Lorentz Transforms from Reflections

Randy S

Abstract Every Lorentz transform that preserves the origin in N-dimensional spacetime can be expressed as the composition of N or fewer reflections. This is a special case of the **Cartan-Dieudonné theorem**. This article shows some examples and uses the theorem to demonstrate that null rotations can be expressed in terms of ordinary rotations and boosts.

Contents

1 Introduction

Let V be an N-dimensional vector space V over the real numbers. The components of a vector $v \in V$ will be denoted v_n, where the index n can take N different values.[1] Partition the set of allowed index-values into two disjoint subsets S and T, and define

$$\langle a, b\rangle \equiv \sum_{n\in S} a_n b_n - \sum_{n\in T} a_n b_n \tag{1}$$

with $a, b \in V$. Equation (1) defines an **inner product** on the N-dimensional vector space. Let $|S|$ and $|T|$ be the number of elements in S and T, respectively, so $|S| + |T| = N$. The pair $|S|, |T|$ will be called the **signature** of the inner product.[2] An **isometry** of (1) is a linear transform $\sigma : V \to V$ that satisfies

$$\big\langle \sigma a, \sigma b\big\rangle = \langle a, b\rangle$$

for all $a, b \in V$. Special cases:

- If either $|S|$ or $|T|$ is equal to 0, then the signature is called **euclidean**, and the group[3] of isometries is called the **orthogonal group**. This is the group of origin-preserving[4] symmetries of N-dimensional flat space.

- If $|S|$ and $|T|$ are both nonzero and one of them is equal to 1, then the signature is called **lorentzian**, and the group of isometries is called the **Lorentz group**. This is the group of origin-preserving symmetries of N-dimensional flat spacetime (article 48968).

For any signature, every isometry of (1) can be expressed as a composition of (N or fewer) reflections. This is the **Cartan-Dieudonné theorem**.[5] This article uses that theorem to illuminate the structure of the Lorentz group.

[1] In this article, every index is written as a subscript, in contrast to articles 09894 and 48968.

[2] Mnemonic: $|S|$ and $|T|$ are the numbers of **space** and **time** dimensions, respectively.

[3] Article 29682 reviews the axioms of group theory.

[4] The condition *origin-preserving* excludes translations.

[5] Lam (2005), chapter 1, theorem 7.1; and Varadarajan (2004), section 5.2. Pages 18-22 in Lam (2005) give a relatively friendly proof.

2 Reflections

Two vectors a, b are called **orthogonal** to each other if $\langle a, b\rangle = 0$. An individual vector $a \in V$ will be called

- **spacelike** if $\langle a, a\rangle > 0$,
- **timelike** if $\langle a, a\rangle < 0$,
- **lightlike** (or **null** or **isotropic**)[6] if $\langle a, a\rangle = 0$.

For every non-lightlike vector a, we can define a corresponding **reflection**, which is the linear transform $\rho(a) : V \to V$ defined by

$$\boxed{\rho(a)v = v - 2\frac{\langle v, a\rangle}{\langle a, a\rangle}a} \tag{2}$$

for all $v \in V$. The vector a is the **direction** of the reflection.[7] We can write $v = v_\perp + v_\parallel$, where the two terms are orthogonal and parallel to a, respectively. Then $\rho(a)v = v_\perp - v_\parallel$. In words: the reflection changes the sign of the component parallel to a.

Every reflection is an isometry:

$$\big\langle \rho(a)x,\ \rho(a)y\big\rangle = \langle x, y\rangle$$

for all $x, y \in V$. To prove this, use the definition (2) together with the linearity property

$$\langle x + x', y + y'\rangle = \langle x, y\rangle + \langle x', y\rangle + \langle x, y'\rangle + \langle x', y'\rangle,$$

which follows from the definition (1).

[6] The names *lightlike* and *null* are more common in the physics literature. The name *isotropic* is common in the mathematics literature.

[7] An ordinary mirror in 3-dimensional space reflects a single direction, namely the direction orthogonal to the mirror. When you look at yourself in a mirror, front and back are switched (this is why you see your eyes instead of the back of your head), whereas left, right, up, and down all remain the same. Words look backward in a mirror because front and back are switched. Left and right are *not* interchanged.

3 Examples of isometries in lorentzian signature

For the rest of this article, suppose that the signature is lorentizan, and take the sets S and T of allowed index-values in (1) to be

$$S = \{1, 2, ..., N-1\} \qquad\qquad T = \{0\}.$$

Call v_0 the **time component** of a vector v, and call v_n with $n \in S$ the **space components**. Given a linear transform $\sigma : V \to V$, let $(\sigma v)_n$ denote the nth component of the transformed vector σv.

One example of an isometry is the (ordinary) rotation[8]

$$\begin{bmatrix} (\sigma v)_1 \\ (\sigma v)_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

with $(\sigma v)_n = v_n$ for $n \notin \{1, 2\}$. Another example is the **null rotation**

$$\begin{bmatrix} (\sigma v)_0 \\ (\sigma v)_1 \\ (\sigma v)_2 \end{bmatrix} = \begin{bmatrix} 1+\theta^2/2 & -\theta^2/2 & \theta \\ \theta^2/2 & 1-\theta^2/2 & \theta \\ \theta & -\theta & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \tag{4}$$

with $(\sigma v)_n = v_n$ for $n \geq 3$. Another example is the boost (or **hyperbolic rotation**)[9]

$$\begin{bmatrix} (\sigma v)_0 \\ (\sigma v)_1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \tag{5}$$

with $(\sigma v)_n = v_n$ for $n \geq 2$. The following sections show that each of these can be expressed as a pair of reflections, which in turn proves that they really are isometries.

[8] I'm using matrix notation here (article 18505).

[9] The functions $\cosh\theta$ and $\sinh\theta$ are defined in article 77597.

4 Planes

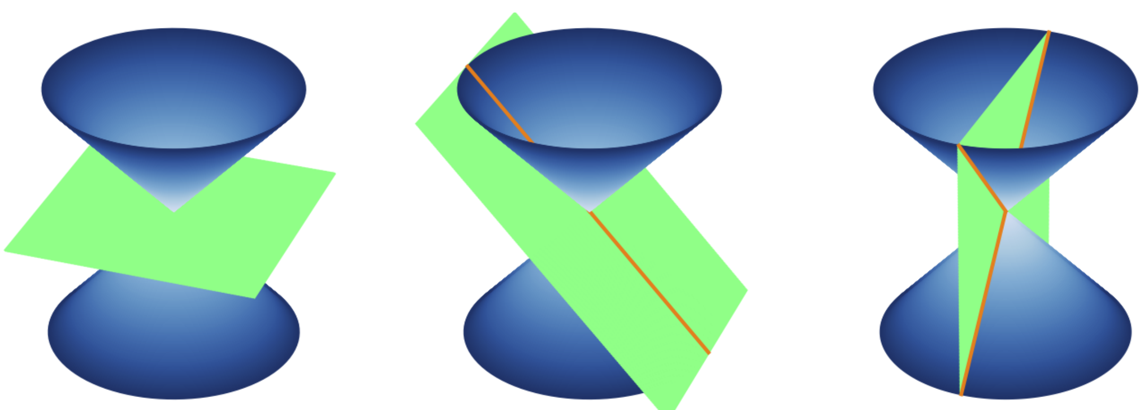
If two nonzero vectors are not parallel to each other, then they span a plane. When the signature is lorentzian, a plane P can be one of three types:

- A **space-space plane** does not contain any lightlike lines. Such a plane can *only* be spanned by two spacelike vectors.[10]

- A **null plane** contains exactly one lightlike line. Such a plane is tangent to the light cone.

- A **time-space plane** contains two lightlike lines. Such a plane contains both timelike and spacelike vectors.

Examples of each type are depicted below. The plane is shown in green. The shaded blue surface is the **light cone**, the set of lightlike vectors from the origin. Red lines indicate where the plane touches the light cone.



The following sections consider isometries $\sigma : V \to V$ that can be expressed as pairs of reflections, $\sigma = \rho(b)\rho(a)$, where a and b are not lightlike.

[10]Each other type of plane can also be spanned by two spacelike vectors, but for a space-space plane, this is the *only* option.

5 Ordinary rotations from reflections

Let a, b be linearly independent vectors that span a space-space plane, so that all linear combinations of a, b are spacelike. Then the isometry $\rho(b)\rho(a)$ is called an **ordinary rotation**, or just **rotation**.

To illustrate this, consider the spacelike vectors

$$\begin{aligned} a &= (a_0,\, a_1,\, a_2,\, a_3,\, ...) = (0,\, 1,\, 0,\, 0,\, ...) \\ b &= (b_0,\, b_1,\, b_2,\, b_3,\, ...) = (0,\, C,\, S,\, 0,\, ...) \end{aligned}$$

with $C \equiv \cos(\theta/2)$ and $S \equiv \sin(\theta/2)$, and the components not shown are all zero. The plane spanned by a and b does not contain any lightlike lines, so $\rho(b)\rho(a)$ is an ordinary rotation. In fact, $\rho(b)\rho(a)$ is the rotation shown in equation (3). To prove this, use $\langle a, a\rangle = 1 = \langle b, b\rangle$ to get[11]

$$\rho(b)\rho(a)v = v - 2\langle v, a\rangle a - 2\langle v, b\rangle b + 4\langle v, a\rangle\langle a, b\rangle b.$$

Evaluate the inner products to get

$$\rho(b)\rho(a)v = v - 2v_1 a + 2(Cv_1 - Sv_2)b. \tag{6}$$

Clearly $\big(\rho(b)\rho(a)v\big)_n = v_n$ for $n \notin \{1, 2\}$, and the 1,2 components of (6) can be written in matrix form like this:

$$\begin{bmatrix} \big(\rho(b)\rho(a)v\big)_1 \\ \big(\rho(b)\rho(a)v\big)_2 \end{bmatrix} = \begin{bmatrix} 2C^2 - 1 & -2CS \\ 2CS & 1 - 2S^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Finally, use the identities

$$2C^2 - 1 = 1 - 2S^2 = C^2 - S^2 = \cos\theta \qquad\qquad 2CS = \sin\theta$$

to recover the rotation (3).

[11] Notice that $\rho(a)$ and $\rho(b)$ don't commute with each other: $\rho(a)\rho(b) \neq \rho(b)\rho(a)$.

6 Boosts from two spacelike reflections

Let a, b be linearly independent vectors that span a time-space plane as defined in section 4. If a and b are both spacelike or both timelike, then the isometry $\rho(b)\rho(a)$ is called a **boost**.[12]

To illustrate this, consider the spacelike vectors

$$\begin{aligned} a &= (a_0,\, a_1,\, a_2,\, a_3,\, ...) = (0,\, 1,\, 0,\, 0,\, ...) \\ b &= (b_0,\, b_1,\, b_2,\, b_3,\, ...) = (S,\, C,\, 0,\, 0,\, ...) \end{aligned}$$

with $C \equiv \cosh(\theta/2)$ and $S \equiv \sinh(\theta/2)$, and the components not shown are all zero. The plane spanned by a and b contains two lightlike lines, so $\rho(b)\rho(a)$ is a boost. In fact, $\rho(b)\rho(a)$ is the boost shown in equation (5). To prove this, use $\langle a, a\rangle = 1 = \langle b, b\rangle$ to get

$$\rho(b)\rho(a)v = v - 2\langle v, a\rangle a - 2\langle v, b\rangle b + 4\langle v, a\rangle\langle a, b\rangle b, \tag{7}$$

just like in the previous section. Evaluate the inner products to get

$$\rho(b)\rho(a)v = v - 2v_1 a + 2(Sv_0 + Cv_1)b. \tag{8}$$

Clearly $\big(\rho(b)\rho(a)v\big)_n = v_n$ for $n \notin \{0, 1\}$, and the 0,1 components of (8) can be written in matrix form like this:

$$\begin{bmatrix} \big(\rho(b)\rho(a)v\big)_0 \\ \big(\rho(b)\rho(a)v\big)_1 \end{bmatrix} = \begin{bmatrix} 2S^2+1 & 2CS \\ 2CS & 2C^2-1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}.$$

Finally, use the identities

$$2S^2 + 1 = 2C^2 - 1 = C^2 + S^2 = \cosh\theta \qquad\qquad 2CS = \sinh\theta$$

to recover the boost (5).

[12] Section 10 addresses the case where one is timelike and one is spacelike.

7 Boosts from two timelike reflections

A time-space plane can be spanned either by a pair of spacelike vectors (as in the previous section) or by a pair of timelike vectors. The latter case doesn't give anything new, though, because the composition of any two timelike reflections may be re-expressed as a composition of two spacelike reflections.

To illustrate this, let a, b be the two spacelike vectors used in the previous example, and let x, y be the two timelike vectors

$$x = (x_0,\, x_1,\, x_2,\, x_3,\, ...) = (-1,\, 0,\, 0,\, 0,\, ...)$$
$$y = (y_0,\, y_1,\, y_2,\, y_3,\, ...) = (C,\, S,\, 0,\, 0,\, ...)$$

with $C \equiv \cosh(\theta/2)$ and $S \equiv \sinh(\theta/2)$ as before. Then

$$\rho(b)\rho(a) = \rho(y)\rho(x). \tag{9}$$

To prove this, evaluate the right-hand side using the same approach that was used in the previous section to evaluate the left-hand side.

More generally, let x, y be any two timelike vectors normalized so that $\langle x, x\rangle$ and $\langle y, y\rangle$ are both equal to -1. This normalization doesn't compromise any generality, because the reflections $\rho(x)$ and $\rho(y)$ are not affected by the magnitudes of $\langle x, x\rangle$ or $\langle y, y\rangle$. Now consider the vectors

$$a \equiv \frac{\langle x, y\rangle\, x + y}{\sqrt{\langle x, y\rangle^2 - 1}} \qquad\qquad b \equiv \frac{\langle x, y\rangle\, y + x}{\sqrt{\langle x, y\rangle^2 - 1}}.$$

These satisfy

$$\langle a, x\rangle = 0 = \langle b, y\rangle \qquad\qquad \langle a, a\rangle = 1 = \langle b, b\rangle.$$

In particular, they are both spacelike. Now, let v be any vector. Let $v_\perp$ be the part that is orthogonal to both x and y (equivalently, to both a and b), and let $v_\|$ be the part that is a linear combination of x and y (equivalently, of a and b). The fact that a and x are orthogonal to each other implies $\rho(a)\rho(-x)v = v_\perp - v_\|$, and the fact that b and y are orthogonal to each other implies $\rho(b)\rho(-y)v = v_\perp - v_\|$. Combine these to get $\rho(b)\rho(a)\rho(-x)\rho(-y)v = v$, which implies (9).

8 Null rotations from reflections

Let a, b be linearly independent vectors that span a null plane as defined in section 4. Then the isometry $\rho(b)\rho(a)$ is a **null rotation**.

To illustrate this, consider the spacelike vectors

$$\begin{aligned} a &= (\theta/4,\ \theta/4,\ -1,\ 0,\ 0,\ ...) \\ b &= (\theta/4,\ \theta/4,\ 1,\ 0,\ 0,\ ...). \end{aligned}$$

The components not shown are all zero. The sum $a+b$ is lightlike, and a linear combination of a and b cannot be timelike, so a and b span a null plane. In fact, $\rho(b)\rho(a)$ is the null rotation shown in equation (4). To prove this, use $\langle a, a\rangle = 1 = \langle b, b\rangle$ to get equation (7) again, and evaluate the inner products to get

$$\begin{aligned} \rho(b)\rho(a)v = v &- \big((\theta/2)(v_1 - v_0) - 2v_2\big)a \\ &- \big((\theta/2)(v_1 - v_0) + 2v_2\big)b \\ &- \big(\theta(v_1 - v_0) + 4v_2\big)b. \end{aligned}$$

Clearly $\big(\rho(b)\rho(a)v\big)_n = v_n$ for $n \notin \{0, 1, 2\}$, and the 0,1,2 components of (8) can be written in matrix form as shown in equation (4).

Notice that the reflections $\rho(a)$ and $\rho(b)$ both leave the lightlike vector $a+b$ invariant, because $\langle a+b,\ a\rangle = \langle a+b,\ b\rangle = 0$.

9 Null rotations from ordinary rotations and boosts

A null rotation can be expressed as a composition of an ordinary rotation and a boost. To prove this, let a and b be two spacelike vectors such that the plane spanned by a and b intersects the light cone in just one null line. Then $\rho(a)\rho(b)$ is a null rotation. Now let c be any other spacelike vector. Applying a reflection twice returns things to the way they were, to we have the obvious identity

$$\rho(a)\rho(c)\rho(c)\rho(b) = \rho(a)\rho(b). \tag{10}$$

If either of the two planes a-c or c-b is tangent to the light cone, then we can use an arbitrarily small perturbation of the direction c to push the plane either toward or away from the interior of the light cone, so that it is no longer tangent to the light cone. This means that we can choose c so that neither $\rho(a)\rho(c)$ nor $\rho(c)\rho(b)$ is a null rotation. Thanks to equation (10), this shows that any null rotation can be expressed in terms of ordinary rotations and boosts.

10 Connected components of the Lorentz group

Consider two vectors a and b, one timelike and one spacelike. The plane spanned by a and b is a time-space plane as defined in section 4, but the isometry $\rho(b)\rho(a)$ is *not* a boost: it is not equivalent to any $\rho(b')\rho(a')$ where a', b' are both timelike or both spacelike. The proof is easy: for a boost, one of the two reflection-directions can be smoothly deformed into the other one (because they're either both timelike or both spacelike), and then the two reflections cancel each other, so a boost can be smoothly deformed to the identity transform. But when one reflection-direction is timelike and one is spacelike, then the they cannot be smoothly deformed into each other without encountering a lightlike direction. Reflection along a lightlike direction is undefined because of the denominator in (2), so this composition of reflections cannot be smoothly deformed to the identity transform.

The Lorentz group is an example of a **Lie group**: it is both a group and a smooth manifold, because any element of the group can be smoothly deformed to obtain (some) other elements of the group. The Lorentz group has four **connected parts**, each consisting of transforms that can all be smoothly deformed into each other. The four components correspond to the four different combinations of these options in a composition of reflections:

- The number of reflections along spacelike directions can be even or odd.
- The number of reflections along timelike directions can be even or odd.

The number of connected components is just one simple aspect of the topology of the Lorentz group. For four-dimensional spacetime, the topology of the Lorentz group is described in more detail on page 89 in Weinberg (1995).

11 Generalizations

This article focused on a special case of the Cartan-Dieudonné theorem, namely a vector space over the *real* numbers with an inner product of *lorentzian* signature. Section 1 already mentioned that the Cartan-Dieudonné theorem works for arbitrary signature. It also works for vector spaces over arbitrary fields (such as the field of complex numbers), with one exception: if the field has characteristic 2 and the signature is $|S| = |T| = 2$, then isometries exist that cannot be expressed as a composition of reflections.[13]

[13]This is stated without proof in Scharlau (1985), chapter 9, theorem 4.12. For fields that don't have characteristic 2, proofs are shown in Lam (2005) (section 1.7, theorem 7.1) and O'Meara (2000) (chapter IV, theorem 43.3).

12 References

Lam, 2005. *Introduction to Quadratic Forms over Fields.* American Mathematical Society

O'Meara, 2000. *Introduction to Quadratic Forms.* Springer

Scharlau, 1985. *Quadratic and Hermitian Forms.* Springer-Verlag

Varadarajan, 2004. *Supersymmetry for Mathematicians: An Introduction.* American Mathematical Society

Weinberg, 1995. *The Quantum Theory of Fields.* Cambridge University Press

13 References in this series

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