or have so little mass that their net influence on object 1 is negligible.
- For $k > 2$, the kth objects are all either so far away or have so little mass that their net influence on object 2 is negligible.
- The influence of object 1 on object 2 is significant.

More precisely, suppose that we can choose a threshold A with units of acceleration such that:

- $\sum_{k>1} M_k/|\mathbf{x}_k - \mathbf{x}_1|^{D-1} \ll A$
- $\sum_{k>2} M_k/|\mathbf{x}_k - \mathbf{x}_2|^{D-1} \ll A$
- $M_1/|\mathbf{x}_1 - \mathbf{x}_2|^{D-1} \gtrsim A$.

Each term in these inequalities is the magnitude of one of the terms in (1). Neglecting terms with magnitude $\ll A$, equations (1) for objects 1 and 2 reduce to

$$\ddot{\mathbf{x}}_1 \approx 0 \qquad\qquad \ddot{\mathbf{x}}_2 \approx M_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^D}. \tag{3}$$

Instead of starting with the original N-object model and restricting to solutions that satisfy this approximation, we can consider an **approximate model** (also called an **effective model**) with only two objects whose equations of motion are

$$\ddot{\mathbf{x}}_1 = 0 \qquad\qquad \ddot{\mathbf{x}}_2 = M_1 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^D} \tag{4}$$

so that the approximation (3) is built into the model itself. Most (if not all) of the models we use in physics are like this – they are mere approximations to something else that is more accurate and comprehensive, even if we haven't discovered it yet. The model that we started with in this article (equations (1)) is like this, too.

6 Preview of the action principle

If $f(\mathbf{x})$ is a function of the location $\mathbf{x}$, then ∇f will denote the gradient of f with respect to $\mathbf{x}$. It has components

$$\nabla f \equiv \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_D}\right).$$

If f depends on the locations of multiple objects, then $\nabla_k f$ will denote the gradient with respect to the location of the kth object (the gradient with respect to $\mathbf{x}_k$).

With this notation, equations (1) can also be written

$$\boxed{M_k\,\ddot{\mathbf{x}}_k(t) = -\nabla_k V,} \tag{5}$$

where V is this function of the locations of all of the objects:[7]

$$\boxed{V \equiv \frac{-1}{D-2}\sum_k\sum_{j<k}\frac{M_j\,M_k}{|\mathbf{x}_j - \mathbf{x}_k|^{D-2}}.} \tag{6}$$

The sum is over all pairs of distinct objects: the condition $j < k$ ensures that each pair is counted only once. On the right-hand side of (5), the gradient $\nabla_k V$ is evaluated first using the definition (6) as though the $\mathbf{x}$s were independent of time, and then the $\mathbf{x}$s are evaluated at time t.

This way of writing the equations of motion forshadows the **action principle**, which is introduced in articles 33629 and 46044.

[7] The minus sign in equation (5) cancels the minus sign in equation (6), so we could make our lives slightly easier by omitting the minus sign from both equations, but we'll retain them to be consistent with a widespread convention.

7 Motivation for the D-dependence

Newton's model is an approximation to general relativity. For $D \geq 3$, deriving Newton's model from the natural D-dimensional version of general relativity leads to the D-dependence that is shown in equation (6) (equivalently, in equation (1)).[8]

Here, instead of showing the derivation from general relativity, I'll highlight just one indication that the the D-dependence shown in equation (6) is mathematically natural. Start with the quantity $1/|\mathbf{x}|^{D-2}$, as in equation (6). For any $D \geq 3$, we have the identity

$$\nabla^2 \frac{1}{|\mathbf{x}|^{D-2}} = 0 \qquad \text{whenever } \mathbf{x} \neq 0 \tag{7}$$

with

$$\nabla^2 \equiv \sum_n \left(\frac{\partial}{\partial x_n}\right)^2.$$

The identity (7) holds only if the D in the exponent is the same as the number of components of $\mathbf{x}$.

Here's a sketch of how that relates to general relativity. In the approximations that lead to Newton's model for $D \geq 3$, general relativity implies an equation of motion like (5) with

$$\nabla^2 V(\mathbf{x}) \propto \rho(\mathbf{x}), \tag{8}$$

where $V(\mathbf{x})$ describes the gravitational field at any point $\mathbf{x}$ in space, and $\rho(\mathbf{x})$ is the mass density (mass per unit volume) at that point in space. Real objects have a finite mass spread over a finite volume, but in applications where the distances between objects are much greater than the sizes of the objects, we can take a limit in which each object's mass is concentrated at a single point. In this limit, equation (8) leads to equation (6). Equation (7) is part of that story, because equation (7) shows that equations (8) and (6) are consistent with each other wherever mass is absent (wherever $\rho(\mathbf{x}) = 0$).

[8] Robinson (2006)

8 Spherically-symmetric body with finite size

In this article, we considered a system of several objects whose sizes are negligible compared to the distances between them. The identity shown in the previous section can be used to derive a result that can be useful when an object's size is *not* negligible. The result derived here is part of the **shell theorem**, which says (among other things) that an object with a spherically-symmetric distribution of mass might as well be pointlike as far as its gravitational effect on outside objects is concerned.

Consider a system of two objects, A and B, where object B has a continuous mass density $\rho(r)$ that depends only on the distance r from its center. This is what **spherical symmetry** means. Suppose that the object has radius R, so $\rho(r) = 0$ for $r > R$. Equations (5)-(6) say that the equation of motion for object A is

$$\ddot{\mathbf{x}} \propto \nabla V$$

with[9]

$$V(\mathbf{x}) \propto \int d^D y \; \rho\Big(|\mathbf{y} - \mathbf{c}|\Big) |\mathbf{x} - \mathbf{y}|^{2-D} \tag{9}$$

where $\mathbf{c}$ is the location of the center of object B. Terms in V representing the gravitational interaction between different parts of object B have been omitted from V because they don't contribute to the gradient of (6) with respect to $\mathbf{x}$, the location of object A. Spherical symmetry implies that the right-hand side of (9) must be a function only of $|\mathbf{x}|$, and the identity shown in the previous section implies $\nabla^2 V = 0$ because object A is outside object B, so we can infer $V \propto 1/|\mathbf{x}|^{D-2}$. This shows that an object with a spherically-symmetric distribution of mass might as well be pointlike as far as its gravitational effect on outside objects is concerned. Notice that the special D-dependence shown in equation (6) (equivalently, in equations (1)) played an essential role in the derivation.

[9] To see this, think of the integral over $\mathbf{y}$ as a sum over a continuum of values of the index j in equation (6), with ρ being the continuous version of M_j.

9 References

Robinson, 2006. "Normalization conventions for Newtons constant and the Planck scale in arbitrary spacetime dimension" https://arxiv.org/abs/gr-qc/0609060

10 References in this series

Article **33629** (https://cphysics.org/article/33629):
"Conservation Laws and a Preview of the Action Principle" (version 2022-02-05)

Article **37431** (https://cphysics.org/article/37431):
"How to Think About Units" (version 2022-02-05)

Article **46044** (https://cphysics.org/article/46044):
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