Discretizing Curved Space(time) for Quantum Field Theory

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Abstract In quantum field theory, the only known well-defined nonperturbative constructions of many important models involve discretizing space or spacetime. In flat space or spacetime, one common scheme uses a hypercubic lattice, which has the virtue of preserving a discrete version of exact translation symmetry. This article describes a more general discretization scheme that is useful when translation symmetry is either unimportant or absent in the nominal smooth space or spacetime.

Contents

1 Introduction

Quantum field theory (QFT) is a general framework that encompasses many different models. The only known well-defined nonperturbative constructions of many important models involve discretizing space or spacetime.[1] We don't have any reason to think that space(time) really is discrete,[2] but we can get away with treating it that way as long as the discretization scale is much finer than the resolution of any real-world observations.

Two different ways of discretizing space(time) are commonly used in quantum field theory:

- Flat space(time) is often discretized using a scheme based on a (hyper)cubic lattice that preserves a discrete version of translation symmetry.[3]

- Another way to discretize space(time), not limited to flat manifolds, is to use a **simplicial complex** (section 8).

Section 4 will describe a generalization that includes both of these common schemes. This generalization will be called a **lattice**. Mathematicians normally reserve the word *lattice* for something more specific (section 7), and the names **lattice QFT** and **lattice gauge theory** may have their roots in that more specific meaning, but now these names are often used with any way of discretizing space or spacetime.[4]

[1]The beginning of section 8.2 in Argyres and Ünsal (2012) says, "Currently the only general non-perturbative definition of QFTs is through a lattice formulation."

[2]We do have good reasons to think that models based on the traditional concept of a spacetime continuum can only be approximately valid in the real world, partly because a quantum model with gravity should exhibit some version of the holographic principle, but discretizing spacetime doesn't accomplish that.

[3]Examples: articles 51376 and 71852

[4]Examples that use the word *lattice* in this more general sense include Christ *et al* (1982); Dijkgraaf and Witten (1990), text after equation 6.28; Gaiotto *et al* (2017), section 1.2; Brower *et al* (2017); and the quote in footnote 1.

2 Perspective

The goal is to discretize a smooth n-dimensional manifold M. This manifold may represent either space or spacetime. To encompass both applications, this article calls it the **underlying manifold**. The discrete structure will be called a *lattice*.[5]

We can think of the lattice as something that stands on its own, but this article will describe it as something that occupies the underlying manifold.

The lattice is more than just a special set of points in the underlying manifold. It uses a hierarchy of k-cells to convey information about how those points are connected to each other. This information is only topological, not geometric. Geometry (quantitative distances, areas, volumes) would be implicit in the coefficients in the action or hamiltonian,[6] which are not specified in this article. When they are specified, each k-cell should be geometrically much smaller than the resolution of any real-world observations that the model is meant to reproduce.

[5] Section 1

[6] The action or hamiltonian for QFT typically involves a prescribed metric tensor, either explicitly (like in article 26542) or implicitly (like in article 52890).

3 Low-resolution limit in curved spacetime

In curved space(time), taking a strict smooth-spacetime limit would require progressively refining the lattice. Most (all?) models in quantum field theory aren't meant to be accurate at arbitrarily fine resolution, though, so we can do something else instead: we can use a fixed lattice that is much finer than the resolution of any practical measurement. Then the role of the smooth-spacetime limit can be replaced by an agreement to consider only observables whose resolution is very coarse compared the lattice scale.[7] Their resolution may still be much finer than the scale on which the curvature of space(time) is significant.

[7]In this article and other articles in this series, the purpose of discretizing space(time) is to provide unambiguous nonperturbative constructions of quantum field models, and the purpose of having an unambiguous nonperturbative construction is to have a solid foundation for everything else – including intuition, theorems, and perturbation theory, not just for computer calculations. Using a lattice with 10^{100} points would not be helpful for computer calculations, but it can still be helpful for those other purposes.

4 A general discrete structure

Let M be an n-dimensional smooth manifold. The discrete structure consists of a countable collection of 0-cells, 1-cells, 2-cells, ..., up to n-cells, with these properties:

- Each **0-cell** is a point in M.
- For each $k \in \{1, 2, ..., n\}$:
  - A ***k*-cell** is topologically a k-dimensional closed[8] ball in M.
  - The boundary of a k-cell is the union of distinct $(k-1)$-cells.
  - Each $(k-1)$-cell is part of the boundary of at least one k-cell.
- If Ω is the set consisting of the 0-cells and the interiors of the k-cells for every $k \geq 1$, then the elements of Ω do not intersect each other, and their union is the manifold M.

This structure will be called a **lattice**. The structure formed by just the 0-cells and 1-cells is also called a **graph**.[9] More language:

- A 0-cell is also called a **site**[10] or a **point**.[11]
- A 1-cell is also called a **link**.[10,11] It's a line segment whose endpoints are 0-cells.
- A 2-cell is also called a **plaquette**.[11] It's a polygon whose sides are links.

For $k \geq 1$, the interior of a k-cell is a k-dimensional manifold, so it can be assigned either of two **orientations**.[12] An oriented link is also called a **directed** link.[13]

[8] A *closed* ball includes its limit points. If $k \geq 1$, then the boundary of a k-dimensional closed ball is a $(k-1)$-dimensional sphere, and its interior is an open ball.

[9] Articles 00951 and 11617

[10] Harlow and Ooguri (2021), section 3.2

[11] Montvay and Münster (1997), section 3.2

[12] Article 91116

[13] The book Montvay and Münster (1997) uses this language in the text above equation (3.51) even though it also uses the language *oriented plaquette* in the text above equation (3.68).

5 Examples

Using the structure described in section 4, a 2-sphere can be described in many different ways, including these:

- It may be described as the surface of a cube, using six 2-cells (squares), twelve 1-cells, and eight 0-cells.
- It may be described as the surface of a tetrahedron, using four 2-cells (triangles), six 1-cells, and four 0-cells.

6 Comparison to cell complexes

The structure described in section 4 is essentially a special case of what topologists call a **cell complex** or **CW complex**.[14] A **simplicial complex** is a special case in which each cell is a simplex,[15] but the structure described in section 4 allows each cell to be a more general polyhedron.

As an example, suppose the underlying manifold M is a 2-sphere. A 2-sphere can be described as a cell complex in either of the two ways listed in section 5, but it can also be described using only two cells: a single 2-cell (the interior of a disk) and a single 0-cell, with an **attaching map** that collapses the disk's boundary onto the 0-cell.[16] In contrast, the structure described in section 4 must use k-cells of every dimension $k \in \{0, 1, 2\}$. This would be inefficient if the goal were to calculate topological invariants, but the structure described in section 4 serves a different purpose, namely to parameterize and organize the **field variables** in the path integral formulation of quantum field theory (or the **field operators** in the hamiltonian formulation where only space is discretized). The details of how it is used for that purpose depend on the model.[17]

[14] Article 93875

[15] Section 8

[16] Hatcher (2001), exercise 0.3

[17] In article 51376, some 1-cells have associated **link variables** (the variables describing the gauge field), but 1-cells whose endpoints are both on the boundary of M do not have associated link variables.

7 A special case with translation symmetry

Sometimes the underlying smooth manifold M has translation symmetry with respect to the given euclidean or lorentzian metric. In that case, using a discretization scheme that preserves a discrete version of that translation symmetry has benefits. A hypercubic lattice is the most common choice when M is topologically a cartesian product of lines (copies of $\mathbb{R}$) and/or circles (copies of S^1).

In math, the word *lattice* is commonly used for a finitely generated free abelian group.[18] A discrete group of translations in a finite-dimensional manifold fits that description. The name *lattice* is also used for the discrete array of points obtained by applying those translations to one point. When the word *lattice* is used for a space(time) discretization scheme in quantum field theory, though, something like the structure described in section 4 is usually implied.[19] In particular, a hypercubic lattice includes k-cells such that each 2-cell is a square bounded by four 1-cells, each 3-cell is a cube bounded by six 2-cells, and so on.

[18] The same word is also commonly used for something else in math (a particular type of partially ordered set).

[19] Section 3.2 in Harlow and Ooguri (2021) says, "In mathematics the term 'lattice' refers to a regular set of points in $\mathbb{R}^n$, but in lattice gauge theory it also includes a graph connecting those points."

8 A common special case: simplicial complex

The structure described in section 4 may be viewed as a generalization of what topologists call a **simplicial complex**.[20,21] This may be used to discretize any underlying smooth manifold, at least when the given metric has euclidean signature.[22]

The structure described in section 4 allows each cell to be a polyhedron that is not necessarily a simplex. It reduces to a simplicial complex if each k-cell is a k-dimensional simplex: each 2-cell is a triangle, each 3-cell is a tetrahedron, and so on.[23] One advantage of this specialization is that the orientation of a k-cell can be defined as an ordering of its vertices (the 0-cells on its boundary) independently of any underlying manifold.[24]

[20] Article 28539

[21] The distinction between CW complexes and simplicial complexes is described in Hatcher (2001) and in `https://math.stackexchange.com/questions/1528005`.

[22] Section 9 will consider lorentzian signature.

[23] Hatcher (2001), text after example 2.5

[24] Hatcher (2001), chapter 2, text before section 2.1

9 Static spacetimes and prismatic cell complexes

Suppose the smooth n-dimensional spacetime manifold and its associated lorentzian-signature metric has these properties:

- The manifold is homeomorphic to $\mathbb{R} \times M_s$. (Every **globally hyperbolic** spacetime satisfies this condition.)
- The metric is **static**, which means it has a one-parameter group of isometries whose orbits are timelike worldlines orthogonal to some spacelike hypersurface.[25] In an appropriate coordinate system, the components of such a metric satisfy $|g_{00}| = 1$, $g_{0k} = 0$, and $\partial_0 g_{jk} = 0$, using 0 as the index of the time coordinate and j, k for the space coordinates.

In that case, we can use a scheme in which the discrete version of M_s is a simplicial complex and the spacetime lattice consists of a discrete set of time translations of that simplicial complex. This type of lattice has been called a **prismatic cell complex** because each n-cell is a prism obtained by translating a spacelike simplex through one time-step.[26] This scheme is convenient for a few reasons:

- It ensures that each link is either timelike or spacelike, never lightlike.
- Wick rotation is straightforward (works just like in articles 63548 and 89053).
- Taking a continuous-time limit of the path integral to derive a hamiltonian formulation is straightforward (works just like in articles 63548 and 89053).

Sections 10-11 will consider how to apply this to the construction of quantum models with scalar fields or gauge fields.

Literature about quantum field theory in discrete versions of non-static spacetimes is scarce,[27] so such spacetimes will not be considered here.

[25] Wald (1984), section 6.1.1

[26] Desbrun *et al* (2005), figure 22

[27] One exception is Cotler and Strominger (2022), which includes warnings about naïve discretizations when the metric is time-dependent.

10 Scalar fields

Consider a static spacetime discretized as described in section 9. The action for a scalar field ϕ in smooth spacetime would be

$$S[\phi] \propto \int d^n x \, \sqrt{|\det g|} \, \big(g^{ab}(\partial_a\phi)(\partial_b\phi) + V(\phi)\big) \tag{1}$$

where g_{ab} are the coefficients of the spacetime metric field. For any n and any metric with euclidean or lorentzian signature, we can choose the discretized action to have the form[28]

$$S[\phi] \propto \sum_{(x,y)} c(x,y)\big(\phi(x) - \phi(y)\big)^2 + \sum_x V\big(x, \phi(x)\big). \tag{2}$$

The first sum is over links, each represented as a pair (x, y) of points. The coefficients $c(x, y)$ and the function $V(x, \phi(x))$ are real-valued. Articles 63548 uses this action for a hypercubic lattice in flat spacetime, where the fact that it approximates the smooth-spacetime action is relatively easy to understand intuitively. In that special case, the function $V(x, \phi(x))$ depends only on $\phi(x)$ and is otherwise independent of x. Allowing V to depend on x (and $c(x, y)$ to depend on x, y) is important for accommodating inhomogeneous and anisotropic metrics. Even if the metric is homogeneous and isotropic, allowing V to depend on x (and $c(x, y)$ to depend on x, y) can still be necessary to compensate for lattice artifacts.[29,30]

[28] Section 2.2 in Brower *et al* (2017) reviews how a discrete version of the calculus of differential forms can be used to deduce the required values of the coefficients $c(x, y)$ when V is a quadratic function of ϕ. Even though they're not about quantum field theory, Gillette (2008) and Gillette (2009) do a nice job of summarizing the key ideas involved in discrete versions of the calculus of differential forms, including clarifying which aspects depend on a metric and what choices are involved. Boissonnat *et al* (2018) highlights a complication that typically occurs in curved spaces.

[29] Brower *et al* (2016) and Brower *et al* (2018)

[30] Even in the simplest case of a scalar field in flat spacetime discretized using a hypercubic lattice, compensating for lattice artifacts requires choosing the values of the coefficients in the action to make the correlation length very large compared to the discretization scale (article 07246), because this is a prerequisite for the existence of useful observables whose resolution is much coarser than the discretization scale.

11 Gauge fields

The action for a gauge field A in smooth spacetime would be

$$S[A] \propto \int d^n x \, \sqrt{|\det g|}\, g^{ab} g^{a'b'} \text{trace}(F_{aa'} F_{bb'}) \tag{3}$$

where F_{ab} are the components of the field-strength 2-form $F = dA + A^2$. In flat spacetime discretized using a hypercubic lattice, the standard action has the form[31]

$$S[A] \propto \sum_{\square} c(\square) \left(1 - \frac{w(\square) + w^*(\square)}{2}\right) \tag{4}$$

where the sum is over oriented plaquettes (2-cells) $\square$. This is called the **Wilson action**. The plaquette variable $w(\square)$ is essentially trace$(e^{\oint_\square A})$. This same form has been proposed – with appropriate values of the real-valued coefficients $c(\square)$ – for use in any static spacetime that is discretized as described in section 9.[32] Section 12 will acknowledge the near-absence of studies in non-flat spacetimes. The case of a hypercubic lattice in flat spacetime has been studied extensively, but for non-flat spacetimes the form (4) and the proposed values of the coefficients should be treated as a reasonably-motivated but not-well-tested conjecture. As a concession, most articles in this series only rely on a few basic properties of the action, including these:

- It depends on the gauge field only through the traced plaquette variables $w(\square)$.
- Its Wick-rotated version[31] (the **euclidean action**) has a finite lower bound.
- When the gauge field is treated as a classical (instead of quantum) field, the discrete action 4 agrees with the smooth-spacetime action (3) in some approximate sense.[33]

[31] Article 89053

[32] Brower *et al* (2017), equation (7.4); summarized in Brower *et al* (2022), $J = 1$ case of equation (2)

[33] Defining this *approximate sense* is one of the purposes of the formalism mentioned in section 12 and in footnote 28 in section 10.

12 The diagonal form of the action

The discrete-spacetime action (4) is **diagonal** in the sense that each term involves only a single 2-cell (plaquette): cross-terms involving distinct 2-cells are absent. We might question whether this diagonal form is sufficient, given that the links of the lattice typically won't be aligned with the axes of that coordinate system.

Its sufficiency might seem even less certain in view of a property of metrics in smooth n-dimensional spacetime. If $n \leq 3$, then any metric with euclidean or lorentzian signature on an n-dimensional manifold can be locally diagonalized by a coordinate transformation,[34,35] so for those values of n, we don't lose any generality by supposing that g_{ab} is nonzero only for $a = b$. However, when $n \geq 4$, most metrics are not diagonal in any coordinate system, not even locally.[34,36]

Some evidence for the sufficiency of the diagonal form of the discretized action is given in the review Desbrun *et al* (2005), which explains how a discrete version of the calculus of differential forms could be used to justify the diagonal form if the field were classical (instead of quantum) and the action were quadratic in A (text after definition 12.5) like the smooth-spacetime action is. The references cited in footnote 32 propose using the same coefficients in equation (4). Beware, though, that the study of quantum field theory in discrete versions of non-flat spacetimes has only barely begun, and definitive answers to some basic questions are not yet available. This is true even for scalar fields,[37] which are simpler than gauge fields.

[34]Tod (2019), section 1

[35]Here, **locally** means each point has a neighborhood in which such a coordinate system exists.

[36]Gauduchon and Moroianu (2020) describes some counterexamples with euclidean signature in $n = 4$.

[37]Brower *et al* (2018) uses the word *conjecture* several times.

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"Complete Gauge Fixing"

Article **07246** (https://cphysics.org/article/07246):
"Phase Structure of Models of Scalar Quantum Fields"

Article **11617** (https://cphysics.org/article/11617):
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