Langlands Duals of Compact Lie Groups

Randy S

Abstract This article provides a table of Langlands duals of compact connected simple Lie groups. References are provided for the definition of **Langlands dual** and for the entries in the table.

Contents

1 Some group theory background

This section summarizes some group theory material that will be used either implicitly or explicitly in the following sections.

A compact connected Lie group is called **semisimple** if its center is finite.[1,2] A Lie algebra with at least 2 dimensions is called **simple** if it has no nonzero ideals.[3] A Lie group is called **simple** if its Lie algebra is simple.[4] A Lie group is called **reductive** if its Lie algebra is a product of an abelian Lie algebra with a direct sum of simple Lie algebras.[5] The compact Lie groups with such Lie algebras are precisely the compact connected Lie groups,[4] so isomorphism classes of compact connected Lie groups correspond (bijectively) to isomorphism classes of complex reductive groups.[6]

Each locally compact abelian group has a **Pontryagin dual**,[7] defined to be the group of continuous group homomorphisms from the original group to $U(1)$.[8] The Pontryagin dual is used in the general theory of Fourier transforms. The groups $U(1)$ and $\mathbb{Z}$ are each other's Pontryagin duals, and $\mathbb{Z}_n$ is its own Pontryagin dual.[9]

A Lie group G is, among other things, a smooth manifold. The **fundamental group**[10] of G, denoted $\pi_1(G)$, is trivial if and only if every closed curve in G is contractible. Roughly, the number of elements of $\pi_1(G)$ is the number of closed curves in G that are not isotopic[11] to each other (cannot be continuously deformed into each other within G).

[1] Niblo *et al* (2014), section 2.2

[2] Example: $SU(n)$ is semisimple, but $U(1)$ is not.

[3] Fulton and Harris (1991), text after exercise 9.2

[4] Article 92035

[5] Fulton and Harris (1991), exercise 9.25, combined with the fact that a semisimple Lie algebra is a direct sum of simple Lie algebras (article 91563)

[6] Kamnitzer (2011), section 5.2

[7] *Pontryagin* is also spelled *Pontrjagin*.

[8] Ben-Zvi (2021), equation (3.34)

[9] `https://ncatlab.org/nlab/show/Pontrjagin+dual`, examples 2.1 and 2.3

[10] Article 61813

[11] Section 4.5 and chapter 8 in Hirsch (1976) both define **isotopic**.

2 The Langlands dual of a Lie group

Let G be a compact connected Lie group, so G is also a reductive group.[12] Every such group has a corresponding **Langlands dual group** or **GNO dual group**,[13,14] denoted $G^\vee$ (or ^LG),[15,16] which is again a compact connected Lie group. The group $G^\vee$ is characterized by these properties[17]

- The root data of G and $G^\vee$ are dual to each other.
- If G is compact and simple, then $Z(G^\vee)$ is naturally isomorphic to $\pi_1(G)$.[18] More generally, $Z(G^\vee)$ is isomorphic to the **Cartier dual**[19,20] of $\pi_1(G)$.

In particular, if G is simply connected, then $G^\vee$ has trivial center.[21] The converse is also true, thanks to the identity[22]

$$(G^\vee)^\vee = G. \tag{1}$$

In physics, one of the most important properties of $G^\vee$ is that its irreducible representations correspond to conjugacy classes of homomorphisms $U(1) \to G$.[23,24]

---

[12] Section 1

[13] Debray (2021), lecture 28; Kapustin and Witten (2007), section 1

[14] Some non-compact Lie groups also have Langlands duals. Examples are shown in Frenkel (2009), page 1010-05; and Debray (2021), text after equation (29.1).

[15] The Langlands dual group is often denoted ^LG in the physics literature (example: Kapustin (2010), section 0.1). According to https://mathoverflow.net/a/475646, mathematicians use the notation ^LG for something else.

[16] Beware that the notation $G^\vee$ is also used for the *Cartier dual* and *Pontryagin dual* of an abelian group G (Ben-Zvi (2021), equations (3.34) and (3.77)).

[17] Ben-Zvi (2021), text above equation (11.124); Debray (2021), text after theorem 28.5 and above equation (29.1); Daenzer and Van Erp (2014), section 2.4 (for complex Lie groups)

[18] Kapustin and Witten (2007), text before equation (6.10) (the condition *compact* is stated in the first paragraph of section 2.1, and the condition *simple* is stated in the text after equation (2.3))

[19] Debray (2021), text after theorem 28.5

[20] In the present context, *Cartier dual* may be replaced with *Pontryagin dual* (https://ncatlab.org/nlab/show/Cartier+duality).

[21] Ben-Zvi (2021), text below equation (11.132); Debray (2021), text above equation (29.1).

[22] Gukov and Witten (2006), text after equation (A.15); Debray (2017), last Remark in lecture 24 (calls $G \to G^\vee$ an *involution*)

[23] Kapustin and Witten (2007), text after equation (6.9)

[24] Section 0.1 in Kapustin (2010) uses this as the definition of $G^\vee$.

3 The Langlands dual of a Lie group: examples

The group $U(n)$ is its own Langlands dual.[25] This table lists the Langlands duals of the compact connected *simple* Lie groups:[26,27,28]

root system of G	root system of $G^\vee$	G	$G^\vee$
A_n	same	$SU(km)/\mathbb{Z}_k$	$SU(km)/\mathbb{Z}_m$
B_n	C_n	$SO(2n+1)$ $\text{Spin}(2n+1)$	$\text{Sp}(n)$ $\text{Sp}(n)/\mathbb{Z}_2$
D_n	same	$SO(2n)$ $\text{Spin}(2n)$ $\text{Spin}(8k)/\mathbb{Z}_2'$ $\text{Spin}(8k)/\mathbb{Z}_2''$ $\text{Spin}(8k+4)/\mathbb{Z}_2'$	$SO(2n)$ $SO(2n)/\mathbb{Z}_2$ $\text{Spin}(8k)/\mathbb{Z}_2'$ $\text{Spin}(8k)/\mathbb{Z}_2''$ $\text{Spin}(8k+4)/\mathbb{Z}_2''$
E_6	same	E_6	$E_6/\mathbb{Z}_3$
E_7	same	E_7	$E_7/\mathbb{Z}_2$
$E_8,\ F_4,\ G_2$	same	self-dual	

The first row includes this important case: the groups $SU(n)$ and $SU(n)/\mathbb{Z}_n$ are each other's Langlands duals.[25] Thanks to the relationship (1), this table includes every compact connected simple Lie group.[29] The B_n-C_n cases are the only ones for which the Lie algebras of G and $G^\vee$ are not isomorphic to each other.[30,31]

Based on this list and the relationships in section 2, we can infer that if G is semisimple (no $U(1)$ factors), then the universal cover of $G^\vee$ is $(G/Z(G))^\vee$.

[25] Kapustin and Witten (2007), table 1

[26] Compiled from table 6.4 in Figueroa-O'Farrill (1998) and table 1 in Kapustin and Witten (2007)

[27] Different authors use different notations for the groups $\text{Sp}(n)$. Article 92035 defines the notation used here, which agrees with Kapustin and Witten (2007).

[28] $G/\mathbb{Z}_n$ is the quotient of G by an $\mathbb{Z}_n$ subgroup of $Z(G)$. In the A_n row, $km = n+1$. In the last three D_n rows, $8k = 2n$. $\mathbb{Z}_2'$ and $\mathbb{Z}_2''$ denote distinct subgroups of the center of $\text{Spin}(\cdot)$ for which the quotient is not isomorphic to $SO(\cdot)$ (article 92035). The groups denoted E_6 and E_7 are the compact 1-connected groups with those root systems.

[29] Article 92035

[30] Kapustin and Witten (2007), text after equation (2.17)

[31] For the A, D, E root systems, Langlands duality is a form of **T-duality** (Daenzer and Van Erp (2014)).

4 References

(Open-access items include links.)

Ben-Zvi, 2021. "Between electric-magnetic duality and the Langlands program" https://jacksontvandyke.com/lecture_notes.html

Daenzer and Van Erp, 2014. "T-duality for Langlands dual groups" *Adv. Theor. Math. Phys.* **18**: 1267–1285, https://arxiv.org/abs/1211.0763

Debray, 2017. "M392C notes: representation theory" https://adebray.github.io/lecture_notes/m392c_RT_notes.pdf

Debray, 2021. "M390C notes: geometric Langlands" https://adebray.github.io/lecture_notes/m390c_GL_2021_notes.pdf

Figueroa-O'Farrill, 1998. "Electromagnetic Duality for Children" https://www.maths.ed.ac.uk/~jmf/Teaching/Lectures/EDC.pdf

Frenkel, 2009. "Gauge theory and Langlands duality" https://arxiv.org/abs/0906.2747

Fulton and Harris, 1991. *Representation Theory: A First Course*. Springer

Gukov and Witten, 2006. "Gauge theory, ramification, and the geometric Langlands program" *Current Developments in Mathematics* **2006**: 35-180, https://intlpress.com/JDetail/1806709570892447746

Hirsch, 1976. *Differential Topology*. Springer-Verlag

Kamnitzer, 2011. "Representation theory of compact groups and complex reductive groups, Winter 2011" https://www.math.toronto.edu/jkamnitz/courses/reptheory/reptheory.pdf

Kapustin, 2010. "Langlands Duality and Topological Field Theory" https://www.icmat.es/seminarios/langlands/school/handouts/kapustin.pdf

Kapustin and Witten, 2007. "Electric-Magnetic Duality And The Geometric Langlands Program" *Communications in Number Theory and Physics* **1**: 1-236, https://link.intlpress.com/JDetail/1805800346326876162

Niblo *et al*, 2014. "Affine Weyl groups and Langlands duality" https://eprints.maths.manchester.ac.uk/2172/1/AffineWeyl11C.pdf

5 References in this series

Article **61813** (https://cphysics.org/article/61813):
"Homotopy, Homotopy Groups, and Covering Spaces"

Article **91563** (https://cphysics.org/article/91563):
"Characterizing the Irreducible Representations of Compact Simple Lie Groups"

Article **92035** (https://cphysics.org/article/92035):
"The Topology of Lie Groups: a Collection of Results"