GNO Configurations of Gauge Fields

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Abstract Article 93302 describes a way to construct 't Hooft operators localized on $(d-3)$-dimensional submanifolds of d-dimensional spacetime in models with a quantum gauge field, where the gauged group G can be any compact connected Lie group. That construction uses a **GNO configuration** – a specially chosen Yang-Mills connection on a principal G-bundle. (A **Yang-Mills connection** is a connection that satisfies the Yang-Mills equations.) This article defines GNO configurations using a connection on a principal $U(1)$-bundle and a homomorphism from $U(1)$ into G. Specifying such a homomorphism is equivalent (modulo conjugation) to specifying an irreducible representation of the **Langlands dual** of the group G.

When $d=3$, these are the same configurations that were historically used to define the asymptotic (far from the centerpoint) structure of **GNO monopoles**, a generalization the idea of a $U(1)$ magnetic monopole to any compact connected G.

Contents

1 Introduction

This article is a supplement to article 93302. Let Γ be a submanifold of spacetime with codimension 3, and let the gauged group G be any compact connected Lie group. Article 93302 defines an 't Hooft operator localized in a tubular neighborhood[1] τ of Γ using a specially constructed principal G-bundle and connection ω, which are the subjects of this article.[2] These special configurations of the gauge field will be called **GNO configurations**.[3]

This article treats spacetime as a smooth manifold, and the connection must be defined everywhere in the base space $B = \tau \setminus \Gamma$.[4] The connection is defined using a homomorphism $\rho : U(1) \to G$ and a principal $U(1)$-bundle whose restriction to any 2-sphere in τ linked once with Γ is nontrivial.[5]

The construction in article 93302 treats spacetime as discrete. It uses a single local potential A to represent the connection ω. In smooth spacetime, a connection on a nontrivial principal bundle over B cannot be represented everywhere in B by a single local potential. Article 93302 exploits the fact that the connection can always be represented by a single local potential within a subset $U \subset B$ that covers all the links,[6,7,8] but having a well-behaved connection defined on the whole base space B is still important. Article 11617 explains why.

[1] Article 53600 defines **tubular neighborhood**.

[2] Section 5 reviews the concept of a connection on a principal G-bundle.

[3] Section 38 will explain the reason for this name.

[4] If a tubular neighborhood τ of Γ is trivial, then existence of a nontrivial principal $U(1)$-bundle over $\tau \setminus \Gamma$ is guaranteed (article 36626). Article 53600 describes some examples in which this condition is satisfied. Article 36626 addresses when such a bundle can be extended to $M \setminus \Gamma$, where M is the spacetime manifold. This is necessary for the resulting 't Hooft operator to have the desired properties in the smooth-spacetime limit, even though the construction in article 93302 only uses bundles over the smaller base space $\tau \setminus \Gamma$.

[5] This condition can be satisfied because Γ is excluded from the base space.

[6] A **link** is the line segment connecting neighboring points in discrete spacetime (article 46333).

[7] Article 11617 explains why such a local potential always exists.

[8] The local potential should be defined on all the links so it may be used to assign an element of G to each link through parallel transport.

2 Notation and conventions

- i denotes either $\sqrt{-1}$ (section 12) or an integer-valued index (section 26).
- d is the number of dimensions of spacetime. In this article, $d \geq 3$.
- d also denotes the exterior derivative (section 16).
- dx_k is the differential (a one-form) of the kth spacetime coordinate.
- $\mathcal{S}$ is a surface (a 2-dimensional manifold) (section 16).
- S is the action (section 25).
- Γ is a $(d-3)$-dimensional submanifold of spacetime on which an 't Hooft operator is nominally localized (section 1).
- τ is a tubular neighborhood of Γ (section 1).
- $*$ is a single point.
- $X \setminus Y$ is what remains of X after deleting $Y \subset X$.
- B is the base space for a principal bundle (section 5).
- ω is a connection on a principal bundle (section 7).
- A is a local potential (section 5).
- G is the gauged group (always a compact connected Lie group in this article).
- 1_G is the identity element of G.
- ρ is a homomorphism from $U(1)$ to G (section 12).
- μ is the matrix-valued GNO charge, also called magnetic charge (sections 12 and 20).

In this article, a local potential A is antihermitian (Lie-algebra-valued) instead of hermitian,[9] so the associated field strength F is also antihermitian.

[9] Article 76708

3 Overview

The construction of a GNO configuration for a compact connected gauged group G uses these ingredients:

- The first ingredient is a principal $U(1)$-bundle over the base space B described in section 1. This is covered in article 36626.
- The second ingredient is a connection $A_{U(1)}$ on that principal $U(1)$-bundle whose field strength is well-distributed, in the sense that it is not especially concentrated along any particular direction transverse to Γ. This condition will be defined precisely with the help of a Riemannian metric on the base space – a metric that has euclidean signature[10] but is not necessarily flat.
- The third ingredient is a homomorphism $\rho : U(1) \to G$.[11] This is used to convert the connection $A_{U(1)}$ to a connection A on a principal G-bundle. The result is a GNO configuration.

Instead of presenting the material in that order, this article uses a different order:[12]

- Part 1 presents the metric-independent material. Section 4 will give an outline for part 1.
- Part 2 presents the metric-dependent material, which is used to define the condition on the field strength that was mentioned above. This will be called the **metric-dependent condition**. Section 23 will give an outline for part 2.

[10] Section 24 will explain why a metric with euclidean signature (instead of lorentzian signature) is used for this.

[11] Kapustin and Witten (2007), section 6.2, text after equation (6.9); reviewed in Atanasov (2018), text below equation (6.12)

[12] This is done because the the metric-dependent material is relatively detail-heavy. Those details are deferred to part 2 to avoid unnecessary distractions along the way to the simpler metric-independent material.

4 Outline for part 1

Part 1 of this article introduces the idea of a GNO configuration without requiring the field strength to be well-distributed. This material does not involve any metric structure for spacetime. The smooth structure is sufficient. The metric-dependent condition will be imposed in part 2.

- Sections 5-6 review some prerequisite math concepts: principal G-bundle, (principal) connection, local potential, and field strength.[13]
- Sections 7-8 review how to represent a local potential and its field strength in terms of components and how local potentials defined in different parts of the base space are related to each other.
- Section 10 reviews the concept of a principal G-bundle associated with a given principal $\breve{G}$-bundle, where $\breve{G}$ is a subgroup of G.
- Sections 12-14 explain how to construct the type of local potential needed in article 93302 (minus the metric-dependent condition) using a connection on a principal $U(1)$-bundle and a homomorphism ρ from $U(1)$ into G.
- Sections 15-18 review a consistency condition for local potentials defined in different parts of the base space to be consistent with a single connection.
- Section 19 shows that homomorphisms ρ and ρ' related to each other by conjugation define the same 't Hooft operator.

Sections 20-22 cover additional material that is not needed in the construction, but it is included to help relate this article to other literature about the subject.

Section 23 will introduce part 2.

[13] The field strength won't be needed until part 2, but it is reviewed in part 1 to emphasize that its definition is metric-independent.

5 Principal bundles, connections, local potentials

In classical field theory, a gauge field is a connection on a principal G-bundle, usually represented by a local potential. This section is a brief reminder of what those words mean.[14]

Let G be a Lie group and B another smooth manifold called the **base space**. In this article, the base space will be a region of spacetime. The **fiber** $\tilde{G}$ is almost the group G: it's the same smooth manifold, and multiplying $\tilde{G}$ by an element of G rearranges the elements of $\tilde{G}$ just like it would rearrange the elements of G. The difference is that the elements of $\tilde{G}$ cannot be multiplied by each other. In particular, the fiber doesn't have any distinguished identity element. A **principal *G*-bundle** over B is a copy of $\tilde{G}$ for each point of B. The **total space** of the bundle is the smooth manifold made of all the fibers, one for each point of B. The bundle is called **trivial** if its total space is equivalent (homeomorphic) to the cartesian product $B \times \tilde{G}$. Nontrivial bundles look like $B \times \tilde{G}$ locally, but not globally.

If a smooth path in the total space is not tangent to any of the fibers, then it can be projected to a smooth path in the base space B by collapsing each fiber to a point. A given path in the base space can be lifted to many different paths in the total space, all of which project to the same path in the base space. A **connection** defines a family of **horizontal** paths in the total space with this property: if p is any path in the base space B and x is any point in the total space that projects to a point on p, then p is the projection of exactly one horizontal path through x. A connection can be represented as a Lie-algebra-valued one-form ω on the total space: horizontal paths are those whose tangent vectors v satisfy $\omega(v) = 0$.

A **section** is a smooth function from the base space to the total space. A section can be defined everywhere on the base space if and only if the bundle is trivial, but we can always define a **local section** on a part U of the base space over which the bundle is trivial. For each point $x \in U$, a local section σ selects one element $\sigma(x)$ of the fiber over that point. The pullback of ω by σ, denoted $A \equiv \sigma^*\omega$, is a one-form on the base space called a **local potential**. Changing the section σ changes A.

[14] Articles 70621 and 76708 review this material in detail.

6 The field strength

This section reviews the concept of the *field strength* of a gauge field.[15]

If a path p in the base space is closed, then any corresponding path in the total space starts and ends in the same fiber, possibly at different points. The **holonomy** around p is the element of G that relates those two points in the same fiber to each other when the path in the total space is horizontal.[16] *Horizontal* is defined by a connection, and we can think of specifying a connection as a way of assigning a holonomy to every closed path in the base space. The **curvature** of a connection is a Lie-algebra-valued 2-form that can be viewed as giving the holonomy around any infinitesimal path.[17]

Given a connection, a local potential A is the one-form given by using a local section σ to pull the connection one-form back to the base space.[18] Similarly, the **field strength** F is the two-form defined by using σ to pull the the curvature two-form back to the base space. The field strength F may be written in terms of the local potential A.

If the bundle is trivial, then a section σ can be defined everywhere on the base space B. If the bundle is not trivial, then a section can only be defined on a subset $U \subset B$ where the bundle is trivial. The local potential A is defined only in U. To represent the connection everywhere, we can cover B with overlapping charts and choose a section in each chart to get a local potential defined in each chart. The field strength F typically depends on a section just like A does, and it is typically defined only where A is defined.[19]

[15] Article 76708 reviews this in more detail.

[16] Section 5

[17] It is Lie-algebra-valued because elements of the Lie algebra may be viewed as infinitesimal deviations from the identity element of the group.

[18] Section 5

[19] The exception is when the group G is abelian: in that case, F turns out to be independent of the section (so it is gauge invariant), and it is defined everywhere on the base space.

7 The components of the field strength

Consider a region U of spacetime in which a connection can be represented by a single local potential. Within U, let x^k denote the kth spacetime coordinate, and let $A_k(x)$ be the kth component of the local potential one-form $A(x)$:

$$A(x) = \sum_k A_k(x)\, dx^k.$$

For each value of k and x, $A_k(x)$ is an element of the Lie algebra of the gauged group G.

Use a faithful matrix representation so the Lie bracket maybe be written as a commutator: $[X, Y] \equiv XY - YX$. Then the components $F_{jk}(x)$ of the field strength 2-form $F(x) = \sum_{j<k} F_{jk}(x)\, dx^j \wedge dx^k$ are related to the components of $A(x)$ by[20,21]

$$F_{jk}(x) = \partial_j A_k(x) - \partial_k A_j(x) + \big[A_j(x), A_k(x)\big]. \tag{1}$$

This may also be written

$$F_{jk}(x) = \big[D_j(x), D_k(x)\big] \tag{2}$$

with

$$D_k(x) \equiv \partial_k + A(x). \tag{3}$$

The matrix differential operator (3) will be called the **covariant derivative**.[22]

[20] Article 76708

[21] Each element of the Lie algebra is represented by an antihermitian matrix, and this article uses a convention in which A_k and F_{jk} are Lie-algebra-valued, so they are antihermitian-matrix-valued. Many sources – including some other articles in this series – include an extra factor of i to make them hermitian (e.g. real-valued when $G = U(1)$).

[22] More precisely, if $\psi(x)$ is an x-dependent column matrix representing an element of the fundamental representation of the gauged group G, then the covariant derivative of $\psi(x)$ has components $D_k(x)\psi(x)$. (Example: $\psi(x)$ could be one spacetime-spinor component of a quark field in quantum chromodynamics. A quark field has $n \times m$ components, where n is the number of components for a spinor representation of the Lorentz group, and m is the number of components for the fundamental representation of the gauged group G.)

8 Gauge transformations

The review in section 5 mentioned that elements of the group G can act on the fiber $\tilde{G}$ by multiplication. When G is nonabelian, the order of multiplication matters: with the convention in article 70621, the action of $g \in G$ on $p \in \tilde{G}$ is $p \mapsto pg$. In words, this is the **right action** of the group G on the fiber $\tilde{G}$.

Recall[23] that changing the local section σ changes the local potential A. This transformation of A is called a **gauge transformation**. One section can be converted to the other using the right-action of G. If the change of section is given by $\sigma(x) \mapsto \sigma(x)g(x)$, then the effect of the gauge transformation (change of local section) on A_k may be expressed as $D_k \to g^{-1}D_k g$.[24] Equation (2) then implies that its effect on F_{jk} is $F_{jk} \to g^{-1}F_{jk}g$. When G is abelian, the field strength F is invariant under gauge transformations.

The name *gauge transformation* is also used for a G-equivariant diffeomorphism of the total space that maps each fiber to itself.[24] A change of section can be implemented as a gauge transformation of that type, but a gauge transformation of that type cannot be implemented using the right-action of the gauged group when G is nonabelian.[24] In this article, *gauge transformation* refers only to a change of section, not to a morphism of the whole bundle, except in section 9 where a useful relationship between them will be highlighted.

[23]Section 5

[24]Article 76708

9 Gauge transformations and transition functions

A principal G-bundle may be assembled from trivial patches glued together using G-valued transition functions where the patches overlap.[25] Given local sections σ_1 and σ_2 defined in charts U_1 and U_2 that overlap each other, the transition function used to glue those two patches together may be chosen so that it converts σ_1 to σ_2 in $U_1 \cap U_2$. In this sense, transition functions and gauge transformations are related to each other.[26]

As an example, consider a principal $U(1)$-bundle over a 2-sphere. Cover the 2-sphere with two hemispheres that overlap at the equator. Transition functions in the same homotopy class define the same bundle over the 2-sphere.[27] The resulting bundle is nontrivial if and only if the transition function is homotopically nontrivial. Given a transition function and the two sections, composing the transition function with a homotopically trivial gauge transformation (in the bundle-morphing sense) doesn't change its homotopy class, so the composition may be used in place of the original transition function. We can choose that homotopically trivial bundle-morphing gauge transformation to make the new transition function convert one section to the other in the region where they overlap.[28]

For the rest of this article, *gauge transformation* refers only to a change of section, not to a morphism of the whole bundle, but the relationship between transition functions and gauge transformations (in the section-changing sense) will be important.

[25] Article 70621

[26] When G is nonabelian, transition functions are defined using the *left* action of G. Article 76708 explains how to relate this to a gauge transformation that uses the right action of G to convert one section to the other.

[27] Article 33600

[28] This section-equalizing function can't be smoothly extended over either hemisphere because it's homotopically nontrivial, which illustrates the fact that a nontrivial principal bundle does not admit a section defined on the whole base space (article 70621).

10 Principal G-bundle from a subgroup-bundle

Consider a principal G-bundle over a base space B. Denote the total space by P and the bundle projection by $\pi : P \to B$. Any such bundle (P, π, B, G) may be constructed by covering B with contractible charts and using G-valued transition functions to specify how the trivial G-bundles over each chart should be glued together.[29]

Given a subgroup $\breve{G} \subset G$,[30] we can construct a principal G-bundle over B using $\breve{G}$-valued transition functions. We could also use those same transition functions to construct a principal $\breve{G}$-bundle over B. Denote the total space of that $\breve{G}$-bundle by $\breve{P}$ and the bundle projection by $\breve{\pi} : \breve{P} \to B$. The bundles (P, π, B, G) and $(\breve{P}, \breve{\pi}, B, \breve{G})$ are related to each other by the fact that they were both constructed using the same $\breve{G}$-valued transition functions. The bundle $(\breve{P}, \breve{\pi}, B, \breve{G})$ might be nontrivial even if the associated G-bundle is trivial.[31]

[29] Article 70621

[30] Sections 12-13 will use a subgroup $\breve{G}$ that is isomorphic to $U(1)$. The notation $\breve{G}$, with a U-shaped accent over G, is motivated by that case.

[31] Example: take $B = S^2$ and $\breve{G} = U(1) \subset G = SU(n)$ and recall that all principal $SU(n)$-bundles over S^2 are trivial (article 33600).

11 Alternative definition without transition functions

Section 10 explained how to construct a principal G-bundle associated with a principal $\breve{G}$-bundle when $\breve{G}$ is a subgroup of G. In the approach used there, both bundles are assembled using ($\breve{G}$-valued) transition functions. Many different choices of transition functions may give the same bundle, so a definition that doesn't rely on transition functions is conceptually more appealing. This section reviews such a definition.

Given any sugbroup $\breve{G} \subset G$ and any principal $\breve{G}$-bundle $(\breve{P}, \breve{\pi}, B, \breve{G})$, we can define an **associated** principal G-bundle (P, π, B, G) using only the relationship $\breve{G} \subset G$.[32] To define it, first construct an associated fiber (not yet principal) bundle like this:[33]

- The total space is defined to be $P = \breve{P} \times_{\breve{G}} G$, which is the cartesian product $\breve{P} \times G$ modulo the equivalence relation $(\breve{p}z, z^{-1}g) \sim (\breve{p}, g)$ for all $z \in \breve{G}$.
- The projection $\pi : P \to B$ is defined by $\pi(\breve{p}, g) = \breve{\pi}(\breve{p})$.

To promote this fiber bundle to a principal G-bundle, define the right action[29] of G on P by

$$(\breve{p}, g)h \equiv (\breve{p}, gh) \qquad \text{for all } h \in G.$$

[32]This is a special case of the concept of an **associated fiber bundle** defined in Michor (2008), paragraph 18.7. The concept of an **associated vector bundle** defined in article 70621 is another special case.

[33]The resulting bundle (P, π, B, G) is the same as what we would get by using $\breve{G}$-valued transition functions to construct both $(\breve{P}, \breve{\pi}, B, \breve{G})$ and (P, π, B, G), as in the previous paragraph. The action of the transition functions on the fiber G is the *left* action (article 70621). The left action of $\breve{G}$ on G is defined by $g \mapsto zg$ for all $g \in G$ and $z \in \breve{G}$, where the product zg is the usual product in the group G.

12 Homomorphisms from $U(1)$ to G

Let G be any compact connected Lie group. Every such group G has at least one subgroup $\breve{G}$ isomorphic to $U(1)$. Let $\rho : U(1) \to G$ be a homomorphism whose image is such a subgroup.[34] The homomorphism ρ converts any given principal $U(1)$-bundle to a principal $\breve{G}$-bundle, which in turn gives a principal G-bundle as explained in sections 10-11.[35] If ρ is nontrivial (if it doesn't map the whole group $U(1)$ to the identity element of G), then its image in G is isomorphic to $U(1)$.

The group $U(1)$ is faithfully and irreducibly represented as the group of complex numbers z with unit magnitude: $|z| = 1$. More generally, the group G can be faithfully and irreducibly represented as a matrix group in which each matrix is unitary. Given these representations, the homomorphism ρ may be described as

$$\rho(e^{i\theta}) = e^{i\mu\theta} \qquad\qquad \theta \in \mathbb{R} \tag{4}$$

with a fixed matrix μ satisfying

$$e^{2\pi i\mu} = 1_G, \tag{5}$$

where 1_G is the identity element of G. Equation (5) expresses the fact that ρ must map the identity element of $U(1)$ to the identity element of G.[36] In a basis where μ is diagonal, this implies that the components of μ are integers, so equation (5) is called the **(Dirac or GNO) quantisation condition** in the literature about GNO configurations.[37]

When $G = U(1)$, the matrix μ has only one component. In that case, the quantisation condition (5) can be expressed as a flux quantisation condition: the flux on a 2-sphere linked with Γ is an integer multiple of 2π.[38]

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[34] The image of any homomorphism $\rho : U(1) \to G$ is either a subgroup isomorphic to $U(1)$ or the trivial subgroup consisting of only one element (the identity element).

[35] ρ doesn't need to be injective: it can map more than one element of $U(1)$ to the identity element of G. Section 21 will exploit this freedom.

[36] This is required because ρ is a homomorphism.

[37] Figueroa-O'Farrill (1998), section 6.2; Tong (2018), text after equation (2.80)

[38] Section 16

13 The local potential

Suppose that we have already chosen a principal $U(1)$-bundle over a base space B and a local potential $A_{U(1)}$ for that bundle defined everywhere in a region $U \subset B$. This section explains how to use a homomorphism $\rho : U(1) \to G$ to convert $A_{U(1)}$ to a local potential A on a principal G-bundle.

A connection on the $\breve{G}$-bundle gives an **induced connection** on the associated G-bundle. The definition won't be reviewed here[39] because we can get the desired result more directly, without defining a connection first. The local potential $A_{U(1)}$ is a one-form on U taking values in the Lie algebra of $U(1)$.[40] If we think of $U(1)$ as the group of complex numbers of the form $e^{i\theta}$ with $\theta \in \mathbb{R}$, then the Lie algebra of $U(1)$ is generated by i, so $A_{U(1)}$ is i times a real-valued one-form. The homomorphism ρ sends the generator i of the Lie algebra of $U(1)$ to a generator $i\mu$ of the Lie algebra of G, where μ is the matrix defined in section 12. The factor of i is included so that μ is a hermitian matrix whenever G is represented as a unitary matrix group. The Lie algebra of the subgroup $\breve{G}$ is generated by $i\mu$. The group homomorphism (4) corresponds to a Lie algebra homomorphism $i \to i\mu$. Applying this homomorphism to the local potential $A_{U(1)}$ gives a local potential

$$A \equiv \mu A_{U(1)} \tag{6}$$

for the G-bundle, as promised at the beginning of this section.

[39]Article 76708 defines the induced connection for an associated vector bundle. The same idea works for any associated bundle.

[40]The letter U by itself denotes a subset of the base space B, and $U(1)$ denotes the abelian Lie group homeomorphic to a circle (article 92035).

14 Example: $SU(n)$

As an example of the recipe described in section 13, suppose G is the matrix group $SU(n)$. A matrix representing an element of $SU(n)$ is unitary and has determinant 1, so μ must be hermitian and traceless. If we choose the homomorphism ρ so that the matrix μ is diagonal, then the condition $e^{2\pi i\mu} = 1$ says that the diagonal components of μ must be integers, and those integers must sum to zero so that μ is traceless. When used in equation (4), any matrix μ satisfying these conditions defines a homomorphism ρ from $U(1)$ to $SU(n)$, so we can use any such μ in equation (6).

15 Consistency with a connection

Section 8 implies that if two local potentials A and A' represent the same connection where their domains of definition overlap, then they must be related by

$$A - A' = h^{-1}\,dh \tag{7}$$

for some G-valued function h. Consider two local potentials A and A' defined by

$$A \equiv \mu A_{U(1)} \qquad\qquad A' \equiv \mu A'_{U(1)} \tag{8}$$

as in equation (6). These two local potentials will satisfy the condition (7) if both of these things are true:

- $A_{U(1)}$ and $A'_{U(1)}$ satisfy the $U(1)$ version of (7),
- the matrix μ satisfies (5).

To derive (7), suppose $A_{U(1)}$ and $A'_{U(1)}$ satisfy

$$A_{U(1)} - A'_{U(1)} = h^{-1}_{U(1)}\,dh_{U(1)}, \tag{9}$$

and choose h to be

$$h = \rho(h_{U(1)}). \tag{10}$$

Use $h_{U(1)} = e^{i\theta}$ in (10) to get $h = e^{i\mu\theta}$, which gives[41]

$$h^{-1}\,dh = \mu\,h^{-1}_{U(1)}\,dh_{U(1)}. \tag{11}$$

Use (8) and (11) in (9) to get (7).

[41]If μ satisfies (5), then h is a well-defined G-valued function.

16 The flux on a 2-sphere

To illustrate the consistency condition (7), this section uses it to derive a basic fact about principal $U(1)$-bundles. Any 2-form F can be integrated over any smooth oriented surface $\mathcal{S}$. If F is the field strength for a connection on a principal $U(1)$-bundle,[42] then the integral $i\int_{\mathcal{S}} F$ will be called the **flux**[43,44,45] on $\mathcal{S}$. This section shows that in a principal $U(1)$-bundle, the flux on any 2-sphere is 2π times an integer that depends only on the bundle and the 2-sphere,[46] independent of the connection.

Consider any 2-sphere $\mathcal{S}$ in the base space B. The base space may have more than 2 dimensions. Chose two distinct points p and p' in $\mathcal{S}$. We can cover the base space with contractible charts such that one chart M^+ covers $\mathcal{S}\setminus p$ and one chart M^- covers $\mathcal{S}\setminus p'$. Any principal $U(1)$-bundle over B restricts to a principal $U(1)$-bundle over $M^+\cup M^-$, so the constraint derived below for all principal $U(1)$-bundles over $M^+\cup M^-$ also applies to all principal $U(1)$-bundles over B.[47]

Start with the trivial principal $U(1)$-bundle over each chart, and construct the bundle over $M^+\cup M^-$ by gluing those trivial bundles together using the transition function h defined by equation (14) where the charts overlap.[48] A connection on this bundle can be represented by a pair of local potentials A^+ and A^- defined in the charts M^+ and M^- respectively, related to each other by the consistency condition (7) in the intersection $M^+\cap M^-$ where h is the transition function.[49] This will be used to derive the property of the flux mentioned above.

When the gauged group is $U(1)$, the field strength 2-form associated with a

[42] Sufficient conditions for a 2-form F to come from a connection on a principal $U(1)$-bundle are given in Moore (2010), section 1.8 (pages 36-37 and page 40).

[43] In a context where M is interpreted as 3-dimensional space (instead of as a 3-dimensional spacetime), this would be the *magnetic* flux.

[44] This article uses a convention in which F is antihermitian, so $i\int_{\mathcal{S}} F$ is a real number.

[45] Article 91116 includes a review of the concept of integrating an n-form over an n-dimensional oriented manifold.

[46] Article 36626

[47] Principal $U(1)$-bundles over B may be subject to other constraints, too.

[48] Article 33600

[49] Section 9

given local potential one-form A is $F = dA$.[50] Equation (7) combined with the fact that the one-form (14) satisfies $d(h^{-1}\,dh) = 0$[51] implies $dA^+ = dA^-$, so in the region where the two charts overlap, we can use either of the two local potentials in the relationship $F = dA$.

Choose a closed curve C on $\mathcal{S}$ that cuts $\mathcal{S}$ into two parts $\mathcal{S}^+$ and $\mathcal{S}^-$ contained in M^+ and M^-, respectively. Choose the orientations of those parts to be consistent with the orientation of $\mathcal{S}$, and choose the orientation of C so that $C = \partial\mathcal{S}^+$. Then $\partial\mathcal{S}^-$ is the same curve but with the opposite orientation. The local potentials A^+ and A^- are defined everywhere on $\mathcal{S}^+$ and $\mathcal{S}^-$, respectively, so we can use Stokes's theorem[52] calculate the flux like this:

$$\begin{aligned}\int_{\mathcal{S}} F &= \int_{\mathcal{S}^+} F + \int_{\mathcal{S}^-} F = \int_{\mathcal{S}^+} dA^+ + \int_{\mathcal{S}^-} dA^- \\ &= \int_{\partial\mathcal{S}^+} A^+ + \int_{\partial\mathcal{S}^-} A^- = \int_C A^+ - \int_C A^- \\ &= \int_C (A^+ - A^-) = \int_C h^{-1}\,dh.\end{aligned}$$

This shows that the result depends only on the transition function that defines the bundle. In particular, it's independent of the connection.

To evaluate the remaining integral, use the fact that the intersection $M^+ \cap M^-$ is homotopic to a circle, so the $U(1)$-valued transition function h may be written $h = e^{i\phi}$ for some real-valued function that goes from 0 to $2\pi n$ upon traveling once around the circle, where n is an integer. The function ϕ isn't smooth at the starting/ending point, and that's okay because we're only using it to evaluate an integral. That gives

$$\int_C h^{-1}\,dh = i\int_C d\phi = 2\pi\,n\,i.$$

This shows that the flux on $\mathcal{S}$ is $-2\pi\,n$, as claimed.

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[50] Article 03838

[51] Section 17

[52] Article 91116

17 A useful one-form

This section introduces a one-form that will be used in the examples in sections 16 and 37.

Use a coordinate system x_1, x_2, x_3 for $\mathbb{R}^3$. The one-form[53]

$$\alpha \equiv i\,\frac{x_1\,dx_2 - x_2\,dx_1}{x_1^2 + x_2^2} \tag{12}$$

is defined everywhere except on the line $x_1 = x_2 = 0$. Straightforward calculation[54] shows that the one-form α is **closed**, which means its exterior derivative is zero: $d\alpha = 0$. The one-form α may also be written

$$\alpha = h^{-1}\,dh \tag{13}$$

where h is the $U(1)$-valued function defined by

$$h(x_1, x_2, x_3) \equiv \frac{x_1 + ix_2}{\sqrt{x_1^2 + x_2^2}}. \tag{14}$$

We can make this look more familiar by defining a quantity ϕ by

$$h = e^{i\phi} \qquad\qquad \text{where } x_1^2 + x_2^2 \neq 0. \tag{15}$$

Substituting this into (13) and manipulating ϕ as though it were a smooth single-valued function would give

$$\alpha = i\,d\phi. \tag{16}$$

The one-form (13) is often written this way, but the notation (16) ϕ is *not* both smooth and single-valued (not even for $x_1^2 + x_2^2 \neq 0$), and α is not proportional to the exterior derivative of any such function.

[53] Here, $i = \sqrt{-1}$.

[54] When calculating exterior derivatives, remember that one-forms anticommute with each other. The exterior derivative of $f(x_1, x_2, x_3)\,dx_k$ is defined to be $df \wedge dx_k$, not $dx_k \wedge df$ (which would equal $-df \wedge dx_k$).

18 The flux on a 2-sphere: example

This section illustrates the general result that was derived in section 16. In this example, the base space is $\mathbb{R}^3$ with one point deleted.

Use a coordinate system (x_1, x_2, x_3) for $\mathbb{R}^3$, and let $*$ be the point $(0,0,0)$. Consider a principal $U(1)$-bundle over the base space $B = \mathbb{R}^3 \setminus *$. Cover the base space with this pair of contractible charts:

- The chart M^+ includes all of B except the half-line $(0,0,x_3)$ with $x_3 \geq 0$.
- The chart M^- includes all of B except the half-line $(0,0,x_3)$ with $x_3 \leq 0$.

In this context, the half-line excluded by one chart is called a **Dirac string**. Let $\mathcal{S}$ be any 2-sphere that encloses the deleted point $*$.[55] The derivation in section 16 showed that the flux on $\mathcal{S}$ may be written

$$\int_C h^{-1}\, dh \tag{17}$$

for some $U(1)$-valued function h defined in $M^+ \cap M^-$, where C is a closed curve that wraps once around the cylinder $M^+ \cap M^- \cap \mathcal{S}$. In this example, we can take h to be the function that was constructed in section 17. The quantity ϕ in equation (15) is not both smooth and single-valued everywhere in $M^+ \cap M^-$, but it is smooth and single-valued almost everywhere in $M^+ \cap M^-$, and that's enough to evaluate the integral (17). The result of the integral is $2\pi\, i$, so the flux is -2π.

If n is an integer, then the nth power of the $U(1)$-valued transition function used above is still $U(1)$-valued. Using this new function as the transition function would define a different principal $U(1)$-bundle (if $n \neq 1$), and the same calculation would show that the flux on $\mathcal{S}$ is $-2\pi n$.

If a 2-sphere $\mathcal{S}$ does not enclose the deleted point, then we can use a single local potential everywhere on $\mathcal{S}$, so Stokes's theorem says that the flux on such a surface $\mathcal{S}$ is zero: $\int_{\mathcal{S}} F = \int_{\mathcal{S}} dA = \int_{\partial\mathcal{S}} A = 0$ because $\partial\mathcal{S} = 0$.

[55] In this article, saying that a 2-sphere $\mathcal{S}$ **encloses the point** $*$ means $\mathcal{S}$ is the boundary of a ball $V \in \mathbb{R}^3$ that includes the point $*$ in its interior.

19 Invariance under conjugation

Choose any element $h \in G$, and let ρ with ρ' be two homomorphisms from $U(1)$ to G that are related to each other by

$$\rho'(g) = h^{-1}\rho(g)h \qquad\qquad \text{for all } g \in G. \tag{18}$$

In group theory, a transformation of this form is called **conjugation** by h. This section shows that if ρ and ρ' are related to each other by conjugation, then they define the same 't Hooft operator.[56]

Recall[57] that a gauge transformation is described by a function h from spacetime to the gauged group G. If h is constant (if it assigns the same element of G to every point in spacetime), then the gauge transformation replaces the original local potential one-form A with $A' \equiv h^{-1}Ah$. In the context of the construction described in section 13, this is the same as replacing the homomorphism ρ by the homomorphism ρ' defined in equation (18). Article 93302 shows that 't Hooft operators are gauge invariant, so this shows that if two homomorphisms ρ and ρ' related to each other by conjugation, then they define the same 't Hooft operator.

[56] Kapustin and Witten (2007), section 6.2, text after equation (6.9)

[57] Section 8

20 Topological charge and GNO charge

Let $\breve{G}$ be the image of the homomorphism ρ, as in the preceding sections. The restriction of the $\breve{G}$-bundle to a 2-sphere linked once with Γ may be nontrivial even if the G-bundle over that 2-sphere is trivial.[58] If the $\breve{G}$-bundle is nontrivial, the 't Hooft operator is said to have a nonzero **GNO charge**[59,60] (or **magnetic charge**).[61] If the G-bundle is also nontrivial, then the 't Hooft operator is also said to have a nonzero **topological charge**[62] (or **'t Hooft charge**).[63] If G is nonabelian, then the GNO charge may be nonzero even if the topological charge is zero.[64]

Instead of merely calling them zero or nonzero, both types of charge may be quantified. The possible values of the topological charge correspond to the elements of $\pi_1(G)$, the first homotopy group of G, because $\pi_1(G)$ classifies principal G-bundles over a 2-sphere.[65] The GNO charge is matrix μ defined in section 12, modulo conjugation[66] – or the list of integers in its diagonal form.[67] If G is nonabelian, then infinitely many different values of the GNO charge correspond to each value of the topological charge.[68]

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[58]Section 1 defined Γ.

[59]Kapustin and Seiberg (2014), section 2; Aitken *et al* (2018), abstract; Ang *et al* (2020), text after equation (1.2)

[60]Kapustin (2006b) uses the name **GNO charge** specifically for the quantity μ (section 4.2); also Kapustin (2006), equation (4.3) and the beginning of section 4.6.

[61]The name **magnetic charge** has been used for various quantities proportional to μ: it has been used for μ (Argyres and Ünsal (2012), text after equation (2.26)), for $\mu/2$ (Goddard *et al* (1977), equation (1.3) and the text after equation (1.1)), and for $\mu/2\pi$ (Figueroa-O'Farrill (1998), equation (6.10) and text after equation (6.7)).

[62]Coleman (1982), section 3.4

[63]Cherkis and Durcan (2009), text after equation (1)

[64]Section 14 describes an example with $G = SU(n)$. In that case, the G-bundle over $\tau \setminus \Gamma$ is trivial because all principal $SU(n)$-bundles over a 2-sphere are trivial (article 33600), but the $\breve{G}$-bundle is not, so the resulting operator has zero topological charge but nonzero GNO charge.

[65]Article 33600

[66]Section 19

[67]Sometimes a different normalization convention is used in which the charge is labeled by the half-integer components of the diagonal matrix $\mu/2$. This amounts to using a different normalization convention for the basis elements of the Lie algebra $\mathfrak{h}$ of a maximal torus H, called a **Cartan subalgebra** of the Lie algebra $\mathfrak{g}$ of G (article 91563).

[68]Sections 21-22; Gomis *et al* (2009), section 2

21 Classifying/labeling GNO configurations

If we start with a principal $U(1)$-bundle having flux 2π on any 2-sphere linked once with Γ, then we can use the conjugacy class of[69] the homomorphism $\rho : U(1) \to G$ to specify a GNO configuration and the corresponding 't Hooft operator.[70]

When $G = U(1)$, the homomorphism ρ is a homomorphism from $U(1)$ to itself. Any such homomorphism has the form $g \mapsto g^n$ for some integer n,[71,72] so when $G = U(1)$, we can use the integer n to label the 't Hooft operator.

Similarly, a homomorphism from $U(1)$ to any compact connected Lie group G is specified by the matrix μ defined in section 12. In a basis where μ is diagonal, the diagonal components of μ are integers. Much of the physics literature uses this list of integers (or $1/2\times$ these integers)[73] to label an 't Hooft operator.

Here's another variation. Choose any **maximal torus** H in G, a subgroup of the form $U(1)\times U(1)\times\cdots$ with the maximum possible number of $U(1)$ factors. This number is called the **rank** of G. Every element of G is in some maximal torus, and any two maximal tori in G are conjugate to each other,[74,75] so any homomorphism $\rho : U(1) \to G$ is conjugate to a homomorphism from $U(1)$ to H. This shows that GNO configurations are classified by principal H-bundles over S^2, where H is a fixed maximal torus for G.[76]

Another way of classifying GNO configurations (or 't Hooft operators) uses the concept of a holomorphic bundle over the complex projective line $\mathbb{C}\mathrm{P}^1$,[77] which is topologically a 2-sphere.[78] That perspective won't be reviewed here.

[69] Section 19

[70] Article 93302 uses this way of labeling 't Hooft operators.

[71] For $G = U(1)$ (or any abelian group), $g \mapsto g^{-1}$ is a homomorphism, so both signs of n are allowed.

[72] This illustrates why section 13 allowed the homomorphism ρ to be non-injective.

[73] Footnotes 61 and 67 in section 20

[74] Article 92035

[75] Section 19 defined *conjugate*.

[76] https://mathoverflow.net/questions/475625

[77] Kapustin and Witten (2007), section 6.2, text between equations (6.9) and (6.10)

[78] https://ncatlab.org/nlab/show/Riemann+sphere

22 GNO configurations and Langlands duality

Section 21 mentioned a few equivalent ways to classify GNO configurations. This section mentions another one: conjugacy classes of homomorphisms $\rho : U(1) \to G$ are classified by irreducible representations of $G^\vee$,[79,80] the **Langlands dual** $G^\vee$ of the gauged group G, and therefore so are GNO configurations (and 't Hooft operators).[81]

Article 40094 provides some information about Langlands dual groups, and article 22721 relates this to the pattern of Wilson and 't Hooft operators.

[79] Kapustin and Witten (2007), section 6.2, text after equation (6.9)

[80] The text around equations (4.3)-(4.4) in Kapustin (2006) relates this to μ.

[81] Atanasov (2018), observation 6.2.2

23 Outline for part 2

The definition of a GNO configuration has a metric-independent aspect and a metric-dependent aspect. Part 1 of this article described the metric-independent aspect. Part 2 will describe the metric-dependent aspect. This is used to impose one more condition on the connection: the field strength should be well-distributed. Here's an outline:

- Section 24 will define the *well-distributed field strength* condition and explain why it is expressed in terms of the euclidean Yang-Mills equations.
- Sections 25-27 review the Yang-Mills equations for an arbitrary compact connected gauged group G.
- Section 28 shows that the local potential A defined in equation 6 satisfies the Yang-Mills equations for G if the local potential $A_{U(1)}$ in that equation satisfies the Yang-Mills equations for $U(1)$.
- Section 29 considers the degree of generality of this family of solutions to the Yang-Mills equations.
- Sections 30-38 show that configurations that satisfy the Yang-Mills equations have *well-distributed field strength* as defined in section 24.

24 The metric-dependent condition

To complete the definition of GNO configurations that was started in part 1, one more condition should be imposed on the connection: the field strength should be well-distributed.[82] In smooth spacetime, this condition can be made precise if a metric with euclidean signature is given. This section introduces the precise condition.

If the metric is flat and Γ is straight,[83] then the condition could be expressed like this: if $\mathcal{S}$ is any 2-sphere that has constant radius from a point p on Γ, then the field strength for a connection on the principal $U(1)$-bundle on $\mathcal{S}$ should have the same rotational symmetry about p that $\mathcal{S}$ has. The field strength may depend on the radius of $\mathcal{S}$, but its magnitude should be the same in all directions from p transverse to Γ – the same everywhere on $\mathcal{S}$. A generic metric does not have any symmetry, though, so we need a more generally-applicable way to express the condition.

The quantum model is meant to exist in spacetime with a metric of lorentzian signature, but the model's construction in discrete spacetime relies on a technical device called *Wick rotation*.[84] When that device is used, the path integral near the smooth-spacetime limit is dominated by configurations that minimize the **euclidean action**. Section 25 will introduce the euclidean action.[85] For now, the important thing is that it is a gauge-invariant function of the connection, and it depends on a spacetime metric (with euclidean signature instead of lorentzian signature).

Sections 30-37 will show that if the metric on M is flat and Γ is straight, then configurations that minimize the euclidean action (among all configurations consistent with a given principal bundle on $M \setminus \Gamma$) have a field strength with

[82] For the construction of 't Hooft operators in article 93302, this prevents the field strength from being concentrated in the space between links and helps ensure that the resulting operator has the desired GNO charge in the smooth-spacetime limit.

[83] Section 1 defined Γ.

[84] Article 89053

[85] Article 89053 introduces the euclidean action in a discrete version of flat spacetime. Section 25 will introduce the euclidean action in smooth spacetime with an arbitrary metric.

3d spherical symmetry about Γ. Any riemannian manifold M and codimension 3 submanifold $\Gamma \subset M$ look approximately flat and straight in a small enough neighborhood of a point on Γ, so an appropriate general definition of *well-distributed field strength* is to require the connection to minimize the euclidean action among all connections consistent with the given principal bundle on $M \setminus \Gamma$.

A necessary condition for a connection ω to minimize the action $S[\omega]$ is that the variational derivative of $S[\omega]$ with respect to ω should be zero when evaluated at a minimum. This condition can be expressed as a system of partial differential equations for the connection (expressed in terms of local potentials) called the **(euclidean) Yang-Mills equations**. Section 26 will derive the Yang-Mills equations for any compact connected gauged group G.

A connection that satisfies the Yang-Mills equations is called a **Yang-Mills connection**.[86] Section 28 will show that if a local potential $A_{U(1)}$ represents a Yang-Mills connection for $U(1)$, then the local potential A defined by (6) represents a Yang-Mills connection for G.

[86] Jiang (2020), definition 2.10

25 The action

Partition the base space B into contractible charts that intersect only at their boundaries, and let Ω denote this set of charts. Within a given chart $U \in \Omega$, a connection ω can be represented by a single local potential.[87] The euclidean action for the gauge field over the base space $B = \cup_{U\in\Omega} U$ is this function of the connection ω:

$$S[\omega] = \sum_{U\in\Omega} S_U[A], \tag{19}$$

where $S_U[A]$ is a function of a local potential A that represents the connection ω in U. The function $S_U[A]$ is[88]

$$S_U[A] = -\kappa \int_U d^d x \; |\det g|^{1/2} \sum_{j,j',k,k'} \text{trace}\left(g^{jj'} g^{kk'} F_{jk} F_{j'k'}\right) \tag{20}$$

where

- F is the field strength for A,
- $\det g(x)$ is the determinant of the metric tensor with components $g_{jk}(x)$,
- $g^{jk}(x)$ are the components of the inverse metric tensor,
- κ is a positive real-valued constant, so the fact that F is antihermitian[89] implies $S_U[A] \geq 0$.[90]

$S_U[A]$ is gauge invariant, so it depends on the As only through the connection ω.

[87] Every principal G-bundle over a contractible chart is trivial.

[88] Article 89053 derives the flat-spacetime case of this from the discrete-spacetime model. Article 11475 illustrates the reason for the factor $|\det g|^{1/2}$ in the context of a simpler model in not-necessarily-flat spacetime.

[89] Section 2

[90] If the coefficient κ is allowed to be a complex number, then equation (20) is also valid for a lorentzian metric (article 89053).

26 The Yang-Mills equations (YME)

The Yang-Mills equations come from extremizing[91] the action among connections on a given principal G-bundle. We can implement this by working in one of the charts $U \in \Omega$ and extremizing with respect to the components of the local potential in that chart. If we write $A_k = \sum_a A_{k,a} T^a$ where the matrices T^a are a basis for the Lie algebra, then the coefficients $A_{k,a}$ are independent variables,[92] so the extremization condition may be expressed like this:

$$\frac{\delta}{\delta A_{k,a}(x)}\, S_U[A] = 0. \tag{21}$$

To derive the Yang-Mills equations, use equation (2) to write the quantity $F_{jk}F_{j'k'}$ in (20) like this:

$$F_{jk}F_{j'k'} = [D_j, D_k][D_{j'}, D_{k'}]$$

and write

$$D_k = I\partial_k + \sum_a T^a A_{k,a}$$

where I is the identity matrix. Use

$$\frac{\delta}{\delta A_{i,a}(x)}\, [D_j(y), D_k(y)][D_{j'}(y), D_{k'}(y)] = [\delta^d(x-y)\delta^i_j T^a, D_k][D_{j'}, D_{k'}] + (\text{3 more terms})$$

where $\delta^i_j T^a$ is the result of applying the variational derivative $\delta/\delta A_{i,a}(x)$ to D_j, and each of the "3 more terms" has the same structure but with the variational derivative applied to one of the other $D_\bullet$ factors. The index-symmetries of (20) imply that all four of these terms make identical contributions to the final result,

[91] Minimization (section 24) is a special case of extremization.

[92] The consistency condition (7) between local potentials in different patches is used to select appropriate solutions of the Yang-Mills equations, not a constraint to be imposed during the derivation of the Yang-Mills equations.

so we only need to work out one of them. The result is

$$\frac{\delta}{\delta A_{i,a}(x)} S_U[A] \propto \int_U d^d y \ |\det g|^{1/2} \sum_{j,j',k,k'} \text{trace}\left(g^{jj'}g^{kk'}[\delta^d(x-y)\delta^i_j T^a, D_k][D_{j'}, D_{k'}]\right)$$
$$\propto \sum_{j',k,k'} \text{trace}\left(T^a\big[D_k, |\det g|^{1/2}\, g^{ij'}g^{kk'}[D_{j'}, D_{k'}]\big]\right).$$
$$= \sum_k \text{trace}\left(T^a\big[D_k, |\det g|^{1/2}\, F^{ik}\big]\right)$$

with

$$F^{jk} \equiv \sum_{j',k'} g^{jj'}g^{kk'}F_{j'k'}. \tag{22}$$

On the first line, the quantities g and D are functions of y. On the second line, they are functions of x. Imposing equation (21) for every value of the index a gives

$$\sum_k \big[D_k, |\det g|^{1/2}F^{jk}\big] = 0. \tag{23}$$

These are the Yang-Mills equations. They may also be written

$$\sum_k \left(\nabla_k F^{jk} + \big[A_k, F^{jk}\big]\right) = 0.$$

where ∇ is the Levi-Civita covariant derivative[93] for the given the spacetime metric.[94] In this form, the metric-dependence is implicit in ∇ and in equation (22).

[93] This derivative is "covariant" with respect to the spacetime metric, not with respect to the gauged group. D_k is "covariant" with respect to the gauged group, not with respect to the spacetime metric. A derivative that incorporates both forms of covariance can be defined, but it won't be needed in this article because the only example worked out in this article uses a flat metric (section 30-36).

[94] Article 03519

27 Example: the YME in flat spacetime

Consider a flat euclidean metric[95] in a coordinate system where

$$\sum_{j,k} g_{jk} dx^j\, dx^k = \sum_k (dx^k)^2.$$

Then equation (23) becomes

$$\sum_j [D_j,\ F_{jk}] = 0. \tag{24}$$

The relationship $F_{jk} = [D_j, D_k]$ (equation 2) implies

$$[D_i,\ F_{jk}] + [D_j,\ F_{ki}] + [D_k,\ F_{ij}] = 0. \tag{25}$$

When $G = U(1)$, the covariant derivative D_k reduces to ∂_k, and then equations (24)-(25) are mathematically the same as Maxwell's equations for a static magnetic field in d-dimensional space. The interpretation is different, though, because but here the d-dimensional manifold is spacetime (so F is the electromagnetic field, not just the magnetic field)[96] with a metric that happens to be euclidean instead of lorentzian for the reason given in section 24. The charge-density term is absent because the model with action (20) doesn't include any charged matter.

[95] Section 24 explains why the metric has euclidean signature even though the manifold represents spacetime.
[96] Article 31738

28 Transferring solutions from $U(1)$ to G

This section shows that a homomorphism $\rho : U(1) \to G$ converts a solution of the $U(1)$ version of the Yang-Mills equations to a solution of the G version of the Yang-Mills equations.

If A is given by equation (6), then every matrix in the Yang-Mills equations is proportional to μ at every point in spacetime, so they all commute with each other. When the commutators are all zero, the covariant derivative D_k reduces to ∂_k, so[97]

$$\sum_k \left[D_k,\ |\det g|^{1/2} F^{jk}\right] = \partial_k \left(|\det g|^{1/2} F^{jk}\right) = \mu\,\partial_k \left(|\det g|^{1/2} F^{jk}_{U(1)}\right)$$

and

$$[D_i,\ F_{jk}] = \partial_i F_{jk} = \mu\,\partial_i (F_{U(1)})_{jk}.$$

This shows that μ factors out of the Yang-Mills equations (24)-(25) when A is given by equation (6).

The transferability of solutions from $U(1)$ to G is a special property of the action (20). If the action were not homogeneous in F, then the resulting equation of motion would not be homogeneous in μ, so the μ-dependence would not factor out.

[97] The last step uses $[A_j, A_k] = 0$, which gives $F_{jk} = [D_j, D_k] = \partial_j A_k - \partial_k A_j = \mu (F_{U(1)})_{jk}$.

29 How general is this family of solutions?

Section 28 showed that applying the homomorphism ρ to a solution of the Yang-Mills equations for $U(1)$ gives a solution of the Yang-Mills equations for G. According to one source,[98] "on $\mathbb{C}\mathrm{P}^1$ every solution of the Yang-Mills equations is equivalent to one of the abelian solutions used to construct 't Hooft operators." If solutions of the form (6) exhaust all solutions of the Yang-Mills equations (modulo gauge transformations) compatible with a given principal G-bundle, then the Yang-Mills equations for G could be used as a sufficient condition for a GNO configuration in addition to being imposed as a necessary condition. This would eliminate the need to start with a $U(1)$ connection. That seems to be consistent with this statement, at least for the case $M = \mathbb{R}^4$:[99] "In a given theory, an 't Hooft operator creates a codimension three singularity for the fields that appear in the classical action. The only restriction on the admissible codimension three singularities created by an 't Hooft operator is that they solve the equations of motion of the theory in $\mathbb{R}^4 \setminus C$." (Their C corresponds to this article's Γ.)

[98]Kapustin and Witten (2007), section 6.2, text before equation (6.10)
[99]Gomis *et al* (2009), section 2

30 Minimizing the action on $\mathbb{R}^d \setminus \mathbb{R}^{d-3}$

Take M to be d-dimensional flat euclidean space $\mathbb{R}^d$, and use a coordinate system $(x_1, ..., x_d)$ in which geodesics are linear. Take $\Gamma \simeq \mathbb{R}^{d-3}$ to be the submanifold defined by $x_1 = x_2 = x_3 = 0$, and consider a principal G-bundle on $M \setminus \Gamma$.[100] Sections 33-36 will show that if $G = U(1)$, then among all configurations of the gauge field consistent with the given bundle, those that minimize the euclidean action have a field strength with 3-dimensional spherical symmetry about Γ, as promised in section 24. Section 38 will use equation (6) to apply this result to any compact connected gauged group G.

First, we need to define the minimum-action condition. The action nominally involves an integral over the non-compact manifold $M \setminus \Gamma$. In a principal $U(1)$-bundle, the flux on any 2-sphere linked once with Γ is fixed, so the field strength grows without bound when approaching Γ arbitrarily closely. As a result, the integral over the whole manifold $M \setminus \Gamma$ is undefined (infinite). To define the minimum-action condition, we can restrict the domain of integration to the compact region $B(r_{\min}, r_{\max}, s) \subset M \setminus \Gamma$ defined by

$$r_{\min}^2 \le x_1^2 + x_2^2 + x_3^2 \le r_{\max}^2 \qquad |x_k| \le s \text{ for } k \in \{4, ..., d\}$$

for positive real numbers $r_{\min}$, $r_{\min}$, and s. The integral over this compact domain is finite, so the minimum-action condition is well-defined for the action in this domain. This regularization scheme is in effect throughout the following sections, but it won't be written explicitly. We can take $r_{\min}$ to be arbitrarily small and take $r_{\min}$ and s to be arbitrarily large, so we are effectively minimizing the euclidean action over the whole non-compact manifold $M \setminus \Gamma$.

First consider the case $d = 3$, so $M = \mathbb{R}^3$ and Γ is a single point $*$. The base space is $B \equiv M \setminus \Gamma \simeq \mathbb{R}^3 \setminus *$. Given any connection on a principal $U(1)$-bundle over this base space, let F denote the corresponding field strength 2-form. Sections 16-18 reviewed the fact that the flux on any 2-sphere linked once with Γ is $2\pi n$ for some integer n that depends only on the bundle. For any given flux on a given

[100] In this case, a tubular neighborhood τ of Γ (section 1) is homeomorphic to $\mathbb{R}^d$, so we can take τ to be all of M.

2-sphere, F may or may not be well-distributed over that 2-sphere. It could all be concentrated in a small neighborhood of one point on the 2-sphere and be zero everywhere else on the 2-sphere. The goal is to show that if a connection on this bundle minimizes the euclidean action, then F has spherical symmetry. Here's an outline:

- For $G = U(1)$, section 31 introduces a condition on F that will be called the *flux condition*. Section 31 also derives a consequence of this condition and relates it to the Yang-Mills equations.

- Section 32 describes a family of 2-forms F on $\mathbb{R}^3 \setminus \Gamma$ that satisfies the Yang-Mills equations and the flux condition and has spherical symmetry. F is not yet required to be the field strength of any connection.

- As a warm-up for section 34, section 33 shows that if $\mathcal{S}$ is a 2-sphere with the standard constant-curvature euclidean metric and if the flux on $\mathcal{S}$ is fixed, then the 2-form F that minimizes the euclidean action on $\mathcal{S}$ has spherical symmetry.

- Section 34 shows that the 2-form F described in section 32 minimizes the action among all 2-forms satisfying the flux condition with a given value of c. Section 34 also shows that the 2-form F described in section 32 is the *only* one that minimizes the action subject to the flux condition.

- Sections 35-37 show that requiring F to be the field strength of a connection selects a subset of those spherically-symmetric configurations, namely the ones for which the flux is 2π times an integer.

These results can be extended to any number of dimensions $d \geq 3$ by taking the connection to be independent of the last $d - 3$ coordinates $x_4, ..., x_d$. Altogether, this shows that among configurations consistent with a given principal $U(1)$-bundle on $\mathbb{R}^d \setminus \mathbb{R}^{d-3}$, those that minimize the euclidean action have spherical symmetry about Γ, as promised in section 24. Section 38 extends the results to any compact connected G.

31 The flux condition and the Yang-Mills equations

Set $B = M \setminus \Gamma$. When $G = U(1)$, the euclidean action (20) may be written

$$-\int_B F \wedge \star F, \tag{26}$$

where F is the field strength 2-form. This is positive for all nonzero F.[101]

Section 25 defined the action as a functional of a connection. For some purposes, we can think of it as a functional of the 2-form F instead, without constraining F to be the field strength of a connection. With no constraints on F, the action would be minimized only by $F = 0$. Section 34 will minimize the euclidean action subject to the **flux condition** $F \in \Phi_c$, where Φ_c is the set of 2-forms for which the flux on every topological 2-sphere enclosing the point Γ is equal to a given real number c.[102] This section derives a consequence of the flux condition and relates it to the Yang-Mills equations.

The flux condition implies $dF = 0$. To deduce this, first use the flux condition to infer that the flux is zero on the boundary ∂V of every compact 3-ball V.[103] Stokes's theorem says

$$\int_{\partial V} F = \int_V dF,$$

so if the left side is zero for all V, then $dF = 0$.

If F minimizes the action subject to a flux condition, then F satisfies the Yang-Mills equations. To deduce this, consider a contractible chart $U \subset B$ of the base space B. Within U, any 2-form that satisfies $dF = 0$ may be written as $F = dA$ for some one-form A defined in U, so minimizing the action subject to a flux condition implies unconstrained minimization of the action with respect to A, which in turn gives the Yang-Mills equations.

[101] To deduce this, write $F = \sum_{j,k} F_{jk}\, dx_j \wedge dx_k$ and recall that F is antihermitian (section 2).

[102] Footnote 55 in section 18 defined *encloses*.

[103] Let $\mathcal{S}$ be a 2-sphere that encloses Γ (footnote 55) and whose intersection with V is a disk. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the 2-spheres obtained by rerouting $\mathcal{S}$ so that it either just barely contains just barely excludes the interior of V, respectively. Then the integral over ∂V is the same as the difference between the integrals over $\mathcal{S}_1$ and $\mathcal{S}_2$.

32 A solution for the field strength

Set $G = U(1)$. This section describes a 2-form F that satisfies the Yang-Mills equations on $U \subset \mathbb{R}^3 \backslash *$ without requiring F to be the field strength for a connection on a principal $U(1)$-bundle.

For any d, the Yang-Mills equations for $U(1)$ in flat d-dimensional spacetime may be written[104]

$$d(\star F) = 0 \qquad\qquad dF = 0 \tag{27}$$

where the d in these equations is the exterior derivative (not the number of dimensions) and $\star\omega$ is the Hodge dual[105] of a differential form ω. The Hodge dual depends on the metric. Let r be the distance of the point x from the origin. The identities

$$d\,\frac{1}{r} = \frac{-dr}{r^2} \qquad dr = \frac{\sum_k x_k\, dx_k}{r} \tag{28}$$

hold for all $r \neq 0$ in any number of dimensions. In 3 dimensions, the identities

$$\star\, dr = \frac{x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2}{r} \qquad \sum_k \partial_k \frac{x_k}{r^3} = 0 \tag{29}$$

also hold. These identities may be used to show that if $d = 3$, then equations (27) are satisfied by[106]

$$F(x) = \frac{i\mu}{2}\left(\star\, d\,\frac{1}{r}\right) \tag{30}$$

for all $r \neq 0$ and for any $\mu \in \mathbb{R}$. Clearly, the integral of this F over any sphere of constant r has magnitude $2\pi\mu$, and then the second of equations (27) implies that is has this same flux on any 2-sphere that encloses the point $r = 0$. This solution of equations (27) is called a **magnetic monopole**.

[104] The components of these two equations are $\sum_j \partial_j F_{jk} = 0$ and $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$, respectively. These are equations (24)-(25) with $G = U(1)$.

[105] Article 91116 reviews the definition of **Hodge dual**.

[106] Kapustin and Witten (2007), equation (6.7)

33 Minimizing the action on a standard sphere

Set $G = U(1)$. This section shows that if the base space is a 2-sphere $\mathcal{S}$ with the standard constant-curvature euclidean metric and if the flux on this sphere is fixed, then the 2-form F that minimizes the euclidean action has constant magnitude.[107] In this section, the 2-form F is not required to be the field strength of any connection.

Any 2-form F on $\mathcal{S}$ may be written $F = F_0 + F_1$ where F_0 has constant magnitude and $\int_{\mathcal{S}} F_1 = 0$. The euclidean action is

$$-\int_{\mathcal{S}} F \wedge \star F = \int_{00} + \int_{01} + \int_{10} + \int_{11}$$

with

$$\int_{jk} \equiv -\int_{\mathcal{S}} F_j \wedge \star F_k.$$

To evaluate the cross-terms $\int_{01}$ and $\int_{10}$, use the fact that $\star F_j$ is a 0-form to get $F_0 \wedge \star F_1 = F_1 \wedge \star F_0$, and use the fact that F_0 is constant to get $\int_{10} = 0$ from the assumed property of F_1. This leaves

$$-\int_{\mathcal{S}} F \wedge \star F = \int_{00} + \int_{11}.$$

The terms $\int_{00}$ and $\int_{11}$ are both nonnegative, and F_1 does not contribute to the flux, so minimizing the euclidean action subject to the given flux condition gives $F = F_0$, which has constant magnitude, as claimed.

Intuitively, the action (a quadratic function of F) is minimized by distributing F over the sphere as evenly as possible. This is analogous to the fact that among lists $f_1, f_2, ..., f_N$ of real numbers that satisfy $\sum_n f_n = 1$, the list that minimizes $\sum_n f_n^2$ is the one with $f_1 = f_2 = \cdots = f_N$.

[107] This is a warm-up for section 34, which will derive an analogous result for the base space $\mathbb{R}^3 \setminus *$.

34 Minimizing the action on $\mathbb{R}^3 \setminus *$

Set $G = U(1)$ and take the base space to be $B = \mathbb{R}^3 \setminus \Gamma$ where Γ is a single point. This section determines the 2-form F that minimizes the euclidean action (26) among all 2-forms satisfying a flux condition $F \in \Phi_c$.[108] In this section, the 2-form F is not required to be the field strength of any connection.

Choose any $F \in \Phi_c$. Write it as $F = F_0 + F_1$, where F_0 is the solution described in section 32 with this same value of the flux on any 2-sphere enclosing the point Γ. Then $F_1 \equiv F - F_0$ is a 2-form with zero flux on every 2-sphere, including 2-spheres that enclose Γ. The goal is to show that if F minimizes the euclidean action subject to a flux condition, then $F_1 = 0$. Just like in section 33, write the action for F as

$$-\int_B F \wedge \star F = \int_{00} + \int_{01} + \int_{10} + \int_{11} \tag{31}$$

with

$$\int_{jk} \equiv -\int_B F_j \wedge \star F_k.$$

To evaluate the cross-terms $\int_{01}$ and $\int_{10}$, write F_0 and F_1 as linear combinations of the coordinate 2-forms $dx_a \wedge dx_b$ to deduce $F_0 \wedge \star F_1 \propto F_1 \wedge \star F_0$, so

$$\int_{01} \propto \int_{10}. \tag{32}$$

Equation (30) gives

$$\int_{10} \propto \int_B F_1 \wedge d\,\frac{1}{r} = \int_B d\,\frac{F_1}{r} \tag{33}$$

because $dF_1 = 0$.[109] Now recall the implied regularization scheme that was established in section 30. Here, the manifold B is 3-dimensional, so the boundary ∂B of the compact region defined there consists of two 2-spheres: an inner 2-sphere $\mathcal{S}_{\min}$,

[108] Section 31

[109] Section 31

and an outer 2-sphere $\mathcal{S}_{\max}$. With that understanding, applying Stokes's theorem to the right side of equation (33) gives

$$\int_{10} \propto \int_{\mathcal{S}_{\max}} \frac{F_1}{r} - \int_{\mathcal{S}_{\min}} \frac{F_1}{r}. \tag{34}$$

The radius r is constant on each of these 2-spheres, and the definition of F_1 implies that the integral of F_1 over any 2-sphere is zero, so the integrals on the right side of (34) are both zero. This gives

$$\int_{10} = 0. \tag{35}$$

Combine equations (31), (32), and (35) to get

$$-\int_B F \wedge \star F = \int_{00} + \int_{11}.$$

The two quantities on the right side are both positive if $F_1 \neq 0$, and F_1 does not affect the value of the flux on any 2-sphere, so taking $F_1 = 0$ minimizes the action among all 2-forms satisfying the given flux condition. Altogether, this shows that the 2-form F_0 described in section 32 minimizes the action among all 2-forms satisfying a flux condition $F \in \Phi_c$.

35 Consistency with a connection

Sections 31-34 viewed the euclidean action as a function of the 2-form F and explored the consequences of minimizing the euclidean action subject only to a flux condition on F, without constraining F to be the field strength for a connection on a principal $U(1)$-bundle. The main result was that when the base space is $\mathbb{R}^3 \setminus *$ with a flat euclidean metric on $\mathbb{R}^3$, the 2-forms F that satisfy that minimization condition have spherical symmetry around the deleted point $*$.

If the action is viewed as a function of a connection on a given principal $U(1)$-bundle, then minimizing the action amounts to selecting a subset of those spherically symmetric configurations, namely the ones for which the flux is 2π times an integer.[110] To demonstrate that, this section constructs local potentials for the bundle in section 30, and then sections 36-37 will show that the corresponding field strength 2-form F has the form (30) with $\mu \in \mathbb{Z}$. ($\mathbb{Z}$ denotes the ring of integers.)

Use a coordinate system (x_1, x_2, x_3) for $\mathbb{R}^3$, and let $*$ be the point $(0, 0, 0)$. Cover base space $\mathbb{R}^3 \setminus *$ with the pair of contractible charts M^+ and M^- defined in section 18. The 2-form F in equation (30) may be written as $F = dA^+$ in M^+ or as $F = dA^-$ in M^-, using these local potential one-forms:[111]

$$A^+ = i\mu \frac{x_1\, dx_2 - x_2\, dx_1}{2(x_3 - r)r} \qquad\qquad A^- = i\mu \frac{x_1\, dx_2 - x_2\, dx_1}{2(x_3 + r)r} \tag{36}$$

with $r \equiv \sqrt{x_1^2 + x_2^2 + x_3^2}$. The local potential A^+ is defined everywhere in M^+, and A^- is defined everywhere in M^-. Section 36 will show that this really does reproduce equation (30). Section 37 will show that these local potentials satisfy the consistency condition (7) where the charts overlap if and only if $\mu \in \mathbb{Z}$. The cases $\mu = \pm 1$ give the **Hopf fibration** (or **Hopf bundle**) that article 03838 described using a different approach.

[110] Footnote 92 in section 26

[111] Nakahara (1990), section 1.3.1

36 Confirmation of the field strength

The field strength may be computed using either local potential within the domain where it's defined. They're both defined in the domain $M^+ \cap M^-$, which includes all of $\mathbb{R}^3 \setminus *$ except the line $x_1 = x_2 = 0$. In that domain, the field strength is[112]

$$\begin{aligned} F = dA^\pm &= -\left(d\,\frac{x_3}{r}\right) \wedge \frac{\mu\alpha}{2} \\ &= i\mu\,\frac{x_1\,dx_2 \wedge dx_3 + x_2\,dx_3 \wedge dx_1 + x_3\,dx_1 \wedge dx_2}{2r^3} \\ &= \frac{i\mu}{2}\left(\star\,d\,\frac{1}{r}\right) \end{aligned} \tag{37}$$

d is the exterior derivative, and $\star\omega$ is the Hodge dual of a differential form ω. The 2-form (37) is invariant under rotations about $r = 0$ (spherical symmetry).

This shows that F has the form (30). Section 32 showed that this F satisfies the Yang-Mills equations, so this shows that each of the local potentials $A^{(\pm)}$ in section 35 satisfies the Yang-Mills equations.

[112]Recall the identity $d\alpha = 0$ (section 17).

37 Quantization of the magnetic charge

This section shows that the local potentials defined in section 35 are consistent with a connection on a principal $U(1)$-bundle if and only if μ is an integer.

Multiply equations (36) by

$$1 = \frac{x_3 \pm r}{x_3 \pm r}$$

and use the identities $(x_3 \mp r)(x_3 \pm r) = -(x_1^2 + x_2^2)$ in the denominators to get

$$A^{\pm} = \frac{\mu}{2}\left(\mp 1 - \frac{x_3}{r}\right)\alpha, \tag{38}$$

where α is the one-form defined in section 17. It is defined throughout the region where both A^+ and A^- are defined. Where the charts overlap (namely where $|x_3| \neq r$), the difference between these two local potentials is

$$A^- - A^+ = \mu\alpha = h^{-\mu}\, dh^{\mu} \tag{39}$$

where h is the $U(1)$-valued function (14) and h^μ is its μth power. This is consistent with (7) if (and only if) μ is an integer,[113] because then h^μ is a $U(1)$-valued transition function that can be used to glue the trivial bundles $M^+ \times U(1)$ and $M^- \times U(1)$ together.[114]

[113] If μ is not an integer, then h^μ is undefined.
[114] Section 9

38 GNO monopoles

Sections 30-36 studied GNO configurations on the base space $\mathbb{R}^3 \setminus *$ when the gauged group is $U(1)$. In that family of examples, the quantity μ is an integer.

Now let G be any compact connected Lie group. Section 13 explained how a homomorphism $\rho : U(1) \to G$ can be used to convert any configuration of a $U(1)$ gauge field to a corresponding configuration of a G gauge field. According to that recipe, we can replace the integer quantity μ in sections 30-36 with the matrix μ in $\rho(e^{i\theta}) = e^{i\mu\theta} \subset G$, and this gives a GNO configuration for G.[115]

With this new μ, the field strength 2-form (30) is still spherically symmetric. When G is nonabelian, a gauge transformation converts it to

$$F(x) = \frac{\mu(x)}{2}\left(\star\, d\,\frac{1}{r}\right) \tag{40}$$

with $\mu(x) = h^{-1}(x)\mu h(x)$ for some G-valued function h. The quantity $\mu(x)$ is typically not spherically symmetric, but (40) is still consistent with a solution $A(x)$ of the Yang-Mills equations.[116]

GNO are the initials of the three authors of a paper[117] that pioneered the study of configurations that approach the form (40) as $r \to \infty$.[118] The configurations they studied are now called **GNO monopoles**.[119] Even though the context is different, the name *GNO* is also applied to the configurations used to define 't Hooft operators,[120] as in the title of this article.

[115] Section 28

[116] Any gauge transformation of a solution of the Yang-Mills equations is still a solution of the Yang-Mills equations.

[117] Goddard *et al* (1977)

[118] They were motivated by interest in the Higgs phase of models with both gauge fields and scalar (Higgs) fields. Those models admit configurations that are nonsingular everywhere in space, with a gauge field that approaches the form (40) only as $r \to \infty$, and the base space in that paper was 3-dimensional space instead of 3-dimensional euclidean spacetime. The mathematical existence of configurations with that asymptotic form leads to the prediction of (very massive) physical magnetically-charged particles (Coleman (1982); Preskill (1984)). The more abstract application to 't Hooft operators came later (Kapustin (2006), Kapustin and Witten (2007)).

[119] Coleman (1982), text around equation (3.62); Borokhov (2004), equation (2.2)

[120] Kapustin (2006), section 4.2

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