Wilson and 't Hooft Operators for the Quantum Electromagnetic Field at a Single Time

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Abstract This article introduces **Wilson operators** and **'t Hooft operators** in compact quantum electrodynamics without matter using the canonical formulation (operators on a Hilbert space) in D-dimensional space at a single time. *Compact* means that gauged group is the compact group $U(1)$. This article relates those operators to the operators representing magnetic and electric flux, which are localized on submanifolds with 2 and $(D-1)$ dimensions, respectively. The flux operators and their equal-time commutation relation are previewed in smooth space first and then treated more carefully using a lattice model. In the lattice model, the basic gauge invariant operators are Wilson operators and 't Hooft operators localized on submanifolds with 1 and $(D-1)$ dimensions, respectively. These are used to define the flux operators, and then smeared versions of the flux operators are used to define other types of Wilson and 't Hooft operators localized on submanifolds with 2 and $(D-2)$ dimensions, respectively.

Contents

1 Introduction

This article introduces **Wilson operators** and **'t Hooft operators**[1] in the context of quantum electrodynamics without matter. These operators are gauge invariant. Other articles introduce these operators for a generic gauged group G. Article 22721 gives an overview. This article considers only the case $G = U(1)$ and uses perspectives and methods that exploit special features of the gauged group $G = U(1)$. The main emphasis is how Wilson operators and 't Hooft operators are related to the magnetic and electric flux operators, respectively. Various commutation relations are derived and their topological flavor is emphasized. This article uses the hamiltonian formulation and ignores time, so only equal-time commutation relations will be considered.

Two different approaches will be used. The first approach (sections 5-8) treats the flux operators in smooth space and doesn't define the them carefully.[2] The second approach (sections 10-23) includes Wilson and 't Hooft operators. The second approach is more careful. It is also messier because it treats space as a lattice,[3] but it validates the "definitions" and results of the first approach.

The family of Wilson operators is parameterized by a continuous real parameter that will be denoted β, and the family of 't Hooft operators is parameterized by a continuous real parameter that will be denoted α. In the context of D-dimensional space M, most of these Wilson and 't Hooft operators are nominally localized on 2-dimensional and $(D-1)$-dimensional submanifolds of M, respectively. For a special discrete set of values of β and α, they are localized on the boundaries of those submanifolds, which have 1 and $D-2$ dimensions, respectively.

[1] The correct way to pronounce *'t Hooft* is described in `https://www.worldscientific.com/page/news/thooft`, and the correct way to write it is described in `https://webspace.science.uu.nl/~hooft101/ap.html`.

[2] Article 26542 also uses this superficial approach.

[3] Article 51376 introduces the lattice model.

2 Outline

- Section 5 reviews the electric and magnetic field operators and their commutation relations in smooth D-dimensional space.
- Section 6 introduces the electric and magnetic flux operators. Section 7 displays their equal-time commutation relation, which is derived in section 8.
- Section 9 is a segue to the rest of the article. The rest of the article refers to a more careful formulation that treats space as a lattice.
- Sections 10-12 use that more careful formulation to define some Wilson and 't Hooft operators nominally localized on submanifolds with 1 and $D-1$ dimensions, respectively.
- Sections 14-17 use those Wilson and 't Hooft operators to define electric and magnetic flux operators and their smeared versions.
- Sections 18-20 derive the equal-time commutation relation of the smeared flux operators and show that it reduces to the one in section 7 in the smooth-space limit.
- Sections 21-23 use the smeared flux operators to construct Wilson and 't Hooft operators nominally localized on submanifolds with 2 and $D-2$ dimensions, respectively.

3 Notation for Wilson and 't Hooft operators

Two families of Wilson operators appear in this article:

- One family $W_n(C)$ (denoted $W^\circ(C)$ in article 22721) is localized on a closed curve C. In this family, the subscript n is an integer.
- The other family $W^\bullet_\beta(S)$ is localized on a surface S. In this family, the subscript β can be any real number.

The two families overlap: in the smooth-space limit, $W^\bullet_n(S) = W_n(\partial S)$ when n is an integer. (Being localized on the boundary ∂S is a special case of being localized on S.)

Given a $(D-1)$-dimensional surface Σ in space and a real number $\delta\theta$, $T_{\delta\theta}(\Sigma)$ denotes the corresponding 't Hooft operator. Other articles use slightly different notations:

- In article 53519, the operator is denoted $T_z(\Sigma)$ with $z = e^{i\,\delta\theta}$ to accommodate a generalization in which the gauged group G can be any compact Lie group. In that generalization, z is an element of the center of the group. When $G = U(1)$, the center is the whole group.
- In article 22721, the same operator is denoted $T^\bullet_z(\Sigma)$ to distinguish it from 't Hooft operators localized on $\partial\Sigma$ (section 23).
- Article 14005 focuses on the case $\delta\theta = 2\pi n$ for an integer n, in which case the operator is localized on the $(D-2)$-dimensional boundary $\partial\Sigma$. That article writes the subscript as n instead of $2\pi n$.

4 More notation and conventions

- The smooth manifold that serves as **space** (a constant-time slice of spacetime) is denoted M. This manifold has D dimensions.[4] In this article, M does not have a boundary, and it may be closed (like a torus).[5]
- An n-dimensional submanifold of a D-dimensional manifold is said to have **codimension** $D - n$.
- If X is a smooth n-dimensional manifold, then ∂X is its $(n-1)$-dimensional boundary.[6]
- C is a closed curve in space that may or may not be the boundary of a surface.
- The symbol S will be used for a 2-dimensional submanifold of M. Its boundary ∂S is a closed 1-dimensional submanifold of M.
- The symbol Σ will be used for a submanifold of M with codimension 1. Its boundary $\partial\Sigma$ is a closed submanifold of M with codimension 2.
- Boldface $\mathbf{x}$ denotes a point in space with coordinates $(x_1, ..., x_D)$.
- ∇_k denotes the partial derivative with respect to the kth spatial coordinate.
- $\delta(\mathbf{x} - \mathbf{y})$ is the distribution defined by $\int_{\mathbf{y}} \delta(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) = f(\mathbf{x})$.[7]
- This article uses the same units convention as article 26542, so the field operators $E_k(\mathbf{x})$ and $B_{jk}(\mathbf{x})$ all have the same units as mass-divided-by-length, the magnetic flux[8] has the same units as Planck's constant $\hbar$, the electric flux[9] has the same units as q^2, and the speed of light is 1.

[4] In practical applications of electrodynamics, space is a topologically trivial 3-dimensional manifold.
[5] A manifold is called **closed** if it is compact and does not have a boundary (article 44113).
[6] Article 44113
[7] Article 58590
[8] Equation (5)
[9] Equation (6)

- The time is not specified because this article is concerned only with the fields at a single time.
- This article uses the term **continuous deformation** as defined in article 09181.
- $\square$ denotes an oriented plaquette, either its surface or its perimeter (to be specified in context).
- $\eta(C, \Sigma)$ is the **intersection number**[10] of a closed curve C with a $(D-1)$-dimensional submanifold Σ.
- $[X, Y] \equiv XY - YX$.
- $\mathbb{R}$ is the field[11] of real numbers.

[10] Article 44113

[11] This sentence uses the word *field* in the algebraic sense. Much of this article uses the word *field* with another one of its standard meanings, as in *magnetic field* and *electric field*.

5 Quantum electric and magnetic field operators

In quantum field theory in smooth spacetime, some observables may be localized in arbitrarily small open regions of spacetime. They can't be strictly localized at individual points,[12] but for some purposes we can treat the field operators as though they could be strictly localized at individual points.[13] Article 26542 uses that compromise to introduce the free quantum electromagnetic field in D-dimensional space, treating the electric and magnetic field operators as though they could be strictly localized at individual points in space. Sections 6-8 will use the same compromise.

The components of the electric and magnetic fields at a point $\mathbf{x}$ in space will be denoted $E_j(\mathbf{x})$ and $B_{jk}(\mathbf{x})$. The assertion that these observables are localized at $\mathbf{x}$ is part of the model's definition.

In the quantum model, $E_j(\mathbf{x})$ and $B_{jk}(\mathbf{x})$ are operators on a Hilbert space,[14] not real-valued functions. In particular, they don't all commute with each other. Their equal-time commutation relations are[15,16,17]

$$\begin{aligned} [E_i(\mathbf{y}), B_{jk}(\mathbf{x})] &= i\hbar q^2(\delta_{ik}\nabla_j - \delta_{ij}\nabla_k)\delta(\mathbf{x}-\mathbf{y}) \\ [E_j(\mathbf{y}), E_k(\mathbf{x})] &= 0 \\ [B_{j',k'}(\mathbf{y}), B_{jk}(\mathbf{x})] &= 0 \end{aligned} \tag{1}$$

where q is the basic unit of electric charge.

This article will manipulate $E_j(\mathbf{x})$ and $B_{jk}(\mathbf{x})$ as though they were ordinary linear operators on a Hilbert space, but we know this isn't strictly true because the right-hand side of equation (1) is not a function in the usual sense. In a more careful formulation that treats space as a discrete lattice, $E_j(\mathbf{x})$ and $B_{jk}(\mathbf{x})$ are ordinary linear operators on a Hilbert space.

[12] Article 44563 uses a simple example (free scalar field) to explain why.

[13] Article 09193 does this for the free quantum scalar field.

[14] Section 10 will sketch how they can be represented as operators on a Hilbert space.

[15] Article 26542

[16] When used as a subscript, i is an index. Otherwise, i is a complex number satisfying $i^2 = -1$.

[17] δ_{jk} is defined to be 1 if $j = k$ and to be 0 otherwise.

6 Quantum electric and magnetic flux

Even though they don't commute with each other, the components of the electric and magnetic field operators may still be packaged as differential forms:[18]

$$E(\mathbf{x}) = \sum_j E_j(\mathbf{x})dx_j \qquad\qquad B(\mathbf{x}) = \frac{1}{2}\sum_{j,k} B_{jk}(\mathbf{x})dx_j \wedge dx_k. \tag{2}$$

The equations of motion in this model are Maxwell's equations without matter. Among these equations, the ones don't involve time derivatives are

$$dB = 0 \tag{3}$$

$$d(\star E) = 0, \tag{4}$$

where the $(D-1)$-form $\star E$ is the Hodge dual of E.[18] A two-form like B may be integrated over any smooth two-dimensional oriented surface S in space. This gives the **magnetic flux operator**

$$B(S) \equiv \int_{\mathbf{x}\in S} B(\mathbf{x}). \tag{5}$$

For any $(D-1)$-dimensional manifold Σ in D-dimensional space, the corresponding **electric flux operator** $E(\Sigma)$ is defined by

$$E(\Sigma) \equiv \int_{\mathbf{x}\in \Sigma} \star E(\mathbf{x}). \tag{6}$$

Equations (3)-(4) imply that $B(S)$ and $E(\Sigma)$ are invariant under continuous deformations[19] of S and Σ that don't affect the boundaries ∂S and $\partial\Sigma$.[18] Operators with this property are called **topological operators**.[20]

[18] Article 91116

[19] Article 09181 explains what *continuous deformation* means.

[20] Article 09181

7 The flux commutation relation in smooth space

Section 8 will deduce the commutation relation

$$\big[E(\Sigma),\, B(S)\big] = i\hbar q^2 \eta(\partial S, \Sigma). \tag{7}$$

The quantity $\eta(\partial S, \Sigma)$ is the **intersection number** of ∂S and Σ, defined to be the difference between the number of times the oriented loop ∂S pierces the oriented manifold Σ in the positive and negative directions, respectively.[21] It depends only on ∂S and the oriented boundary $\partial\Sigma$ of Σ.[21] Equation (7) says that $E(\Sigma)$ and $B(S)$ commute with each other if and only if the boundaries ∂S and $\partial\Sigma$ are not linked. If they are linked, then $E(\Sigma)$ and $B(S)$ don't commute with each other.[22]

[21] Article 44113

[22] If the boundaries ∂S and $\partial\Sigma$ intersect each other, then the commutator is undefined, just like (1) is undefined when $\mathbf{y} = \mathbf{x}$.

8 The flux commutation relation: derivation

To deduce (7), choose a unit vector $\mathbf{u}$ that is normal to Σ at $\mathbf{y}$. Use (1) and (2) to get

$$\big[\mathbf{u}\cdot\mathbf{E}(\mathbf{y}), B(\mathbf{x})\big] = -i\hbar q^2(\mathbf{u}\cdot d\mathbf{x})\wedge d\mathbf{x}\cdot\nabla\delta(\mathbf{x}-\mathbf{y}).$$

The distribution $\delta(\mathbf{x}-\mathbf{y})$ may be approximated arbitrarily well by an ordinary function, so we can manipulate it in some ways as though it were an ordinary function. In particular, we can use

$$d\mathbf{x}\cdot\nabla\delta(\mathbf{x}-\mathbf{y}) = d\delta(\mathbf{x}-\mathbf{y}),$$

where the right-hand side is the exterior derivative of $\delta(\mathbf{x}-\mathbf{y})$, as though it were an ordinary function. Integrate $\mathbf{x}$ over S to get

$$\big[\mathbf{u}\cdot\mathbf{E}(\mathbf{y}), B(S)\big] = -i\hbar q^2\int_S(\mathbf{u}\cdot d\mathbf{x})\wedge d\delta(\mathbf{x}-\mathbf{y})$$

and then use

$$(\mathbf{u}\cdot d\mathbf{x})\wedge d\delta(\mathbf{x}-\mathbf{y}) = -d\big(\mathbf{u}\cdot d\mathbf{x}\,\delta(\mathbf{x}-\mathbf{y})\big)$$

together with Stokes theorem to get

$$\big[\mathbf{u}\cdot\mathbf{E}(\mathbf{y}), B(S)\big] = i\hbar q^2\int_{\partial S} d(\mathbf{u}\cdot\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}).$$

Now consider a small piece σ of Σ – small enough that it may be treated as orthogonal to $\mathbf{u}$ everywhere, and small enough so that it doesn't contain any more than one intersection with ∂S. Now integrate $\mathbf{y}$ over σ to get

$$\int_\sigma d^{D-1}y\;\big[\mathbf{u}\cdot\mathbf{E}(\mathbf{y}), B(S)\big] = i\hbar q^2\int_{\partial S} d(\mathbf{u}\cdot\mathbf{x})\,\delta(\mathbf{u}\cdot\mathbf{x}-y_u)$$

where y_u is the component of $\mathbf{y}$ parallel to $\mathbf{u}$. Both sides still depend on y_u, the only component of $\mathbf{y}$ that wasn't integrated.[23] Now evaluate the integral over $\mathbf{u}\cdot\mathbf{x}$

[23] σ is a codimension 1 hypersurface, so an integral over σ integrates all but one component of $\mathbf{y}$.

to get

$$\int_\sigma d^{D-1}y\ \big[\mathbf{u}\cdot\mathbf{E}(\mathbf{y}),B(S)\big] = i\hbar q^2 \begin{cases} \pm 1 & \text{if } \mathbf{u}\cdot\mathbf{x} = y_u \text{ somewhere on } \partial S \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The sign is $+1$ or -1 if $\mathbf{u}\cdot\mathbf{x}$ increases or decreases (respectively) along the integration path. In other words, the sign depends on the direction in which ∂S pierces σ. Sum the contributions (8) from all of the pieces σ of Σ to get the final result (7).

9 Segue to the rest of the article

A model of the free electromagnetic field (with no interactions) might admit a mathematically rigorous construction in smooth space, but learning that consruction would have a low value-to-cost ratio because it probably can't be extended to include interactions with matter. This article uses a different approach. When the gauged group G is the compact group $U(1)$, an easier mathematically rigorous construction is available that treats space as a lattice.[24] We may continue to use the smooth-space picture when it's convenient, but we can view it as an intuitive approximation to the lattice-based construction (with a very fine lattice so that the approximation is excellent), and we can consult the lattice-based construction for guidance whenever the approximation runs into trouble.

Sections 10-20 will explain how the lattice $U(1)$ model introduced in article 51376[25] relates to the smooth-space formalism used in section 5-8.[26]

[24] Article 51376

[25] Article 89053 describes the corresponding path integral formulation.

[26] The lattice model is mathematically natural in the sense that its construction does not require fixing the gauge. Article 00951 explains what **fixing the gauge** means.

10 The Hilbert space

This section introduces a Hilbert space on which the quantum observables of interest may be expressed as linear operators. The idea will be introduced first using smooth-space intuition, and then its precise discrete-space formulation will be briefly reviewed.[27]

As a warm-up, suppose that space is the topologically trivial manifold $\mathbb{R}^D$, like article 26542 assumed. Intuitively, the commutation relations (1) for the electric and magnetic field operators could be implemented by using a Hilbert space in which a state-vector is a complex-valued function $\Psi[A]$ of a collection of variables $A_1(\mathbf{x})$, $A_2(\mathbf{x})$, ..., $A_D(\mathbf{x})$. The field operators in (1) could be represented as

$$\begin{aligned} E_k(\mathbf{x})\Psi[A] &= i\hbar q^2 \frac{\partial}{\partial A_k(\mathbf{x})}\Psi[A] \\ B_{jk}(\mathbf{x})\Psi[A] &= \big(\nabla_j A_k(\mathbf{x}) - \nabla_k A_j(\mathbf{x})\big)\Psi[A]. \end{aligned} \tag{9}$$

This would imply the commutation relations (1) at $t = 0$.[28] The variables $A_k(\mathbf{x})$ may be interpreted as the components of a *local potential* for the gauge field.[29]

Instead of using (9), this article uses a generalization that allows the spatial manifold M to have a nontrivial topology. Many manifolds M admit nontrivial principal $U(1)$-bundles, and a connection on such a bundle cannot be represented everywhere by a single local potential A.[30,31] The generalization sketched below handles this.

Given a principal G-bundle over a smooth base space M and a connection on that bundle, we can define a map w that sends each closed loop $C \subset M$ to the element $w(C)$ of G that is applied by parallel transport around C, called the **holonomy** around C.[30] The principal bundle and the connection are both uniquely

[27] Article 53519 introduces the discrete-space formulation in more detail.

[28] The representation (9) also implies equations (3) and (4). In particular, a function Ψ is gauge invariant if and only if $d(\star E)\Psi = 0$. Article 53519 makes this clear by treating space as a lattice.

[29] Article 76708 defines **local potential**.

[30] Article 76708

[31] Any D-dimensional torus with $D \geq 2$ admits nontrivial principal $U(1)$ bundles (article 33600).

(up to equivalence) characterized by this map.[32] We can think of an element Ψ of the Hilbert space roughly as a function of all of the complex variables $w(C)$, one for each closed curve C.[33,34] The variables (holonomies) $w(C)$ are not all independent of each other: if two curves C_1 and C_2 share a segment with opposite orientations, then $w(C_1)w(C_2) = w(C)$ where C is the union of C_1 and C_2 with the shared segment removed.

The Hilbert space used in article 53519 is essentially a precise lattice version of this rough idea. In the lattice version, $w(C)$ is given by the product of G-valued link variables for the links that form the curve C. An element Ψ of the Hilbert space is defined to be a (normalizable) complex-valued gauge invariant function of the link variables. The condition *gauge invariant* implies that it really only depends on holonomies around closed curves instead depending separately on individual link variables.[35] The inner product between two elements Ψ_1 and Ψ_2 of the Hilbert space is defined by integrating their product $\Psi_2^*\Psi_1$ over all the link variables. The smooth-space version of that integral is roughly a sum over principal G-bundles and an "integral" over connections on those bundles.[32]

This article focuses on the special case $G = U(1)$. Sections 11-12 will introduce the basic Wilson and 't Hooft operators as linear operators on the Hilbert space described above. Sections 16-20 will explain how those unitary operators can be used to define flux operators $B(S)$ and $E(\Sigma)$ that satisfy the commutation relation (7).

[32] Article 11617

[33] Section 2 in Witten (1997b) explains this more carefully.

[34] We can think of the curve C as an "index" used to distinguish the different variables $w(C)$ from each other.

[35] The virtue of expressing the construction in terms of link variables is that they are all independent of each other, whereas the holonomies are not.

11 Wilson operators

As in section 10, let $w(C)$ denote the holonomy around a closed curve C in the spatial manifold M. The holonomy $w(C)$ is an element of the gauged group, which is $U(1)$ in this article. Think of $U(1)$ as the group of unit-magnitude complex numbers with multiplication as the group operation. Section 10 described an element Ψ of the Hilbert space as a function of the holonomies $w(C)$. If Ψ is one such function and C is a specific closed curve, then the product $w(C)\Psi$ is another such function, so multiplication by $w(C)$ defines an operator $W_1(C)$ acting on this Hilbert space. If z is any element of $U(1)$ and n is any integer, then z^n is again an element of $U(1)$, so multiplication by $\big(w(C)\big)^n$ also defines an operator $W_n(C)$ on the Hilbert space:

$$W_n(C)\Psi = \big(w(C)\big)^n\Psi. \tag{10}$$

These operators are called **Wilson operators**. The integer n labels representations of the gauged group $U(1)$, because if $z : U(1) \to \mathbb{C}$ is a faithful irreducible representation of $U(1)$, then replacing z with its nth power z^n gives another representation of $U(1)$.

This also works when the gauged group G is a nonabelian compact Lie group, with one adjustment: the factor $\big(\text{holonomy}(C)\big)^n$ is generalized to the trace of the matrix that parallel transport assigns to the loop C in a given matrix representation of G. This makes it invariant under gauge transformations, as required. When $G = U(1)$, the trace is trivial because representations of $U(1)$ all use matrices of size 1×1, and a single integer n specifies the representation.

If n were not an integer, then $\big(w(C)\big)^n$ would be ambiguous and would not be an element of $U(1)$. This is why the family of Wilson operators $W_n(C)$ is parameterized by an integer. Section 21 will define another family of Wilson operators parameterized by a real number instead. Those Wilson operators are nominally localized on a surface instead of on a closed curve.

12 ’t Hooft operators

Think of the lattice as a special set of points in smooth D-dimensional space, and choose a connected $(D-1)$-dimensional submanifold Σ with these properties: Σ does not touch any points in the lattice, its intersections with links are all transverse, and its boundary $\partial\Sigma$ does not intersect any links. Then the intersection number $\eta(C,\Sigma)$ is defined for any curve C made of links.

Let Σ be a $(D-1)$-dimensional hypersurface, possibly with a boundary. Think of the state Ψ as a function of all of the complex-valued quantities $w(C)$ for all closed curves C. For each real number α, define the **’t Hooft operator** $T_\alpha(\Sigma)$ to be the linear transformation that replaces each of Ψ’s inputs $w(C)$ according to

$$w(C) \mapsto w(C)e^{i\alpha\,\eta(C,\Sigma)} \tag{11}$$

where $\eta(C,\Sigma)$ is the intersection number defined in section 7. This map respects the interrelationships among the holonomies $w(C)$ mentioned in section 10, so (11) defines a linear operator on the Hilbert space that was described in section 10.

13 Commutation relation

The operators $W_n(C)$ and $T_\alpha(\Sigma)$ clearly satisfy the commutation relation

$$\boxed{W_n(C)T_\alpha(\Sigma) = e^{i\alpha n\,\eta(C,\Sigma)}\,T_\alpha(\Sigma)W_n(C).} \tag{12}$$

This commutation relation holds exactly in the lattice model. Given that $W_n(C)$ is localized on C, microcausality[36] combined with the commutation relation (12) implies that $T(\Sigma)$ cannot be regarded as localized on $\partial\Sigma$ for most values of α, even though equation (11) says that the operator is invariant under continuous deformations of Σ that preserve its boundary $\partial\Sigma$.

The intersection number is an integer, so (11) implies that $T_\alpha(\Sigma)$ is the identity operator whenever α is an integer multiple of 2π. Section 21 will define another family of 't Hooft operators that remain nontrivial when α is an integer multiple of 2π, and for those values of α they are localized in a neighborhood of the boundary $\partial\Sigma$.[37]

[36]Section 1

[37]Article 22721 also addresses the question of when a topological operator like $T(\Sigma)$ may be regarded as genuinely localized on its boundary.

14 Why smearing is needed

Sections 16-17 will reintroduce the electric and magnetic flux operators, this time with legitimate definitions in discrete space. They are nominally localized on submanifolds with $D-1$ and 2 dimensions, respectively, but in the smooth-space limit, operators associated lower-dimensional submanifolds of smooth space are typically undefined as operators on a Hilbert space. We can fix this by *smearing* the operator – replacing the original operator with a weighted sum of translated versions of the operator over a region whose size remains finite in physical units when the smooth-space limit is taken. This is important because when the magnetic flux operator is defined the way section 17 will define it, the commutation relation (7) holds only in the smooth-space limit, not as an exact relation in discrete space. For the flux operators in 3-dimensional space, smearing only in space (without smearing in time) is sufficient.[38]

[38] Article 10690

15 Smearing

For any operator $\mathcal{O}$, let $\mathcal{O}(\delta\mathbf{x})$ be the result of displacing $\mathcal{O}$ in space as specified by the vector $\delta\mathbf{x}$. The **smeared operator** $\tilde{\mathcal{O}}$ is

$$\tilde{\mathcal{O}} \equiv \epsilon^D \sum_{\delta\mathbf{x}} f(\delta\mathbf{x})\mathcal{O}(\delta\mathbf{x}) \tag{13}$$

where $f(\cdot)$ is a real-valued smearing function and the sum is over a lattice of displacements $\delta\mathbf{x}$ with lattice spacing ϵ. This is the lattice version of integrating over a continuum of displacements $\delta\mathbf{x}$. The smearing function f must satisfy

$$\epsilon^D \sum_{\delta\mathbf{x}} f(\delta\mathbf{x}) = 1. \tag{14}$$

The notation $\tilde{\mathcal{O}}$ does not explicitly indicate exactly what smearing function was used. That detail is not important in the smooth-space limit, as long as the function has the right general properties. Article 10690 introduces some of the required properties, and here's another one: the support of a smearing function should be a D-dimensional ball whose radius r is much larger than the lattice spacing but much smaller than the resolution of any relevant physical measurements. The quantity r will be called the **smearing radius**.

16 Definition of the electric flux operator

The ’t Hooft operators $T_\alpha(\Sigma)$ introduced in section 12 are defined for all $\alpha \in \mathbb{R}$, so we can define an operator $E(\Sigma)$ implicitly by

$$T_\alpha(\Sigma) = e^{i\alpha E(\Sigma)/q^2} \qquad\qquad \text{for all } \alpha \in \mathbb{R} \tag{15}$$

or explicitly (but equivalently) by

$$E(\Sigma) \equiv -iq^2 \left[\frac{d}{d\alpha} T_\alpha(\Sigma)\right]_{\alpha=0}. \tag{16}$$

This is the **electric flux operator**. It reduces to the representation shown in equations (9) when space is topologically trivial so that the holonomies can all be written as $w(C) = e^{i\int_C A/\hbar}$ for a single local potential A.

Smearing the operator (16) as defined in section 15 gives the **smeared electric flux operator**

$$\tilde{E}(\Sigma) \equiv \epsilon^D \sum_{\delta\mathbf{x}} f(\delta\mathbf{x}) E\left(\Sigma_{\delta\mathbf{x}}\right). \tag{17}$$

The smearing function $f(\cdot)$ is real-valued, and $\Sigma_{\delta\mathbf{x}}$ denotes the result of displacing Σ in space through the vector $\delta\mathbf{x}$.

17 Definition of the magnetic flux operator

Compared to the electric flux operator, defining a magnetic flux operator requires a different approach. Section 11 only defined the Wilson operator $W_n(C)$ for integer values of n, and we can't take a derivative with respect to an integer-valued variable. We might try to define an operator $B(S)$ through the relationship

$$W_n(\partial S) = e^{in\,B(S)/\hbar} \qquad \text{for all } n \in \mathbb{Z}, \tag{18}$$

but this doesn't quite define $B(S)$ uniquely because adding an integer multiple of $2\pi\hbar$ to $B(S)$ doesn't affect the right-hand side of (18). Commutators like the left side of (7) are immune to such ambiguities, but such commutators are not the only things that matter, so this section will define a magnetic flux operator that doesn't have that ambiguity.[39] Section 21 will use this to define a larger family of Wilson operators without the restriction to integer values of n.

In smooth space, even though we can't take a derivative of $W_n(C)$ with respect to the integer-valued variable n, we can effectively take a "derivative" with respect to the loop C, because C can be arbitrarily small when space is smooth. That would give this definition of the magnetic field operator:

$$B_{jk}(\mathbf{x}) \equiv \lim_{a(S)\to 0} \frac{W_1(\partial S) - W_{-1}(\partial S)}{2i\,a(S)}\,\hbar \tag{19}$$

where S is a tiny disk with area $a(S)$ in the j-k plane centered on the point $\mathbf{x}$.

The lattice model that was used for guidance in sections 10 and 11-12 can be used again here to make that idea mathematically sound. In the lattice model, the smallest possible surface S is a square whose perimeter consists of four links. Such a square is called a **plaquette**, denoted $\square$. The lattice version of (19) is

$$\overline{B}(\square) \equiv \frac{W_1(\square) - W_{-1}(\square)}{2i}\,\hbar. \tag{20}$$

[39]The definition used here is also used in article 51376 (version 2025-01-26 or later). Article 40191 introduces a more sophisticated way to define the magnetic flux operator in discrete space. That more sophisticated definition has properties that are more aesthetically satisfying, but describing the definition takes more effort.

On the left side of equation (20), the symbol $\square$ denotes the surface of the plaquette. On the right side, it denotes the plaquette's perimeter instead. Equation (20) defines a magnetic flux operator for a single plaquette. Dividing by $a(\square)$ would give the lattice version of the magnetic field operator $B_{jk}(\mathbf{x})$. The definition (20) is extended to an arbitrary oriented surface S by summing over the oriented plaquettes of which S is composed:

$$\overline{B}(S) \equiv \sum_{\square \in S} \overline{B}(\square). \tag{21}$$

This is the **magnetic flux operator** for S. Smearing this operator as defined in section 15 gives the **smeared magnetic flux operator**

$$\tilde{B}(S) \equiv \epsilon^D \sum_{\delta\mathbf{x}} f(\delta\mathbf{x}) \overline{B}(S_{\delta\mathbf{x}}). \tag{22}$$

The smearing function $f(\cdot)$ is real-valued, and $S_{\delta\mathbf{x}}$ denotes the result of displacing the surface S in space through the vector $\delta\mathbf{x}$.

18 Flux commutation relation without smearing

This section derives the commutation relation between the (unsmeared) flux operators defined in sections 16-17.

Sandwich both sides of (20) inside $(T_\alpha(\Sigma))^{-1}\cdots T_\alpha(\Sigma)$ and use (12) to get

$$\big(T_\alpha(\Sigma)\big)^{-1}\overline{B}(\Box)T_\alpha(\Sigma)=\frac{\hbar}{2i}\left(e^{i\alpha\,\eta(\Box,\Sigma)}W_1(\Box)-e^{-i\alpha\,\eta(\Box,\Sigma)}W_{-1}(\Box)\right) \tag{23}$$

Take a derivative of (23) with respect to α and use the definition (16) to get

$$\big[E(\Sigma),\,\overline{B}(\Box)\big]=i\hbar q^2\,\eta(\Box,\Sigma)\,\overline{I}(\Box) \tag{24}$$

with

$$\overline{I}(\Box)\equiv\big(W_1(\Box)+W_{-1}(\Box)\big)/2. \tag{25}$$

Let S be an oriented surface made of oriented plaquettes, and suppose the boundaries of S and Σ are far away each other. (This will be important in section 19.) Combining (24) with the definition (21) of $\overline{B}(S)$ gives

$$\big[E(\Sigma),\,\overline{B}(S)\big]=\sum_{\Box\in S}i\hbar q^2\,\eta(\Box,\Sigma)\,\overline{I}(\Box).$$

Now use the trivial identity $\overline{I}(\Box)=1+\big(\overline{I}(\Box)-1\big)$ and the not-so-trivial identity

$$\sum_{\Box\in S}\eta(\Box,\Sigma)=\eta(\partial S,\Sigma)$$

to get

$$\big[E(\Sigma),\,\overline{B}(S)\big]=i\hbar q^2\,\eta(\partial S,\Sigma)+\sum_{\Box\in S}i\hbar q^2\,\eta(\Box,\Sigma)\,\big(\overline{I}(\Box)-1\big). \tag{26}$$

This would match (7) if the last term on the right side were absent. Section 19 will show that if the electric flux operator is replaced by its smeared version, then the last term becomes negligible the smooth-space limit.

19 Flux commutation relation with smearing

Equations (14), (17), and (26) give

$$\big[\tilde{E}(\Sigma),\, \overline{B}(S)\big] = i\hbar q^2\,\eta(\partial S,\Sigma) + i\hbar q^2\,\phi(S) \tag{27}$$

with

$$\phi(S) \equiv \sum_{\square\in S}\big(\overline{I}(\square)-1\big)\phi(\square) \tag{28}$$

$$\phi(\square) \equiv \epsilon^D \sum_{\delta\mathbf{x}} f(\delta\mathbf{x})\eta(\square,\Sigma_{\delta\mathbf{x}}). \tag{29}$$

Section 20 will show that $\phi(S)$ becomes negligible when approaching the smooth-space limit. In equation (27), only the electric flux operator is smeared. Smearing the magnetic flux operator is not necessary for showing that $\phi(S)$ becomes negligible, but it is necessary for keeping the operator well-defined as an operator on the Hilbert space in that limit. Smearing both flux operators and using (14) and the fact that $\phi(S)$ becomes negligible gives the **smeared flux commutation relation**

$$\boxed{\big[\tilde{E}(\Sigma),\, \tilde{B}(S)\big] \approx i\hbar q^2\,\eta(\partial S,\Sigma).} \tag{30}$$

The approximation becomes exact in the smooth-space limit, so this reproduces the expected commutation relation (7).

The field operators $E_j(\mathbf{x})$ and $B_{jk}(\mathbf{x})$ can be defined by taking Σ and S to be infinitesimal and dividing those flux operators by their areas. Equation (30) implies that these field operators satisfy the commutation relation (1) in the smooth-space limit. This is the foundation for article 26542, which uses (1) to introduce photons.

20 Why the last term is negligible

This section shows that the quantity $\phi(S)$ in equation (27) becomes negligible when approaching the smooth-space limit.

First consider the quantity $\phi(\square)$ for a single plaquette $\square$, as defined in the second of equations (29). Suppose the intersection number $\eta(\square, \Sigma_{\delta\mathbf{x}})$ is ± 1 when $\delta\mathbf{x} = \mathbf{0}$. The manifold Σ is understood to be extended over a region much larger than the smearing region (which in turn is much larger than a single plaquette) and to be practically flat on that scale, so the intersection number will be zero for most shifts $\delta\mathbf{x}$.

To quantify this, use (14) to get $f(\delta\mathbf{x}) \sim 1/r^D$, where r is the smearing radius. The total number of terms in the sum is $\sim (r/\epsilon)^D$. The smearing radius is large compared to the discretization scale ϵ, and only a $(D-2)$-dimensional subset of the shifts can have a nonzero intersection number with a single plaquette (because a plaquette is 2-dimensional with size $\sim \epsilon$ in those 2 dimensions), so the number of nonzero terms in the sum is $\sim (r/\epsilon)^{D-2}$. Use this in the definition (29) of $\phi(\square)$ to get

$$\phi(\square) \sim \epsilon^D \times \left(\frac{r}{\epsilon}\right)^{D-2} \times \frac{1}{r^D} = \left(\frac{\epsilon}{r}\right)^2,$$

which goes to zero in the limit $\epsilon/r \to 0$.

For an example, suppose that $\square$ is in the x_1-x_2 plane with a corner at $x_1 = x_2 = 0$ and that (near $\square$) Σ is defined by $x_1 = 0$ and $x_2 \geq \epsilon/2$. Then the intersection number will be zero for all shifts $\delta\mathbf{x}$ whose δx_1 or δx_2 component has a magnitude larger than $\sim \epsilon$. This leaves only a $(D-2)$-dimensional subset of shifts that can keep the intersection number nonzero for that plaquette, so $\phi(\square) \sim \epsilon^2$, which goes to zero in physical units in the smooth-space limit. This is illustrated in figure 1.

The definitions (10) and (25) imply that the operator $\overline{I}(\square) - 1$ has norm 1, so the fact that $\phi(\square)$ is negligible for each plaquette $\square$ implies that the last term in (27) is negligible overall, as claimed.

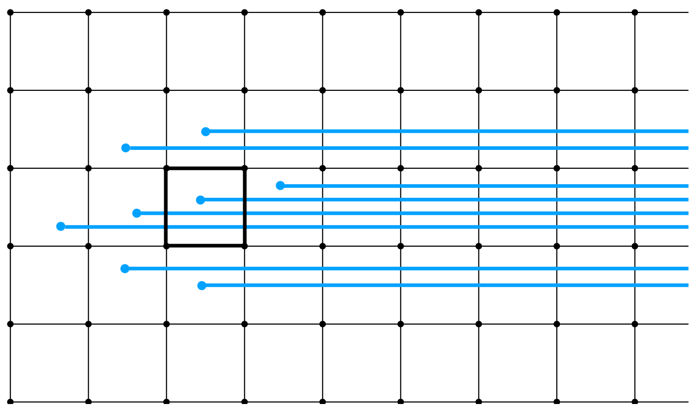

Figure 1 – In this picture, space is a two-dimensional lattice ($D = 2$). Black dots are points in the lattice, and thin black line segments are links. Each blue line represents one of the manifolds $\Sigma_{\delta\mathbf{x}}$ in equation (29), only part of which is visible in this picture. (The configuration extends past the right-hand side of the picture.) Since space is two-dimensional, each $\Sigma_{\delta\mathbf{x}}$ is one-dimensional and its boundary is a pair of points, one of which is outside the scope of these pictures. The one in the picture's scope is shown as a blue dot. The thick black outline highlights a single plaquette $\square$. The intersection number $\eta(\square, \Sigma_{\delta\mathbf{x}})$ is nonzero only for manifolds $\Sigma_{\delta\mathbf{x}}$ pierced exactly once by the plaquette's perimeter. In this example, that condition is satisfied for only one shift (one of the blue lines). For all the others, the intersection number is zero. To approach the smooth-space limit, the smearing region must become enormous compared to the size of a single plaquette. In that situation, only a tiny fraction (approaching zero) of the manifolds $\Sigma_{\delta\mathbf{x}}$ give a nonzero intersection number.

21 Pre-smeared Wilson and 't Hooft operators

The smeared flux operators $\tilde{E}(\Sigma)$ and $\tilde{B}(S)$ defined in equations 17 and 22 can be used to define these pre-smeared unitary operators:[40,41]

$$\tilde{T}_\alpha(\Sigma) \equiv e^{i\alpha\tilde{E}(\Sigma)/q^2} \qquad \alpha \in \mathbb{R}, \tag{31}$$

$$\tilde{W}^\bullet_\beta(S) \equiv e^{i\beta\tilde{B}(S)/\hbar} \qquad \beta \in \mathbb{R}. \tag{32}$$

The flux operators $\tilde{E}(\Sigma)$ and $\tilde{B}(S)$ have the same units as q^2 and $\hbar$, respectively,[42] so the coefficients α, β are unitless. $T_\alpha(\Sigma)$ will be called a **pre-smeared 't Hooft operator**, and $\tilde{W}^\bullet_\beta(S)$ will be called a **pre-smeared Wilson operator**.

The filled-circle superscript $\bullet$ on $\tilde{W}^\bullet_\beta(S)$ is used to indicate that this operator is localized on a surface S, in contrast to the Wilson operator $W_n(C)$ defined in section 11 that is localized on a closed curve C.

Article 40191 describes a different definition of the surface-localized Wilson operator $W^\bullet_\beta(S)$ that does not rely on first defining a pre-smeared flux operator. The definition described in that article takes more effort to describe, but it has nicer mathematical properties: without taking a strict smooth-space limit, it already satisfies the lattice Bianchi identically exactly, it reduces exactly to $W^\circ(\partial S)$ when β is an integer, and it satisfies an exact version of the commutation relation that will be highlighted in the next section. In a strict smooth-space limit, the surface-localized Wilson operators defined that way should become indistinguishable from the ones defined in this article.[43]

[40] Article 10690 defines **pre-smeared**.

[41] The superscript $\bullet$ (a filled-in circle) distinguishes this surface-dependent operator from the curve-dependent Wilson operator in section 11. In article 22721, the Wilson operator defined in section 11 is distinguished by using the unfilled superscript $\circ$.

[42] Section 4

[43] To make this work, both operators should be post-smeared (article 10690, and then the pre-smearing radius used in (32) should be sent to zero (in physical units) so that both operators are *only* post-smeared.

22 Pre-smeared unitary commutation relation

This section uses the commutation relation (30) of the smeared flux operators to derive a commutation relation for the pre-smeared 't Hooft and Wilson operators. This commutation relation assumes that the boundary ∂S does not come within a distance r of the boundary $\partial\Sigma$, where r is the smearing radius.[44]

If X and Y both commute with their commutator $[X, Y]$, then the identity

$$e^X e^Y = e^{[X,Y]} e^Y e^X \tag{33}$$

holds.[45,46] Use (33) together with the commutation relation (30) to deduce

$$\boxed{\tilde{W}^\bullet_\beta(S)\tilde{T}_\alpha(\Sigma) \approx e^{i\alpha\beta\,\eta(\partial S,\Sigma)}\,\tilde{T}_\alpha(\Sigma)\tilde{W}^\bullet_\beta(S).} \tag{34}$$

This becomes exact in the smooth-space limit. The result (34) will be called the **pre-smeared unitary commutation relation** because it uses the pre-smeared unitary operators (32).

The definitions (32) and the approximation (34) depend on the smearing functions, but after taking a strict smooth-space limit we can make the support of the smearing functions arbitrarily small so that the flux operators are localized in arbitrarily small neighborhoods of Σ and S.

Altogether, we have used the lattice formulation to derive two distinct unitary commutation relations. One of these (equation (12)) is already exact on a finite lattice but is restricted to integer values of β. The other one (equation (34)) holds for arbitrary real values of β but becomes exact only in the smooth-space limit.

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[44] The smearing radius r was introduced in section 15.

[45] To deduce this, define $a \equiv [X, Y]$, where a is a number. Operators X, Y with this property can be represented as differential operators on a Hilbert space of functions $f(y)$ of a real variable y, with $Xf(y) \equiv a \times \partial f(y)/\partial y$ and $Yf \equiv y \times f(y)$. Then $e^X f(y) = f(y + a)$ and $e^Y f(y) = e^y f(y)$, which implies (33).

[46] Müger (2020) reviews the **Dynkin formula** associated with the **BCH theorem**, which may be viewed as a generalization of (33) (theorems 1.3 and 7.6) but with a caveat about convergence (theorem 2.14) that isn't satisfied in the present case.

23 Smearing only the boundary

The electric flux operator $E(\Sigma)$ is unaffected by continuous deformations of the manifold Σ as long as the boundary $\partial\Sigma$ is not changed.[47] The analysis in sections 19-20 is consistent with that because the quantity $\eta(\square,\Sigma_{\delta\mathbf{x}})$ in (29) is nonzero only if $\partial\Sigma_{\delta\mathbf{x}}$ is linked with $\square$. This implies that smearing Σ only in a neighborhood of the boundary $\partial\Sigma$ is sufficient. This is illustrated in figure 2.

When the α in $\tilde{T}_\alpha(\Sigma)$ is an integer multiple of 2π, the unsmeared unitary commutation relation (13) implies that $\tilde{T}_\alpha(\Sigma)$ commutes with the Wilson operators $W_n(C)$ that are localized on curves C, even if the intersection number $\eta(C,\Sigma)$ is nonzero. That allows interpreting $\tilde{T}_{2\pi k}(\Sigma)$ as being localized on the boundary Σ when k is an integer.[48] This is true also for the unsmeared 't Hooft operator $T_{2\pi k}(\Sigma)$, but in that case it's just the identity operator. The pre-smeared 't Hooft operator $\tilde{T}_{2\pi k}(\Sigma)$ is distinct from the identity operator even when k is an integer, and it is localized on the boundary $\partial\Sigma$.[49]

This leads to an alternative description of boundary-localized operators $\tilde{T}_{2\pi k}(\Sigma)$. In the smooth-space limit, calculations involving Wilson and 't Hooft operators normally avoid configurations where the boundaries ∂S and $\partial\Sigma$ intersect each other. For those calculations, if we use a smearing function for $\partial\Sigma$ that is supported only on the boundary of the smearing region,[50] then lattice sites/links in the interior of that region don't need to be considered at all – they might as well be excised from the lattice, because Wilson loops won't be allowed to intersect that region anyway. Figure 3 illustrates the idea. This is an example of a more general idea introduced in article 02242.

[47] Section 6

[48] The identity $\eta(\partial S,\Sigma)=\eta(\partial\Sigma,S)$ may be used in the commutation relation (34) to write in a more intuitive way when the 't Hooft operator is localized on the boundary $\partial\Sigma$.

[49] In article 22721, 't Hooft operators localized on $\Gamma\equiv\partial\Sigma$ are denoted denoted $T^\circ(\Gamma)$. Article 93302 constructs these operators for any compact connected gauged group G. In that generalization, these operators are labeled by a homomorphism from $U(1)$ into G. When $G=U(1)$, the integer k defines a homomorphism $g\mapsto g^k$.

[50] This construction involves two boundaries: before smearing, we had the original $(D-2)$-dimensional boundary $\partial\Sigma$ of the $(D-1)$-dimensional manifold Σ, and now we have the $(D-1)$-dimensional boundary of the region over which $\partial\Sigma$ is smeared.

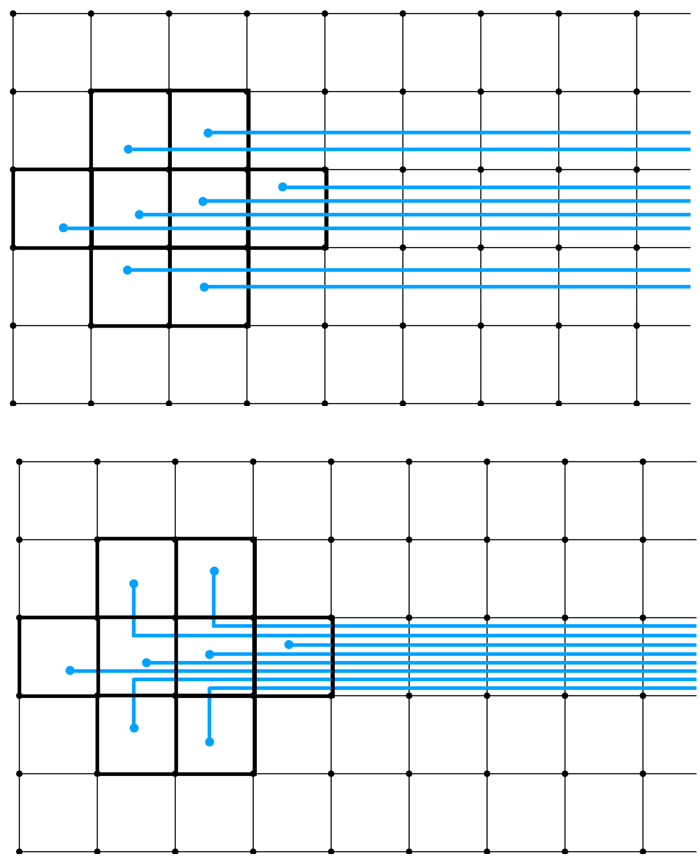

Figure 2 – The top picture is a copy of figure 1, now highlighting every plaquette that links with at least one of the boundaries $\partial\Sigma_{\delta\mathbf{x}}$. The configuration shown in the top picture is equivalent to the one shown in the bottom picture. Space is only two-dimensional in these pictures, but they illustrate the fact that to define a pre-smeared 't Hooft operator, smearing the boundary $\partial\Sigma$ is sufficient. In the bottom picture, $\partial\Sigma$ is smeared over the region tiled by the thicker black squares, and the rest of Σ is not smeared at all (it's width is less than the distance between lattice sites). Smearing the interior of Σ would not matter because the operator is not affected by continuous deformations of Σ anyway as long as the boundary $\partial\Sigma$ remains fixed. If the α in $\tilde{T}_\alpha(\Sigma)$ is an integer multiple of 2π, then $\tilde{T}_\alpha(\Sigma)$ doesn't affect any link variables outside the region where $\partial\Sigma$ is smeared, so it's manifestly localized in the region where $\partial\Sigma$ is smeared.

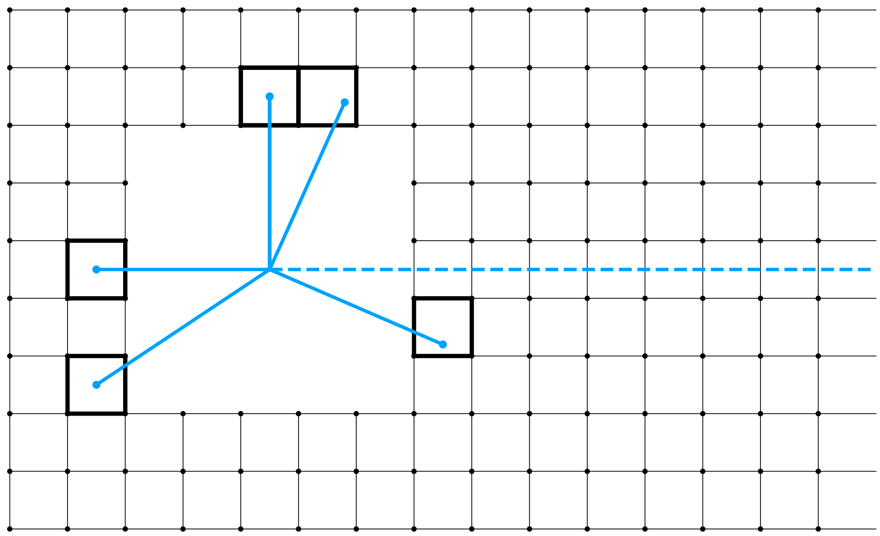

Figure 3 – This example (again in two-dimensional space) illustrates the same idea as the bottom picture in figure 2, but now the endpoint of each $\Sigma_{\delta\mathbf{x}}$ is on the perimeter of a region that is excised from the lattice, as described at the end of section 23. The dashed line represents the bundle of lines entering the picture from the right with a total α of 2π. This is an example of a more general idea introduced in article 02242.

24 References

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