Classical Scalar Fields and Local Conservation Laws

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Abstract This article uses models of scalar fields to introduce some basic concepts in classical field theory. These scalar-field models are **toy models**: they are not meant to have any realisic applications, but they provide a mathematically easier way of introducing some concepts that are also important in realistic models.

Contents

1 Introduction

Most of the models we use in physics are not meant to be complete. We often use a model of electromagnetism that ignores gravity, or a model of gravity that ignores electromagnetism. Such models are often good enough, and they are easier to learn. To make things even easier, we can study **toy models**. A toy model is used to explore concepts that are also important in realistic models, but using ingredients that are easier to handle mathematically. The models studied in this article are toy models.

This article focuses on toy models of **classical scalar fields**. Here's some context to help explain what that means:

- In **classical physics**, all observables (measurable things) are compatible with each other. This means that we can treat them all as having well-defined values, whether or not we are actually measuring them. This is in contrast to quantum physics, where observables are not all compatible with each other.[1]

- Observables are expressed in terms of a model's **dynamic variables**, whose **equations of motion** specify which behaviors are allowed. In **field theory**, the dynamic variables are **fields**. A field is defined everywhere in spacetime. Contrast this to article 50710, where the dynamic variables are the locations of pointlike objects.

- A **scalar field** is the simplest type of field. In a coordinate representation (article 09894), a classical scalar field is an ordinary function of the spacetime coordinates.

Even though models of classical scalar fields are only toy models, they can be used to introduce several concepts that are also important in electrodynamics and general relativity. In this article, they will be used to introduce the action principle and conservation laws,[2] with emphasis on the stress-energy tensor.

[1] Quantum physics is more accurate, but classical physics is often a good approximation.

[2] The action principle and conservation laws for localized objects was introduced in article 46044.

2 Notation

Use x as an abbreviation for the list of spacetime coordinates:[3]

$$x \equiv \{x^0,\, x^1,\, x^2,\, ...,\, x^D\}.$$

The abbreviation $\int dx\; f(x)$ will be used for an integral over all $1+D$ coordinates. This article uses a coordinate system in which the equation for proper time τ is (article 48968)

$$d\tau = \eta_{ab}\, dx^a\, dx^b \tag{1}$$

with x-independent coefficients η_{ab}. The summation convention is used, so a sum is implied over any index that occurs both as a superscript and as a subscript within the same term. In particular, sums over a and b are implied in equation (1).

For any x-independent metric η with lorentzian signature, equation (1) defines the geometry of flat spacetime. Such a metric can be reduced to the standard form

$$\eta_{ab} \begin{cases} 1 & \text{if } a = b = 0, \\ -1 & \text{if } a = b > 0, \\ 0 & \text{if } a \neq b \end{cases} \tag{2}$$

by a coordinate transform, but most of this article assumes only that η is an x-independent metric with lorentzian signature, so that the components η_{ab} of the metric η and the components η^{ab} of the inverse metric η^{-1} are not necessarily equal to each other.

The partial derivative of $f(x)$ with respect to x^a will be denoted $\partial_a f(x)$, and the standard abbrevation

$$\partial^a \equiv \eta^{ab}\partial_b$$

will also be used. For any functions f, g, the abbreviations

$$\partial^2 f \equiv \eta^{ab}\partial_a\partial_b f \qquad\qquad (\partial f)\cdot(\partial g) \equiv \eta^{ab}(\partial_a f)(\partial_b g)$$

will be used.

[3] In this equation, each superscript is an index, not an exponent.

3 Single scalar field

The simplest type of **scalar field** $\phi(x)$ is a single real-valued function of the spacetime coordinates x. Equivalently, it's a function of space that can change with time. Because it depends on time, we can think of a specific function $\phi(x)$ as describing a specific **behavior** of the field. A realistic model should specify which behaviors are physically possible. In a toy model, we can say it this way instead: the model should specify which behaviors are allowed by the model. Different models may allow different behaviors, even if they all involve the same type(s) of fields.

The usual way to specify which behaviors are allowed is to impose **equations of motion**. As an example, consider a model of a single scalar field $\phi(x)$ governed by the single equation of motion

$$\partial^2\phi(x) + V'\big(\phi(x)\big) = 0, \tag{3}$$

where $V(r)$ is a fixed function of a single real variable r, and $V'(r)$ is its derivative. Equation (3) says which behaviors the model allows: a behavior $\phi(x)$ is allowed if and only if it satisfies the equation of motion (3). Equation (3) can be abbreviated

$$\partial^2\phi + V'(\phi) = 0.$$

Different choices for the function $V(r)$ define different models. In the special case $V(r) = m^2r^2/2$ for some constant m, the equation of motion (3) reduces to the **Klein-Gordon equation** $\partial^2\phi + m^2\phi = 0$. When further specialized to $V(r) = 0$, it reduces to the **wave equation** $\partial^2\phi = 0$. These special cases are important partly because the equations can be solved more easily, and partly because they also show up in the context of other types of fields. In particular, each component of the electromagnetic field satisfies the wave equation when charges and currents are absent.

4 Translation symmetry

Some behaviors satisfy the equation of motion, and some don't. Some of the ones that do are related to each other by symmetry. In this context, a **symmetry** is a transformation that, when applied to any allowed behavior, gives another allowed behavior.

One simple example of a such a transformation is a **translation** in spacetime: if $\phi(x)$ satisfies the equation of motion (3), then so does $\tilde{\phi}(x) \equiv \phi(x+c)$ for any constant offset c. (Like x, the symbol c here stands for a list of $1+D$ real numbers, one for each spacetime coordinate.) To prove this, define $\tilde{x} \equiv x+c$ and let $\tilde{\partial}_a$ denote the partial derivative with respect to $\tilde{x}^a$. Then $\partial f(\tilde{x}) = \tilde{\partial} f(\tilde{x})$ for any function f. Use this identity to get

$$\partial^2\tilde{\phi}(x) + V'\big(\tilde{\phi}(x)\big) = \tilde{\partial}^2\phi(\tilde{x}) + V'\big(\phi(\tilde{x})\big). \tag{4}$$

The right-hand side is zero because we chose ϕ to satisfy the equation of motion (3), so the left-hand side must also be zero because this equation (4) is an identity. This proves that $\tilde{\phi}$ satisfies the equation of motion.

Translation symmetry is a property of equation (3). For an example of an equation of motion that doesn't have translation symmetry, consider

$$\partial^2\phi(x) + V'\big(\phi(x)\big) = J(x) \tag{5}$$

for some prescribed function $J(x)$. In this case, the identity (4) doesn't help, because if ϕ satisfies (5), then the right-hand side of (4) is equal to $J(\tilde{x})$, but if $\tilde{\phi}$ satisfied (5), then the left-hand side of (4) would be equal to $J(x)$. This shows that if $J(x) \neq J(\tilde{x})$, then ϕ and $\tilde{\phi}$ cannot both satisfy the equation of motion. In other words, if $J(x) \neq J(x+c)$, then equation (5) does not have translation symmetry in the direction c.

5 Lorentz symmetry, part 1

The equation of motion (3) has another symmetry: if $\phi(x)$ is any solution of (3), then $\tilde{\phi}(x) = \phi(\tilde{x})$ is another solution if $\tilde{x}$ and x are related to each other by a Lorentz transformation, as explained in detail below.

To reduce index-clutter, this section uses a matrix notation (article 18505) in which x is a column matrix and Λ is a square matrix, so that Λx is another column matrix. An x-independent matrix Λ represents a **Lorentz transformation** if (article 48968)

$$\Lambda^T \eta \Lambda = \eta, \tag{6}$$

where η is the matrix with components η_{ab} (equation (1)). If $\tilde{x} = \Lambda x$, then the relationship between $\tilde{\partial}$ and ∂ can be deduced using the general identity

$$d\tilde{x}^T \tilde{\partial} = dx^T \partial,$$

again using a matrix notation in which dx and ∂ are both column matrices. This identity holds for arbitrary coordinate transformations, and substituting $\tilde{x} = \Lambda x$ on the left-hand side gives

$$dx^T \Lambda^T \tilde{\partial} = dx^T \partial,$$

which implies

$$\Lambda^T \tilde{\partial} = \partial. \tag{7}$$

Multiply (6) on the left by $\Lambda \eta^{-1}$ and on the right by $\Lambda^{-1} \eta^{-1}$ to deduce

$$\Lambda \eta^{-1} \Lambda^T = \eta^{-1}, \tag{8}$$

and combine this witih (7) to get

$$\partial^2 \equiv \partial^T \eta^{-1} \partial = \tilde{\partial}^T \Lambda \eta^{-1} \Lambda^T \tilde{\partial} = \tilde{\partial}^T \eta^{-1} \tilde{\partial} \equiv \tilde{\partial}^2.$$

Altogether, this shows that $\tilde{\partial}^2 = \partial^2$ for any Lorentz transformation $\tilde{x} = \Lambda x$. The next section uses this result to show that the equation of motion (3) has Lorentz symmetry.

6 Lorentz symmetry, part 2

Let $\phi(x)$ be any solution of the equation of motion (3), and let Λ be a matrix representing a Lorentz transformation as in the previous section. Then $\tilde{\phi}(x) \equiv \phi(\Lambda x)$ is another solution of the equation of motion (3). In other words, equation (3) has **Lorentz symmetry**.

To prove this, write $\tilde{x} \equiv \Lambda x$ and let $\tilde{\partial}_a$ denote the partial derivative with respect to $\tilde{x}^a$, as before. Use the identity $\tilde{\partial}^2 = \partial^2$, derived in the previous section, to get

$$\partial^2\tilde{\phi}(x) + V'\big(\tilde{\phi}(x)\big) = \tilde{\partial}^2\phi(\tilde{x}) + V'\big(\phi(\tilde{x})\big). \tag{9}$$

The right-hand side is zero because we chose ϕ to satisfy the equation of motion (3), so the left-hand side must also be zero because this equation (9) is an identity. This proves that $\tilde{\phi}$ satisfies the equation of motion, as claimed.

7 The concept of a local conservation law

A **conserved current** is a quantity with components J^a constructed from the fields that satisfies $\partial_a J^a(x) = 0$ whenever the fields satisfy their equations of motion. This is called a **local conservation law**, or just a **conservation law**. Given a conserved current, the corresponding **conserved charge**

$$Q \equiv \int d^D x \ J^0$$

satisfies the (not local) conservation law $\partial_0 Q = 0$. This will be illustrated in section 9. The names *current* and *charge* come from electrodynamics, where the conservation laws for *electric* current and charge are a special case of the general pattern described here.

8 Stress-energy tensor

One of the most important quantities in classical field theory is the **stress-energy tensor** $T^{ab}(x)$. This is a tensor field that is constructed from the model's basic fields. In the model governed by the equation of motion (3), T^{ab} is constructed from the scalar field ϕ, and it satisfies the **local conservation law** $\partial_a T^{ab} = 0$. The system's total energy, momentum, and angular momentum, which are the conserved quantities associated with translation symmetry and rotation symmetry,[4] can all be expressed in terms of T^{ab}. Explicitly, for the model defined by (3), the stress-energy tensor is[5]

$$T^{ab} = (\partial^a\phi)(\partial^b\phi) - \eta^{ab}\left(\frac{(\partial\phi)\cdot(\partial\phi)}{2} - V(\phi)\right). \tag{10}$$

Its key properties are

- It is symmetric: $T^{ab} = T^{ba}$.
- It satisfies $\partial_a T^{ab} = 0$ whenever ϕ satisfies the equation of motion (3).

The first property is obvious by inspection. To prove the second property, start by calculating $\partial_a T^{ab}$ for arbitrary ϕ. After some cancellations, the remainder is

$$\partial_a T^{ab} = \big(\partial^2\phi + V'(\phi)\big)\partial^b\phi,$$

which is clearly zero for any function ϕ that satisfies (3).

Using the language introduced in the previous section, the stress-energy tensor T^{ab} may be regarded as a collection of conserved currents, one for each value of the second index b. Section 15 will explain how these conserved currents are related to the model's translation symmetry that was highlighted in section 4. The next section recognizes the system's total energy and momentum as the conserved charges associated with these conserved currents.

[4] For localized objects, the connection between symmetries and conservation laws was explained in articles 33629 and 12342. For fields, the connection is explained in section 14.

[5] More accurately: T^{ab} are the *components* of the stress-energy tensor.

9 Energy and momentum

The distinction between energy and momentum corresponds to a distinction between time- and space-coordinates. Such a distinction is coordinate-dependent. To facilitate comparison to more familiar concepts of energy and momentum, this section uses a coordinate system in which the metric has the standard form shown in equation (2), so that x^0 is a "time" coordinate and the others are "space" coordinates.[6] For a given value of the time coordinate x^0, suppose that $\phi(x)$ is zero outside a bounded region of space. Then we can define

$$P^b \equiv \int dx\ T^{0b}, \tag{11}$$

where the integral is over the space coordinates $x^1, x^2, ..., x^D$. If ϕ satisfies the equation of motion (3), then the time-derivative of P^b is

$$\begin{aligned} \frac{d}{dx^0}P^b = \int dx\ \partial_0 T^{0b} &= \int dx\ \sum_{k\geq 1}\partial_k T^{kb} && \text{(because } \partial_a T^{ab}=0\text{)} \\ &= 0 && \text{(integration-by-parts).} \end{aligned}$$

Integration-by-parts gives zero because we assumed that $\phi = 0$ outside a bounded region. Altogether, this says that P^b is independent of time whenever the field $\phi(x)$ satisfies the equation of motion.

The quantity P^0, the $b = 0$ component of P^b, is called **energy**, and P^k with $k \geq 1$ are the components of the **momentum**. Use equations (2), (10), and (11) to get these explicit expressions for the energy and momentum of the scalar field:

$$P^0 = \int d^Dx\ \frac{\dot{\phi}^2 + (\nabla\phi)^2}{2} + V(\phi) \qquad\qquad P^k = -\int d^Dx\ \dot{\phi}\nabla_k\phi,$$

with $\dot{\phi} \equiv \partial_0\phi$ and $\nabla_k\phi \equiv \partial_k\phi$.

[6] Article 48968 explains the reason for the scare-quotes, which are omitted in the rest of this section.

10 The action

The model's equations of motion tell us which behaviors $\phi(x)$ are allowed. If the model satisfies the action principle, then all of the equations of motion can be encoded in a compact expression called the action.

Start with the **lagrangian**,[7] which is a function of the field $\phi(x)$ and its derivatives $\partial_a\phi(x)$ at a single point x:

$$L\big(\phi(x),\, \partial_0\phi(x),\, ...,\, \partial_D\phi(x)\big).$$

This can be abbreviated

$$L\big(\phi(x), \partial\phi(x)\big). \tag{12}$$

Since it's not much extra work, we might as well consider a more general model involving N scalar fields $\phi_1, \phi_2, ..., \phi_N$. Then the lagrangian is a function of all of these fields and their derivatives:

$$L\big(\phi_1(x),\, ...,\, \phi_N(x),\, \partial_0\phi_1(x),\, ...,\, \partial_D\phi_1(x),\, ...,\, \partial_0\phi_N(x),\, ...,\, \partial_D\phi_N(x)\big),$$

which can again be abbreviated as (12), with the understanding that ϕ is now an abbreviation for the whole collection of fields.

For any given region R of spacetime, the **action** is the integral of L over all $x \in R$:

$$S_R[\phi] = \int_R dx\ L\big(\phi(x), \partial\phi(x)\big). \tag{13}$$

For any given behavior ϕ and any given region R, the action S_R is a single real number. This number depends on the region R and on the part of the field ϕ that is inside the region R. A function like this is often called a **functional** (that's a noun, not an adjective), because its inputs ϕ are functions. Square brackets are used in the expression $S_R[\cdots]$ to remind us that the inputs are functions.

[7] This is often called the **lagrangian density**, and then its integral over the spatial coordinates is called the lagrangian, whose integral over the time coordinate is the action. In field theory, the lagrangian density is sometimes just called the lagrangian, like I'm doing here.

11 The action principle

The equations of motion are recovered from the action S by applying the **action principle**. The action principle is a statement about how the action is affected by variations of the fields. Article 46044 explains what **variation** means. It can be defined like this:[8]

$$\delta S_R[\phi] := \lim_{\epsilon\to 0} \frac{S_R[\phi+\epsilon\delta\phi]-S_R[\phi]}{\epsilon}. \tag{14}$$

More concisely,

$$\delta S_R[\phi] = S_R[\phi+\delta\phi]-S_R[\phi] \tag{15}$$

where the right-hand side is evaluated **to first order** in $\delta\phi$, which means that $(\delta\phi)^2$ is neglected. The **action principle** may be stated like this:

> A given behavior ϕ is allowed if and only if, for every region R, the condition
>
> $$\delta S_R[\phi] = 0$$
>
> holds for all variations $\delta\phi$ that are zero in a neighborhood of the boundary of R.

The next section shows how to recover the equations of motion from this principle.

[8] This is called a **Gâteaux variation**.

12 Derivation of the equations of motion

To derive the equations of motion from the action principle, start with

$$\delta S_R = \int_R dx\ \delta L. \tag{16}$$

To obtain a useful expression for δL, temporarily treat the fields ϕ and their derivatives $\partial\phi$ as independent variables (article 46044):

$$\delta L = \sum_n \left[\frac{\delta L}{\delta\phi_n(x)}\delta\phi_n(x) + \frac{\delta L}{\delta\,\partial_a\phi_n(x)}\delta\,\partial_a\phi_n(x)\right]. \tag{17}$$

As usual, a sum over the spacetime index a is implied. Substitute this into (16) and use integration-by-parts to get[9]

$$\begin{aligned}\delta S_R = &\sum_n \int_R dx\ \left[-\partial_a\frac{\delta L}{\delta\,\partial_a\phi_n} + \frac{\delta L}{\delta\phi_n}\right]\delta\phi_n \\ &+ \sum_n \int_R dx\ \partial_a\left[\frac{\delta L}{\delta\,\partial_a\phi_n(x)}\delta\phi_n(x)\right].\end{aligned} \tag{18}$$

The action principle says that the behavior ϕ is allowed if and only if, for every region R, the quantity (18) is zero for all variations $\delta\phi$ that are zero in a neighborhood of the boundary of R. For such variations, the boundary term (the second line in (18)) is zero. Since the variations $\delta\phi_n(x)$ are all independent (for each n and x), the action principle implies that the behavior ϕ is allowed if and only if it satisfies the **Euler-Lagrange equations**

$$\partial_a\frac{\delta L}{\delta\,\partial_a\phi_n(x)} = \frac{\delta L}{\delta\phi_n(x)} \tag{19}$$

for all n and all x. These are the model's equations of motion, written in terms of the lagrangian L.

[9] This step uses $\delta\,\partial\phi = \partial\,\delta\phi$.

13 Example

First consider a model with just one scalar field ϕ. The lagrangian

$$L = \frac{\eta^{ab}\big(\partial_a\phi(x)\big)\big(\partial_b\phi(x)\big)}{2} - V\big(\phi(x)\big) \tag{20}$$

gives[10]

$$\frac{\delta L}{\delta\,\partial_a\phi(x)} = \partial^b\phi(x) \qquad\qquad \frac{\delta L}{\delta\phi(x)} = -V'\big(\phi(x)\big),$$

so the Euler-Lagrange equation (19) reduces to the original equation of motion (3) in this case.

More generally, in a model with multiple scalar fields, the lagrangian

$$L = \sum_n \frac{\eta^{ab}\big(\partial_a\phi_n(x)\big)\big(\partial_b\phi_n(x)\big)}{2} - V\big(\phi_1(x),\,...,\,\phi_N(x)\big) \tag{21}$$

gives

$$\frac{\delta L}{\delta\,\partial_a\phi_n(x)} = \partial^b\phi_n(x) \qquad\qquad \frac{\delta L}{\delta\phi_n(x)} = -V_n\big(\phi_1(x),\,...,\,\phi_N(x)\big),$$

where V_n denotes the partial derivative of V with respect to the nth field. Now the Euler-Lagrange equations (19) are

$$\partial^2\phi_n + V_n(\phi_1, ..., \phi_N) = 0,$$

with one equation for each value of the index n. These are the equations of motion for the multi-field system specified by the lagrangian (21). Interactions between the fields are encoded in the fact that V depends on all of them. The fact that the interaction terms V_n in the equations of motion are different partial derivatives of the same function V implies that the way ϕ_j influences ϕ_k is related in a special way to the way ϕ_k influences ϕ_j. As explained in article 46044, this is a precise version of the vague popular statement "For every action, there is an equal and opposite reaction."

[10] Recall (section 2) that $\partial^b = \eta^{ba}\partial_a$, and $\eta^{ab} = \eta^{ba}$ because metric tensors are symmetric.

14 Local conservation laws from symmetries

Under any infinitesimal variation of the fields $\phi_n(x)$, whether or not the fields satisfies the equations of motion, the resulting variation of the action is given by equations (16)-(18).

We could define a symmetry to be any variation $\delta\phi_n$ that gives $\delta S_R = 0$ for all behaviors ϕ_n, but that definition would exclude translation symmetry because the region R is finite. We can remove that limitation by defining a **symmetry** to be any variation $\delta\phi_n$ for which δS_R can be written

$$\delta S_R = \int_R dx\ \partial_a \Lambda^a(x) \tag{22}$$

for all behaviors ϕ_n, where $\Lambda^a(x)$ depends only on the $\phi_n(x)$s and $\partial\phi_n(x)$s at the event x, regardless of R. Thanks to the fundamental theorem of calculus, the right-hand side of (22) does not depend on the system's behavior except at the boundary of the region R, which can be pushed arbitrarily far away.

Now, suppose that the behavior $\phi_n(x)$ satisfies the equations of motion (19) and *also* suppose that $\delta\phi_n$ is a symmetry as defined by equation (22). Combining equations (18), (19), and (22) gives

$$\int_R dx\ \partial_a \left[\sum_n \frac{\delta L}{\delta\,\partial_a\phi_n}\delta\phi_n - \Lambda^a\right] = 0. \tag{23}$$

The action principle requires this to hold for all regions R, so it implies

$$\partial_a \left[\sum_n \frac{\delta L}{\delta\,\partial_a\phi_n}\delta\phi_n - \Lambda^a\right] = 0. \tag{24}$$

This is a local conservation law. It holds whenever both of these conditions are satisfied: the behavior $\phi_n(x)$ satisfies the equations of motion (19), and the variation $\delta\phi_n(x)$ is a symmetry in the sense defined by equation (22).[11]

[11] This is the field-theory version of a result that was derived in article 12342 for localized objects.

15 Conserved currents for translation symmetry

Let's apply the preceding result to translation symmetry in the single-field model (section 4). We'll see that this is also a symmetry in the sense defined in (22). The infinitesimal version of the translation $\phi(x) \to \phi(x+c)$ is

$$\delta\phi(x) = c^a \partial_a \phi(x), \tag{25}$$

where now c is infinitesimal. The effect of this infinitesimal translation on the action is

$$\delta S_R = \int_R dx\ c^a \partial_a L,$$

so the quantity Λ^a in equation (22) is

$$\Lambda^a = c^a L. \tag{26}$$

This shows that the translation (25) is a symmetry in the sense defined in (22). Use (25) and (26) in the single-field version of (24) to get

$$\partial_a \left[\frac{\delta L}{\delta\, \partial_a \phi} c^b \partial_b \phi - c^a L \right] = 0, \tag{27}$$

which can also be written

$$\partial_a T^{ab} = 0 \tag{28}$$

with

$$T^{ab} = \frac{\delta L}{\delta\, \partial_a \phi} \partial^b \phi - \eta^{ab} L. \tag{29}$$

Use the lagrangian (20) in this expression to recover the earlier expression (10) for T^{ab}. As promised in section 8, this shows that the stress-energy tensor T^{ab} may be regarded as a collection of conserved currents associated with translation symmetry. Specifically, $T^{ab}c_b$ is the conserved current associated with translation symmetry in the direction defined by c.

The general derivation in section 14 shows how local conservation laws are related to symmetries, but we also can check more directly that the stress-energy tensor (29) satisfies (28). Start by evaluating the derivative to get

$$\partial_a T^{ab} = \left(\partial_a \frac{\delta L}{\delta\, \partial_a \phi}\right) \partial^b \phi + \frac{\delta L}{\delta\, \partial_a \phi} \partial_a \partial^b \phi - \frac{\delta L}{\delta\, \phi} \partial^b \phi - \frac{\delta L}{\delta\, \partial_c \phi} \partial^b \partial_c \phi.$$

This assumes translation symmetry, because it assumes that L does not depend on x except through the field ϕ. The second and fourth terms cancel identically, leaving

$$\partial_a T^{ab} = \left(\partial_a \frac{\delta L}{\delta\, \partial_a \phi}\right) \partial^b \phi - \frac{\delta L}{\delta\, \phi} \partial^b \phi,$$

which is zero whenever ϕ satisfies the equation of motion (19). This is the local conservation law (28).

The expression (29) is not uniquely determined by the general result (24). The specific expression (29) leads to a symmetric result (10) in that special case, but when other systems of fields are considered (such as the electromagnetic field), it often leads to a result that is *not* symmetric: $T^{ba} \neq T^{ab}$. We can make it symmetric by adding another conserved term,[12] but a more satisfying approach is to use the **Hilbert** stress-energy tensor, which is automatically symmetric.[13] Article 11475 introduces the Hilbert stress-energy tensor, and article 32191 explains how it relates to the specific expression (29).

[12] To learn more about this, the keywords are **Belinfante-Rosenfeld tensor**.

[13] After the definition of the Hilbert stress-energy tensor is modified to accommodate spinor fields, it leads to a symmetric result whenever the fields satisfy their equations of motion, as long as the model has **local Lorentz symmetry** (`https://physics.stackexchange.com/q/678322`).

16 Lorentz symmetry, part 3

Section 6 showed that the equation of motion (3) has Lorentz symmetry, in the sense that if $\phi(x)$ is a solution, then so is $\tilde{\phi}(x) \equiv \phi(\Lambda x)$ if the matrix Λ satisfies equation (6). Using the same ingredients, we can show that if the lagrangian is (20), then the action (13) satisfies

$$S_R[\tilde{\phi}] = S_{\Lambda R}[\phi].$$

For an infinitesimal Lorentz transformation (that is, for Λ very close to the identity matrix), the associated variation

$$\delta S_R = S_{\Lambda R}[\phi] - S_R[\phi]$$

depends on the field's behavior only at the boundary of R. This is another way of expressing the condition (22),[14] so the model with lagrangian (20) has Lorentz symmetry in the sense defined by (22).

More explicitly, consider the infinitesimal version of a Lorentz transformation:

$$\Lambda = 1 + \theta B + O(\theta^2), \tag{30}$$

where B is a matrix and θ is an infinitesimal parameter. Substitute this into equation (6) to get

$$B^T \eta + \eta B = 0.$$

Since η is symmetric, this says that ηB is antisymmetric. Use (30) in the definition of $\tilde{\phi}$ to get

$$\tilde{\phi}(x) \equiv \phi(\Lambda x) = \phi(x) + \theta\big(\partial\phi(x)\big)^T Bx + O(\theta^2),$$

so the variation of ϕ due to an infinitesimal Lorentz transformation is

$$\delta\phi(x) = \theta\big(\partial\phi(x)\big)^T Bx.$$

[14] The single-index quantity Λ^a in equation (22) is distinct from the matrix Λ used here to represent a Lorentz transformation.

By inserting a factor of the identity matrix $\eta^{-1}\eta$, this can also be written

$$\delta\phi(x) = \theta\big(\eta^{-1}\partial\phi(x)\big)^T \eta B x.$$

We already established that the factor ηB is antisymmetric. Conversely, given any antisymmetric matrix A, the matrix B defined by $A = \eta B$ generates a Lorentz transformation, so we might as well consider

$$\delta\phi(x) = \theta\big(\eta^{-1}\partial\phi(x)\big)^T A x$$

for arbitrary antisymmetric A. The components of $\eta^{-1}\partial$ are ∂^a (section 2), so we can also write this as

$$\delta\phi(x) = \theta A_{bc}\big(\partial^b\phi(x)\big)x^c \tag{31}$$

with $A_{bc} = -A_{cb}$. For the specific lagrangian (20), this implies[15]

$$\begin{aligned}
\delta L(x) &= \eta^{ab}\big(\partial_a\phi(x)\big)\big(\partial_b\,\delta\phi(x)\big) - V'\big(\phi(x)\big)\delta\phi(x) \\
&= \eta^{ab}\big(\partial_a\phi(x)\big)\partial_b\big(\theta A_{cd}\big(\partial^c\phi(x)\big)x^d\big) - V'\big(\phi(x)\big)\theta A_{bc}\big(\partial^b\phi(x)\big)x^c \\
&= \theta A_{bc}\big(\partial^b L(x)\big)x^c + \big(\partial^b\phi(x)\big)\theta A_{cd}\big(\partial^c\phi(x)\big)\partial_b x^d \\
&= \theta A_{bc}\big(\partial^b L(x)\big)x^c.
\end{aligned}$$

(Warning: this is only true for a scalar field. If L involves other types of fields, then δL will have additional terms!) That result for δL gives this explicit expression for the quantity Λ^a in equation (22):

$$\Lambda^a = \theta\eta^{ab}A_{bc}L(x)x^c. \tag{32}$$

This confirms that the model has Lorentz symmetry in the sense defined in (22).

[15]The second term on the third line is zero because A_{cd} is antisymmetric.

17 Conserved currents for Lorentz symmetry

According to the general result derived in section 14, the symmetry (31) should have an associated local conservation law. Use equations (31) and (32) in (24) to get

$$\partial_a(A_{bc}T^{ab}x^c) = 0 \tag{33}$$

whenever ϕ satisfies (3), where T^{ab} is the same stress-energy tensor as before. In the language of section 7, $A_{bc}T^{ab}x^c$ is the conserved current associated with Lorentz transformations in the plane defined by the antisymmetric matrix A.

We can also check the conservation law (33) directly, using the fact that $\partial_a T^{ab} = 0$ whenever ϕ satisfies the equation of motion (3). This gives

$$\partial_a A_{bc}T^{ab}x^c = A_{bc}T^{ab}\partial_a x^c$$

whenever ϕ satisfies (3). Now use the identity $\partial_a x^c = \delta_a^c$ to see that this is equal to $A_{bc}T^{cb}$, which is zero because T^{bc} and A_{bc} are symmetric and antisymmetric, respectively. Altogether, this confirms the local conservation law (33).

Warning: when other types of fields are involved (like the electromagnetic field), the T^{ab} defined by (29) may not be symmetric. Article 32191 shows that we can choose it to be symmetric when the fields satisfy their equations of motion, though, so the conservation law still holds.

18 Conservation laws from rotations and boosts

As in section 9, if we use a coordinate system in which the metric has the standard form shown in equation (2), then the quantity

$$Q \equiv \int d^D x\; A_{bc} T^{0b} x^c$$

is conserved in the sense that its derivative with respect to the "time" coordinate x^0 is zero:

$$\frac{d}{dx^0} Q = \int d^D x\; \partial_0 A_{bc} T^{0b} x^c = \int d^D x\; \sum_{k \geq 1} \partial_k A_{bc} T^{kb} x^c = 0,$$

using equation (33) and integration-by-parts. If the antisymmetric matrix A corresponds to an ordinary rotation in a space-space plane, then Q is the system's **angular momentum** in that plane. (This is the *definition* of angular momentum.)

If the antisymmetric matrix A corresponds to a Lorentz boost in a time-space plane (say the 0-k plane), then the associated conserved quantity doesn't have a special name, but we can relate it to something familiar. In this case, the only nonzero components of A_{bc} are $A_{0k} = -A_{k0}$, so

$$Q \propto \int d^D x\; (T^{00} x^k - T^{0k} x^0) = \left(\int d^D x\; T^{00} x^k \right) - x^0 P^k$$

where P^k is the momentum defined by (11). Since $\int d^D x\; T^{00}$ is the total energy (section 9), we can interpret T^{00} as the energy density, so the first term on the right-hand side is the **center-of-energy**. The conservation law $dQ/dx^0 = 0$ then says that the system's center-of-energy moves with constant velocity, because the momentum P^k is constant (section 9). This is a relativistic analog of the nonrelativistic result for the center-of-mass that was derived in article 12342 and illuminated in article 33629.

19 Extra symmetries of the wave equation

When $V = 0$, the equation of motion (3) reduces to the wave equation:

$$\partial^a \partial_a \phi(x) = 0. \tag{34}$$

This special case has additional symmetries, including these:

- If $\phi(x)$ is a solution, then so is $\tilde{\phi}(x) = \phi(x) + C$ for any constant C. The corresponding conserved current is $J^a \equiv \partial^a \phi$, which clearly satisfies $\partial_a J^a = 0$ whenever ϕ satisfies the equation of motion (34).

- If $\phi(x)$ is a solution, then so is $\tilde{\phi}(x) = C\phi(x)$ for any constant C. This doesn't leave the action invariant, though, not even modulo a boundary term, so it doesn't satisfy the premise that led to the conservation law (24).

- If $\phi(x)$ is a solution, then so is $\tilde{\phi}(x) = \phi(Cx)$, where Cx denotes the result of multiplying each of the coordinates by a positive constant C. The effect of this transformation on the action is the same as the previous example except in 2d spacetime, in which case it does satisfy the premise that led to the conservation law (24). This is called **scale symmetry**.

To check directly that the action is invariant under scale symmetry, make the replacement $\phi(x) \to \phi(Cx)$ in the action $\propto \int d^2x\ (\partial\phi)^2$ and then change the integration variable from x to Cx. In 2d spacetime, the factors of C from the derivatives cancel the factors of C from the integration measure.

20 Conserved current for scale symmetry

To specialize the conservation law (24) to the case of the scale symmetry highlighted in the previous section, we need an expression for the quantity Λ^a in (22). The infinitesimal version of a scale transformation is

$$\delta\phi(x) = x^a\partial_a\phi(x). \tag{35}$$

The lagrangian is (20) with $V = 0$, so the effect of the variation (35) on the lagrangian is

$$\delta L = (\partial\phi)\cdot\partial(x\cdot\partial\phi) = \frac{1}{2}\partial_a\big(x^a(\partial\phi)^2\big).$$

The first expression for δL holds in any number of dimensions, but it can be written as the second expression only in 2d spacetime. This gives

$$\Lambda^a = \frac{1}{2}x^a(\partial\phi)^2$$

in 2d spacetime. Use this and (35) in (24) to see that the corresponding conserved current is

$$J^a = \eta_{bc}T^{ab}x^c, \tag{36}$$

where T^{ab} is the stress-energy tensor (10) specialized to $V = 0$. To confirm directly that this current is conserved, use the stress-energy conservation result $\partial_a T^{ab} = 0$ to get[16]

$$\partial_a J^a = \eta_{bc}T^{ab}\partial_a x^c = \eta_{bc}T^{ab}\delta^c_a = \eta_{ba}T^{ab}. \tag{37}$$

This is true in any number of dimensions for any conserved T^{ab} if J^a is given by (36). Now suppose the lagrangian is (20) with $V = 0$, which leads to the equation of motion (34), and suppose that spacetime is two-dimensional. In this special case, the trace of the stress-energy tensor (10) is zero, so equation (37) implies that the current is conserved.

[16] The right-hand side of (37) is called the **trace** of the stress-energy tensor.

21 Example of an internal symmetry

An **internal symmetry** is one for which $\delta\phi(x)$ depends only on the field at x, not on the field at any other points in spacetime. This is in contrast to the symmetries considered so far, which all involved **spacetime symmetries** – that is, transformations that relate the values of $\phi(x)$ at different points in spacetime.

For an example of an internal symmetry, consider a system of N scalar fields with lagrangian (21) and with a V that depends only on the combination $\sum_j \phi_j^2$. Such a V is invariant under $\phi_j \to \sum_k M_{jk}\phi_k$ whenever the matrix M satisfies $M^T M = 1$. These transformations form an **orthogonal group**, denoted $O(N)$. The infinitesimal version of such a transformation is

$$\delta\phi_j = \sum_k \epsilon A_{jk}\phi_k$$

where A_{jk} is an antisymmetric matrix (article 18505) and ϵ is an infinitesimal factor. We assumed that V is invariant under the transformation, so $\delta V = 0$. Use this to get $\delta L = 0$, which implies $\Lambda^a = 0$ in (22), so the corresponding conserved current is

$$J^a = \sum_{j,k} (\partial^a \phi_j) A_{jk}\phi_k,$$

with one such current for each antisymmetric matrix A. To check directly that this current is conserved, use the antisymmetry of A to get

$$\partial_a J^a = \sum_{j,k} (\partial^2 \phi_j) A_{jk}\phi_k.$$

If the fields satisfy the equations of motion, then this implies

$$\partial_a J^a \propto \sum_{j,k} V_j(\phi_1, ..., \phi_N) A_{jk}\phi_k. \tag{38}$$

Since $\delta V = \sum_k A_{jk} V_j \phi_k$ identically, the symmetry condition $\delta V = 0$ implies that (38) is zero.

22 Complex scalar field

In the special case of two scalar fields ($N = 2$), the model described in the previous section has lagrangian

$$L = \frac{(\partial\phi_1)^2 + (\partial\phi_2)^2}{2} - V$$

where V depends only on the combination $\phi_1^2 + \phi_2^2$. In this case, we can also express the model in terms of a single **complex scalar field**

$$\varphi \equiv \frac{\phi_1 + i\phi_2}{\sqrt{2}}.$$

In terms of φ, the lagrangian is

$$L = (\partial\varphi^*) \cdot (\partial\varphi) - V \tag{39}$$

where V depends only on the magnitude $|\varphi|^2$. If we define

$$\frac{\delta}{\delta\varphi} \equiv \frac{1}{\sqrt{2}} \left(\frac{\delta}{\delta\phi_1} - i\frac{\delta}{\delta\phi_2} \right)$$

and define $\delta/\delta\varphi^*$ to be the complex conjugate of this, then we have the intuitive identities

$$\frac{\delta}{\delta\varphi}\varphi = 1 = \frac{\delta}{\delta\varphi^*}\varphi^* \qquad\qquad \frac{\delta}{\delta\varphi}\varphi^* = 0 = \frac{\delta}{\delta\varphi^*}\varphi.$$

The original equations of motion

$$\partial_a \frac{\delta L}{\delta\, \partial_a\phi_j} = \frac{\delta L}{\delta\phi_j}$$

may now be written as the single equation of motion

$$\partial_a \frac{\delta L}{\delta\, \partial_a\varphi} = \frac{\delta L}{\delta\varphi}.$$

23 Conserved current from $U(1)$ symmmetry

Consider the $O(N)$ transformation described in section 21, specialized to $N = 2$. Using the complex-field formulation from the previous section, the same transformation can be written

$$\varphi \to e^{i\theta}\varphi,$$

which clearly leaves the the lagrangian (39) invariant. These symmetry transformations form the group $U(1)$, the simplest example of a **unitary group**. The groups $U(1)$ and $O(2)$ are isomorphic to each other: they are different ways of expressing the same thing. The infinitesimal version of the $U(1)$ transform is

$$\delta\varphi = i\varphi\,\delta\theta. \tag{40}$$

The steps that led to the conservation law (24) can be re-done for the complex case, with the result

$$\partial_a\left(\frac{\delta L}{\delta\,\partial_a\varphi}\delta\varphi + \frac{\delta L}{\delta\,\partial_a\varphi^*}\delta\varphi^* - \Lambda^a\right) = 0.$$

For the variation (40), the quantity Λ^a defined by (22) is zero, so the conserved current corresponding to the $U(1)$ symmetry is

$$J^a = i(\partial^a\varphi^*)\varphi - i(\partial^a\varphi)\varphi^*.$$

24 References in this series

Article **09894** (https://cphysics.org/article/09894):
"Tensor Fields on Smooth Manifolds" (version 2023-11-12)

Article **11475** (https://cphysics.org/article/11475):
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