Energy and Momentum at All Speeds: Derivation

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Abstract Article 77597 introduced equations that describe the relationships between an object's energy E, momentum $\mathbf{p}$, mass m, and velocity $\mathbf{v}$ in special relativity. This article shows how those equations can be derived from something deeper.

Contents

1 Introduction

Article 77597 introduced these relationships between an isolated object's energy E, momentum $\mathbf{p}$, (rest) mass m, and velocity $\mathbf{v}$ in special relativity:

$$m^2 = E^2 - \mathbf{p}^2 \tag{1}$$

$$\mathbf{v} = \frac{\mathbf{p}}{E}. \tag{2}$$

This article motivates equations (1)-(2) by showing how they can be derived from an action principle,[1] where the object is part of a larger system with which it may interact.

Equations (1)-(2) assume that we're using a coordinate system in which the equation for an object's proper time τ is

$$d\tau^2 = g_{ab}\, dx^a\, dx^b \tag{3}$$

using the notation conventions introduces in article 48968, with

$$g_{ab} = \text{diag}(1, -1, -1, -1). \tag{4}$$

These are the components of the metric of flat spacetime, the context for special relativity.

[1] Article 98002

2 Origin of equations (1)-(2), preview

Article 77597 focused on the energy and momentum of an isolated object for simplicity, but energy and momentum are not really useful concepts for an isolated object. They become more useful in a more complicated system, because then their conservation can tell us something about the system's behavior without requiring a full solution of the equations of motion. This article starts with the conservation law $\partial_a T^{ab} = 0$, where T^{ab} is called the **stress-energy tensor**. This conservation law holds in flat spacetime, in any coordinate system where the components of the metric are (4), for any system (however complicated it may be) that respects the symmetries of flat spacetime. In terms of the stress-energy tensor, the components P^a of the system's total momentum (which includes the total energy as the component P^0) are

$$P^a = \int d^3x \; T^{0a} \tag{5}$$

where the integral is over the "spatial" coordinates.[2] This article shows that for any isolated object within the system, that object's contribution to P^a is

$$p^a = m\frac{dX^a}{d\tau} \tag{6}$$

where m is the mass of that object, X^a are the coordinates along the object's worldline, and τ is the object's proper time. Equations (3), (4), and (6) imply

$$(p^0)^2 - \sum_k (p^k)^2 = m^2 \qquad\qquad \frac{p^k}{p^0} = \frac{dX^k}{dX^0}.$$

Use the notation $E \equiv p^0$ to see these equations are the same as equations (1) and (2), where p^k are the components of $\mathbf{p}$ and dX^k/dX^0 are the components of the velocity $\mathbf{v}$.

[2] Under Lorentz transformations, the P^a transform just like the components of $(E, \mathbf{p})$ as described in article 77597.

3 Origin of equations (1)-(2), part 1

To derive equation (6), we need an explicit expression for the stress-energy tensor. The general definition of the stress-energy tensor starts with the **action**, a function of the dynamic variables (such as the objects' coordinates) that summarizes all of the model's equations of motion. The action S depends on the spacetime metric g_{ab}, and the stress-energy tensor is defined by[3]

$$T^{ab}(x) = \frac{-2}{\sqrt{|g|(x)}} \frac{\delta S}{\delta g_{ab}(x)} \tag{7}$$

where $|g|$ is the determinant of the metric. For a system involving an object of mass m in a spacetime with a generic metric, the action is[4]

$$S = -m \int d\lambda \, \sqrt{g_{ab}(X)\dot{X}^a \dot{X}^b} + \cdots \tag{8}$$

where "$\cdots$" are terms representing other objects and fields and the interactions between them. For this derivation, we are only interested in the individual object's contribution to T^{ab}, so we don't need to write those other terms explicitly. The Xs in this part of the action are the coordinates along the object's worldline, as functions of the parameter λ, and $\dot{X}^a$ is the derivative of X^a with respect to λ.

To use the definition (7), we need to write the action in terms of the metric $g_{ab}(x)$ as a function of the coordinates x *everywhere* in spacetime, not just on the object's worldline. We can do this by writing (8) as

$$S = -m \int d^4x \, \int d\lambda \, \sqrt{g_{ab}(x)\dot{X}^a \dot{X}^b} \, \delta^4\big(x - X(\lambda)\big) + \cdots \tag{9}$$

where the coordinates x are unconstrained.

[3] Article 11475, or equation (20.27) on page 389 in Blau (2021)

[4] This is motivated in section 4

4 Equation of motion for an object

To help motivate the expression (8) for the action, recall the action principle: a given behavior is allowed (in the simplified world described by this model) if and only if it is an extremum of the action. This criterion leads to the Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\delta S}{\delta \dot{X}^a} = \frac{\delta S}{\delta X^a}, \tag{10}$$

which are the equations of motion for the object of interest. Using the action (8) gives

$$\frac{\delta S}{\delta \dot{X}^a} = -\frac{m}{2}\frac{g_{ab}\dot{X}^b}{\sqrt{g_{ab}\dot{X}^a\dot{X}^b}} + \cdots$$

and

$$\frac{\delta S}{\delta X^a} = -\frac{m}{2}\frac{(\partial_a g_{bc})\dot{X}^a\dot{X}^b}{\sqrt{g_{ab}\dot{X}^a\dot{X}^b}} + \cdots$$

Specializing to the flat metric (4) gives $\partial_a g_{bc} = 0$, and taking the worldline's parameter λ to be its proper time τ gives $g_{ab}\dot{X}^a\dot{X}^b = 1$. With these simplifications, the equation of motion (10) reduces to

$$-\frac{m}{2}g_{ab}\ddot{X}^b = \cdots\ ,$$

and we can use the fact that the metric is invertible to further reduce this to

$$m\ddot{X}^b = \cdots\ .$$

This looks like the familiar equation of motion for an object with mass m subject to the influences described by the "$\cdots$" on the right-hand side, but with the object's motion parameterized by its own proper time instead of coordinate time.

5 Origin of equations (1)-(2), part 2

Use equation (9) in the definition (7) of the stress-energy tensor to get

$$T^{ab}(x) = \frac{m}{\sqrt{|g|}} \int d\lambda\ \frac{\dot{X}^a \dot{X}^b}{\sqrt{g_{ab}(x)\dot{X}^a \dot{X}^b}}\, \delta^4\big(x - X(\lambda)\big) + \cdots . \tag{11}$$

This integral is invariant under reparameterizations of the worldline, so we can specialize the parameter λ to be the object's proper time τ. Then the δ-factor together with definition of proper time makes the square-root factor equal to 1. Now specialize to flat spacetime, using the coordinate system with $|g| = 1$. Altogether, equation (11) reduces to

$$T^{ab}(x) = m \int d\tau\ \dot{X}^a \dot{X}^b\, \delta^4\big(x - X(\tau)\big) + \cdots . \tag{12}$$

Use (12) in (5) to get

$$P^a(x^0) = m \int d\tau\ \dot{X}^0 \dot{X}^a\, \delta\big(x^0 - X^0(\tau)\big) + \cdots ,$$

after using the δ-factor to evaluate the integral over the "spatial" coordinates. To finish, use the identity

$$\int d\tau\ \dot{X}^0\, \delta\big(x^0 - X^0(\tau)\big) f(\tau) = f(\tau)\Big|_{X^0(\tau)=x^0}$$

where the subscript on the right-hand side meanes to evaluate $f(\tau)$ at the value of τ for which $X^0(\tau) = x^0$. The final result is

$$P^a(x^0) = m\dot{X}^a\Big|_{X^0(\tau)=x^0} + \cdots .$$

Again, the "$\cdots$" represents the contributions of other objects, fields, and their interactions. If we consider only the contribution of this one object, then we are left with (a more careful version of) equation (6). This completes the derivation.

6 References

Blau, 2021. "Lecture Notes on General Relativity (Last update November 15, 2021)" http://www.blau.itp.unibe.ch/GRLecturenotes.html

7 References in this series

Article **11475** (https://cphysics.org/article/11475):
"Classical Scalar Fields in Curved Spacetime" (version 2022-02-05)

Article **48968** (https://cphysics.org/article/48968):
"The Geometry of Spacetime" (version 2022-01-16)

Article **77597** (https://cphysics.org/article/77597):
"Energy and Momentum at All Speeds" (version 2022-02-18)

Article **98002** (https://cphysics.org/article/98002):
"The Action Principle in Classical Electrodynamics" (version 2022-02-18)