The Action Principle in Newtonian Physics

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Abstract Article 33629 introduced conservation laws in newtonian physics. In that article, the action principle was expressed by requiring that all of the forces be encoded in a single function (the potential energy). This article introduces a more powerful form the action principle: it requires that the whole system of equations of motion be derivable from a single lagrangian.

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1 Introduction

Consider a system of objects, each of negligible size compared to the distances between them. Let m_k denote the mass of the k-th object, let $\mathbf{x}_k(t)$ denote the position of the k-th object in D-dimensional space at time t, let $\dot{\mathbf{x}}_k$ and $\ddot{\mathbf{x}}_k(t)$ denote its first and second derivatives with respect to t.

Any conceivable behavior of this system may be described by specifying the position of each object as a function of time. However, not all conceivable behaviors are physically possible. The equations of motion specify which behaviors are "physically" possible (according to the model). As in article 33629, suppose that the equations of motion are

$$m_k\ddot{\mathbf{x}}_k(t) = -\nabla_k V\big(\mathbf{x}_1(t),\, \mathbf{x}_2(t),\, ...\big), \tag{1}$$

where V is a function of the objects' positions at time t, such as

$$V \propto \sum_{j\neq k} \frac{m_j m_k}{|\mathbf{x}_j - \mathbf{x}_k|^N},$$

and $\nabla_k V$ is the gradient of V with respect to $\mathbf{x}_k$. This article shows how the whole system of equations of motion (1) can be written in terms of a single function called the **lagrangian**, denoted L, which will be described in section 2. Integrating the lagrangian over an arbitrary finite time interval I gives the **action**

$$S_I = \int_I dt\ L, \tag{2}$$

and the whole system of equations of motion (1) is written simply as

$$\boxed{\frac{\delta S_I}{\delta \mathbf{x}_k(t)} = 0} \tag{3}$$

for all $t \in I$ and all I. This is the **action principle**. The following sections explain what this means and how it reproduces equations (1).

2 The action and the lagrangian

The lagrangian L that gives the equations of motion (1) is

$$L = \frac{1}{2}\sum_k m_k \dot{\mathbf{x}}_k^2(t) - V\big(\mathbf{x}_1(t),\, \mathbf{x}_2(t),\, ...\big). \tag{4}$$

This may be abbreviated

$$\boxed{L = \frac{1}{2}\sum_k m_k \dot{\mathbf{x}}_k^2 - V.} \tag{5}$$

The lagrangian depends on the locations *and velocities* of the objects at a given instant in time. The action is the integral of L over a finite time interval I:[1]

$$S_I = \int_I dt\; \left(\frac{1}{2}\sum_k m_k \dot{\mathbf{x}}_k^2(t) - V\big(\mathbf{x}_1(t),\, \mathbf{x}_2(t),\, ...\big)\right). \tag{6}$$

The action takes a collection of functions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, ... as input and returns a single real number as output, namely the real number we get by evaluating the integral in equation (6).[2] We can think of the action as a function that assigns a real number to each behavior. The action is defined for *every conceivable* behavior, not just for behaviors that are physically allowed.

[1] The subscript I is usually omitted, and the action is treated formally as though it were an integral over all time. This article will be more careful.

[2] The action is sometimes called a **functional**, because its input is a list of functions instead of a list of numbers.

3 Recovering the equations of motion

This section explains the meaning of the **variational derivative**[3]

$$\frac{\delta S_I}{\delta \mathbf{x}_k(t)} \tag{7}$$

in equation (3) and then shows how (3) reproduces (1).

The idea is that (7) is the "derivative" of S_I with respect to $\mathbf{x}_k(t)$. To make this precise, we can think of the time t as a discrete parameter,[4] restricted to integer multiples of some tiny increment ϵ. Then the derivative $\dot{\mathbf{x}}_k(t)$ becomes a difference,

$$\dot{\mathbf{x}}_k(t) \equiv \frac{\mathbf{x}_k(t+\epsilon) - \mathbf{x}_k(t)}{\epsilon},$$

and the integral that defines S_I becomes a sum over the discrete list of times $t \in I$,

$$\int_I dt\; L(t) \equiv \epsilon \sum_{t\in I} L(t).$$

With these replacements, the action S_I is an ordinary function of a long but *finite* list of positions $\mathbf{x}_k(t)$, namely one for each object at each time $t \in I$. The list of positions is finite because the list of times $t \in I$ is finite.

Now the quantity (7) is just the usual gradient of S_I with respect to $\mathbf{x}_k(t)$. With this definition, straightforward calculation (appendix A) gives

$$\frac{\delta S_I}{\delta \mathbf{x}_j(t)} = -m_j \ddot{\mathbf{x}}_j(t) - \nabla_j V\Big(\mathbf{x}_1(t), \mathbf{x}_2(t), ...\Big) \tag{8}$$

for all $t \in I$ and all I, where $\ddot{\mathbf{x}}_j$ is a discrete version of the second derivative with respect to t. Use this in the action principle (3) to recover the equations of motion (1).

[3] To find more information about this, the keywords are **calculus of variations**.

[4] This device of artificially discretizing a continuous parameter avoids a distracting mathematical digression.

4 Relation to the colloquial "action principle"

A loose popular translation of the action principle says "For every action, there is an equal and opposite reaction." We can make that statement more meaningful by relating it to the real action principle.

Recall equation (3), which is a set of equations governing the system's behavior (the equations of motion), all expressed in terms of a single quantity S_I (the action). The fact that partial derivatives commute with each other implies the identity

$$\frac{\delta}{\delta \mathbf{x}_j(t)}\frac{\delta}{\delta \mathbf{x}_k(s)}S_I = \frac{\delta}{\delta \mathbf{x}_k(s)}\frac{\delta}{\delta \mathbf{x}_j(t)}S_I. \tag{9}$$

To translate this into words, think of the k-th equation (3) as governing the behavior of the k-th object. Then the left-hand side of (9) is an indication of how the behavior of the k-th object depends on the j-th object, and the right-hand side of (9) is an indication of how the behavior of the j-th object depends on the k-th object. The identity (9) says that these two influences are equal to each other. This gives a precise meaning to the loose popular translation of the action principle.

5 Reformulation of the action principle

The next section shows how to derive the equations of motion from a more general version of the action. To prepare for that generalization, this section introduces another way of writing equation (3).

Suppose we have a function $f(x, y)$ of just two variables x, y. Instead of working with the partial derivatives

$$\frac{\partial f}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y},$$

sometimes we can make things easier by working with the combination

$$\delta f \equiv \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y.$$

Intuitively, this describes the change δf in the function f that results from making small changes δx and δy in the variables x and y, in the limit where the changes are so small that higher-order terms like $(\delta x)^2$ and $(\delta x)(\delta y)$ are negligible.

Similarly, instead of calculating the variational derivative (7), we can work with the overall **variation**

$$\delta S_I \equiv \int_I dt \ \sum_k \frac{\delta S_I}{\delta \mathbf{x}_k(t)} \cdot \delta \mathbf{x}_k(t). \tag{10}$$

Intuitively, this describes the change δS_I in the action that results from making small changes $\delta \mathbf{x}_k(t)$ in the behaviors of the objects, in the limit where the changes are so small that higher-order terms are negligible. More formally, the quantity (10) is a convenient way of packaging all of the partial derivatives (7). When they are packaged this way, the action principle (3) can be expressed like this:

> Within the time interval I, a behavior $\mathbf{x}_k(t)$ is physically allowed if and only if it satisfies $\delta S_I = 0$ for all variations $\delta \mathbf{x}_k(t)$ that are zero at the endpoints of I.

6 The Euler-Lagrange equations, part 1

Now, suppose the action S_I is

$$S_I = \int_I dt\ L\big(\mathbf{x}_1(t),\, \mathbf{x}_2(t),\, ...,\, \dot{\mathbf{x}}_1(t)\, \dot{\mathbf{x}}_2(t)\, ...\big) \tag{11}$$

where the lagrangian $L(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{y}_1, \mathbf{y}_2, ...)$ is an ordinary function. In the integrand of S_I, the second half of L's inputs are populated by the time-derivatives of the first half of L's inputs, but the function L is defined without assuming any such relationship between its inputs (section 7).

The action principle highlighted in section 5 involves variations that are zero at the endpoints of the time-interval I. The next section shows that the variation (10) of the action is

$$\delta S_I \equiv \int_I dt\ \sum_k \left(\frac{\partial L}{\partial \mathbf{x}_k(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}_k(t)}\right)\cdot \delta\mathbf{x}_k(t) \tag{12}$$

for all variations $\delta\mathbf{x}_k(t)$ that are zero at the endpoints of I. The functions $\mathbf{x}_k(t)$ describe an arbitrary behavior of the system of objects, whether or not the behavior is allowed, and the variations $\delta\mathbf{x}_k(t)$ describe a small deviation from that behavior. The deviations are arbitrary, except for the restriction that they be small so that higher-order terms may be neglected as explained in section 5.

Now impose the action principle, which says that the behavior $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, ... is allowed if and only if $\delta S_I = 0$ for all time-intervals I. Compare the condition $\delta S_I = 0$ to the identity (12) to see that the action principle is equivalent to the condition

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}_k(t)} = \frac{\partial L}{\partial \mathbf{x}_k(t)}.} \tag{13}$$

These equations (one for each k) are the **Euler-Lagrange** equations. They are the equations of motion for a system defined by the given lagrangian L. They are the conditions that a behavior $\mathbf{x}_k(t)$ must satisfy in order to be allowed.

7 The Euler-Lagrange equations, part 2

To derive equation (12), recall the identity

$$\frac{d}{dx}L(x,x) = \left[\frac{\partial}{\partial y}L(y,z) + \frac{\partial}{\partial z}L(y,z)\right]_{y=x,\,z=x} \tag{14}$$

where L is any function of two variables. The generalization to more than two variables should be clear. Using the multi-variable version of this identity together with equation (11), the variation of S_I is

$$\delta S_I = \int_I dt\ \sum_k \left(\frac{\partial L}{\partial \mathbf{x}_k(t)}\cdot\delta\mathbf{x}_k(t) + \frac{\partial L}{\partial \dot{\mathbf{x}}_k(t)}\cdot\delta\dot{\mathbf{x}}_k(t)\right). \tag{15}$$

As in (14), the partial derivatives of L are evaluated as though $\mathbf{x}_k$ and $\dot{\mathbf{x}}_k$ were independent variables. Only *after* evaluating these partial derivatives do we take $\dot{\mathbf{x}}_k$ to be the time-derivative of $\mathbf{x}_k$. In the last term of (15), we can use integration-by-parts to remove the time-derivative from $\delta\dot{\mathbf{x}}_k$ (appendix B). Use this to see that the identity (15) may also be written

$$\begin{aligned}\delta S_I = &\int_I dt\ \sum_k \left(\frac{\partial L}{\partial \mathbf{x}_k(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}_k(t)}\right)\cdot\delta\mathbf{x}_k(t)\\ &+\int_I dt\ \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{x}}_k(t)}\cdot\delta\mathbf{x}_k(t)\right].\end{aligned} \tag{16}$$

The last term has the form

$$\int_I dt\,\frac{d}{dt}\Lambda(t).$$

Thanks to the fundamental theorem of calculus, this is equal to $\Lambda(t_1)-\Lambda(t_0)$, where t_0 and t_1 are the initial and final endpoints of the interval I. The action principle highlighted in section 5 involves variations that are zero at the endpoints of the time-interval I, so last term in (16) is zero. This leaves (12).

8 Example

With the lagrangian

$$L = \frac{1}{2}\left(\sum_k m_k \dot{\mathbf{x}}_k^2\right) - V(\mathbf{x}),$$

we have

$$\frac{\partial L}{\partial \dot{\mathbf{x}}_k} = m_k \dot{\mathbf{x}}_k \qquad\qquad \frac{\partial L}{\partial \mathbf{x}_k} = -\frac{\partial V}{\partial \mathbf{x}_k} \equiv -\nabla_k V.$$

Use this to see that for this lagrangian L, the Euler-Lagrange equation (13) is equivalent to the original equations of motion (1).

A Appendix: derivation of (8)

The action S_I in equation (8) depends on several functions of time, namely the D components of $\mathbf{x}_k$ for each object-index k. To explain how equation (8) is derived, this appendix works it out explicitly for a simplified action

$$S_I = \int_I dt\, \left(\frac{m\dot{x}^2}{2} - V(x)\right)$$

that depends on only *one* function of time, namely $x(t)$. After discretizing the integration variable t, the derivative becomes a finite difference and the integral becomes a sum:

$$S_I = \epsilon \sum_t \left(\frac{m}{2}\left(\frac{x(t+\epsilon)-x(t)}{\epsilon}\right)^2 - V\big(x(t)\big)\right).$$

To evaluate the variational derivatives $\delta S/\delta x(t)$, think of t as an index, so that $x(t)$ is a list of ordinary real variables, one for each value of the index t. Then straightforward calculation gives

$$\frac{1}{\epsilon}\frac{\delta S_I}{\delta x(t)} = m\frac{2x(t)-x(t+\epsilon)-x(t-\epsilon)}{\epsilon^2} - \nabla V\big(x(t)\big).$$

In the continuum limit, we can absorb the factor ϵ on the left-hand side into the definition of the variational derivative, and we recognize the first term on the right-hand side as the discrete version of $-\ddot{x}(t)$, so we get

$$\frac{\delta S_I}{\delta x(t)} = -m\ddot{x}(t) - \nabla V\big(x(t)\big).$$

The generalization to multiple functions $x(t)$ is straightforward.

B Appendix: discretized integration-by-parts

Section 3 defined the variational derivatives by temporarily discretizing time, but section 7 used integration-by-parts. This appendix shows that integration-by-parts still works when the integral is a discrete sum.

Consider an integral of the form

$$\int_I dt\ \dot{x}(t)y(t).$$

After discretizing the integration variable t, the derivative becomes a finite difference and the integral becomes a sum:

$$\int_I dt\ \dot{x}(t)y(t) \to \epsilon \sum_{t\in I} \frac{x(t+\epsilon)-x(t)}{\epsilon} y(t), \tag{17}$$

where now t is a discrete index. The term

$$\sum_{t\in I} x(t+\epsilon)y(t)$$

can also be written

$$\sum_{t\in I} x(t)y(t-\epsilon) + \big(x(t_1+\epsilon)y(t_1) - x(t_0)y(t_0-\epsilon)\big)$$

where t_0 and t_1 are the endpoints of the interval I. Using this identity, the right-hand side of (17) may be written

$$-\epsilon \sum_{t\in I} x(t)\frac{y(t)-y(t-\epsilon)}{\epsilon} + \big(x(t_1+\epsilon)y(t_1) - x(t_0)y(t_0-\epsilon)\big).$$

In the continuum limit $\epsilon \to 0$, this becomes

$$-\int_I dt\ x(t)\dot{y}(t) + x(t_1)y(t_1) - x(t_0)y(t_0),$$

which matches the result of integrating-by-parts on the left-hand side of (17).

C References in this series

Article **33629** (https://cphysics.org/article/33629):
"Conservation Laws and a Preview of the Action Principle" (version 2022-02-05)