Conformal Isometries in the Embedding Space Formalism

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Abstract This article introduces the group of **conformal isometries** in flat space(-time) with an arbitrary number of dimensions and arbitrary signature, with emphasis on the **embedding space formalism**. The embedding space formalism relates conformal isometries in N dimensions to ordinary isometries in $N+2$ dimensions. When $N \geq 3$, this gives the full group of conformal isometries. This article also introduces the concept of **conformal completion** (also called **conformal compactification**) and describes the conformal completion of Minkowski spacetime, including its topology and how it relates to conformal isometries.

Contents

1 Some conventions

Boldface will be used for a collection of coordinates, as in $\mathbf{x} = (x_1, x_2, ...)$. In sections 3-4, the index on a coordinate will be written as a superscript, as in $\mathbf{x} = (x^1, x^2, ...)$, and the standard summation convention will be used: a sum is implied over any variable index that appears both as a superscript and as a subscript in the same term. Example:

$$g_{ab}(\mathbf{x})dx^a\,dx^b \qquad \text{is an abbreviation for} \qquad \sum_{a,b} g_{ab}(\mathbf{x})dx^a\,dx^b.$$

Given a smooth manifold $\mathcal{M}$, article 48968 explains how geometry can be defined on $\mathcal{M}$ by specifying a **line element**, an expression of the form $g_{ab}(\mathbf{x})dx^a\,dx^b$ where $g_{ab}(\mathbf{x})$ are the components of the metric tensor and $\mathbf{x} = (x^1, x^2, ...)$ are coordinates on the manifold. Sections 5, 9, and 15 will introduce special notations for working with diagonal metrics whose diagonal components are ± 1.

This article is about a relationship between conformal isometries of an N-dimensional manifold $\mathcal{M}$ and ordinary isometries of a higher-dimensional manifold in which $\mathcal{M}$ is embedded. Lowercase letters like $\mathbf{x}$ will be used for coordinates in the N-dimensional manifold, and uppercase letters like $\mathbf{X}$ will be used for coordinates in the higher-dimensional manifold.

The **signature** of the metric is a pair of integers (p, q) specifying how many positive and negative eigenvalues[1] it has, respectively, at any given point.[2] The higher-dimensional space will usually[3] have $N + 2$ dimensions, and then the signature (P, Q) of the higher-dimensional metric is related to the signature (p, q) of the lower-dimensional metric by $P = p + 1$ and $Q = q + 1$.

Most of this article uses the word **space** to refer to a manifold that is topologically trivial and that has a flat metric of any signature, not necessarily euclidean. The more specific word **spacetime** will sometimes be used when the signature is lorentzian.

[1]When the metric is diagonal, the eigenvalues are just the diagonal components.

[2]The signature is the same at all points.

[3]In section 9, the higher-dimensional space has only $N + 1$ dimensions.

2 Ordinary isometries: definition

Any geometry, with any signature and not necessarily flat, can be described by specifying the line element[4]

$$g_{ab}(\mathbf{x})dx^a\,dx^b,$$

where $g_{ab}(\mathbf{x})$ are the components of the metric tensor. An **(ordinary) isometry** is a diffeomorphism $\mathbf{x} \to \hat{\mathbf{x}}$ for which

$$g_{ab}(\hat{\mathbf{x}})d\hat{x}^a\,d\hat{x}^b = g_{ab}(\mathbf{x})dx^a\,dx^b. \tag{1}$$

Poincaré transformations are isometries of Minkowski spacetime.

Recall[5] that a **diffeomorphism** is a smooth re-arrangement of the manifold's points. The notation $\mathbf{x} \to \hat{\mathbf{x}}$ means "the point that had coordinates $\mathbf{x}$ is moved to the point that has coordinates $\hat{\mathbf{x}}(\mathbf{x})$, expressed as functions of the original coordinates $\mathbf{x}$." The effect of a generic diffeomorphism $\mathbf{x} \to \hat{\mathbf{x}}$ on the line element is[4]

$$g_{ab}(\mathbf{x})dx^a\,dx^b \to g_{ab}(\hat{\mathbf{x}})d\hat{x}^a\,d\hat{x}^b = g_{ab}(\hat{\mathbf{x}})(\partial_c\hat{x}^a)(\partial_d\hat{x}^b)dx^c\,dx^d$$

with $\partial_a \equiv \partial/\partial x^a$. A diffeomorphism deforms each path in spacetime. This typically changes the paths' geometric qualities. Equation (1) says that the deformation qualifies as an isometry if it doesn't change the paths' geometric qualities – if each spacelike path remains spacelike with the same proper length, each timelike path remains timelike with the same proper duration, and each lightlike path remains lightlike. This is distinct from the related concept that article 00418 calls a **fieldomorphism**, which doesn't doesn't deform paths but does replace tensor fields with new ones. In particular, a fieldomorphism of the metric field doesn't deform paths, but it does assign new geometric qualities to them.[6]

[4] Article 48968

[5] Article 93875

[6] The definitions of *isometry* and *conformal isometry* could be recast in terms of fieldomorphisms (and then a conformal isometry would be a special type of Weyl transformation, as defined section 6), but basing the definition directly on diffeomorphisms is simpler. Physicists often don't clearly distinguish between diffeomorphisms and fieldomorphisms, using the same name *diffeomorphism* for both. We can often accomplish the same goals using either one, even though they are conceptually distinct.

3 Conformal isometries: definition

A **conformal isometry**[7,8] is a diffeomorphism[9] $\mathbf{x} \to \hat{\mathbf{x}}$ for which

$$g_{ab}(\hat{\mathbf{x}})d\hat{x}^a\, d\hat{x}^b = \Omega^2(\mathbf{x})g_{ab}(\mathbf{x})dx^a\, dx^b, \tag{2}$$

which implies

$$g_{cd}(\hat{\mathbf{x}})(\partial_a\hat{x}^c)(\partial_b\hat{x}^d) = \Omega^2(\mathbf{x})g_{ab}(\mathbf{x}). \tag{3}$$

Equation (2) says that a conformal isometry preserves the line element up to a scale that can be different at different points. A conformal isometry preserves angles but does not necessarily preserve distances. The definition uses a positive factor $\Omega^2(\mathbf{x}) > 0$ to ensure that conformal isometries preserve the causal structure of spacetime. In particular, a conformal isometry of a spacelike hypersurface is another spacelike hypersurface.

The definition of *conformal isometry* makes sense in any spacetime, flat or curved, but this article focuses on conformal isometries of flat spacetime. Most of the analysis is formulated for arbitrary signatures, which includes lorentzian signature (Minkowski spacetime) as a special case.

[7] This is the name used in Wald (1984), appendix C, above equation C.3.13. The name *conformal transformation* is also used for this, but see footnote 16.

[8] Conformal isometries are often called **conformal transformations**. This article doesn't use that name because it's confusingly overloaded: it has several different meanings, two of which are (highlighted in footnote 16) are both important in this article.

[9] These diffeomorphisms are not necessarily defined everywhere in the original spacetime, but they are defined everywhere in its *conformal completion* (section 7).

4 Low-dimensional exceptions

If the manifold is N-dimensional with $N \geq 2$, then most diffeomorphisms don't satisfy the condition (2). The case $N = 1$ is exceptional: in one-dimensional space, *every* diffeomorphism satisfies this condition, because each index takes only one value, so we can divide both sides of equation (3) by $g_{ab}(\mathbf{x})$ to get an unambiguous expression for $\Omega^2(\mathbf{x})$. This shows that when $N = 1$, the group of conformal isometries of is always infinite-dimensional: it's the same as the group of diffeomorphisms.

The group of conformal isometries of two-dimensional Minkowski spacetime is infinite-dimensional, too. To see why, start with the usual expression for the line element of two-dimensional Minkowski spacetime: $dt^2 - dx^2$, with coordinates (t, x). This can also be written as $du\, dv$ with $u \equiv t + x$ and $v \equiv t - x$. This is the product of two $N = 1$ line elements, so diffeomorphisms that don't mix the coordinates u and v with each other are automatically conformal isometries.[10]

In contrast, if a flat space has signature (p, q) with either $p \geq 2$ or $q \geq 2$ (or both), the group of conformal isometries is finite-dimensional.[11] This includes two-dimensional euclidean space.[12]

[10]Two-dimensional Minkowski spacetime is exceptional in another way, too: it admits a diffeomorphism $(t, x) \to (x, t)$ that switches timelike and spacelike directions. Thi is excluded from the definition of *conformal isometry* by writing the factor as Ω^2, which is positive.

[11]Di Francesco *et al* (1997), chapters 4 and 5

[12]The "infinitesimal conformal transformations" that are often considered in two-dimensional euclidean space do not qualify as (finite) conformal isometries (Di Francesco *et al* (1997), chapter 5). Some authors include them in what they call the *conformal group*, but that's a deviation from the standard meaning of *group*.

5 Conformal isometries in flat spacetime

Now suppose that the metric tensor $g_{ab}(\mathbf{x})$ has the form

$$g_{ab}(\mathbf{x}) = \eta_{ab} \equiv \begin{cases} \pm 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The geometry defined by such a metric tensor is flat. In this case, the abbreviations

$$\eta(\mathbf{x}, \mathbf{y}) \equiv \sum_{a,b} g_{ab}\, x^a y^b \qquad\qquad \eta(\mathbf{x}) \equiv \eta(\mathbf{x}, \mathbf{x}) \tag{5}$$

will be used. The signs in (4) are arbitrary, so $\eta(\mathbf{x})$ can be negative or zero or even if $\mathbf{x} \neq \mathbf{0}$. With this notation, the line element is

$$\eta(d\mathbf{x}),$$

and a conformal isometry is a diffeomorphism $\mathbf{x} \to \hat{\mathbf{x}}$ for which

$$\boxed{\eta(d\hat{\mathbf{x}}) = \Omega^2(\mathbf{x})\, \eta(d\mathbf{x}).} \tag{6}$$

In flat spacetime, one simple example of a conformal isometry is the **dilation** (or **dilatation**)[13]

$$\mathbf{x} \to \lambda \mathbf{x} \tag{7}$$

with λ independent of $\mathbf{x}$. This qualifies as a conformal isometry because its effect on the line element has the form (6) with scale factor $\Omega(\mathbf{x}) = \lambda$.

Ordinary isometries are conformal isometries, too, with $\Omega(\mathbf{x}) = 1$. This includes translations

$$\mathbf{x} \to \mathbf{x} + \mathbf{c}$$

[13] https://english.stackexchange.com/questions/160496

and (if $\eta(\mathbf{c}) \neq 0$) reflections

$$\mathbf{x} \to \mathbf{x} - 2\,\frac{\eta(\mathbf{x}, \mathbf{c})}{\eta(\mathbf{c})}\,\mathbf{c}. \tag{8}$$

It also includes rotations and Lorentz boosts, both of which may be expressed as compositions of reflections.[14]

A more interesting example is the **inversion**

$$\mathbf{x} \to \bar{\mathbf{x}} \equiv \frac{\mathbf{x}}{\eta(\mathbf{x})}. \tag{9}$$

This transformation is defined only where the denominator is nonzero,[15] but this defect can be repaired by extending the manifold (section 7). Where it's defined, its effect on the line element is

$$\eta(d\bar{\mathbf{x}}) = \frac{\eta(d\mathbf{x})}{\big(\eta(\mathbf{x})\big)^2}, \tag{10}$$

so this transformation qualifies as a conformal isometry (6) with scale function

$$\Omega(\mathbf{x}) = \frac{1}{\eta(\mathbf{x})}.$$

To derive (10), use the identity

$$d\bar{\mathbf{x}} = \frac{d\mathbf{x}}{\eta(\mathbf{x})} - 2\frac{\eta(\mathbf{x},\, d\mathbf{x})}{\big(\eta(\mathbf{x})\big)^2}\,\mathbf{x}.$$

The terms involving $\eta(\mathbf{x},\, d\mathbf{x})$ cancel when this is substituted into the left-hand side of (10).

[14] Article 39430

[15] If the metric has euclidean signature, then it's defined everywhere except the origin $\mathbf{x} = \mathbf{0}$. If the metric has lorentzian signature (spacetime), then it's defined everywhere except points on the origin's light cone.

Any composition of conformal isometries is clearly another conformal isometry. In particular, the sequence

$$\text{inversion} \;\rightarrow\; \text{translation} \;\rightarrow\; \text{inversion}$$

defines a conformal isometry called a **special conformal transformation**. Explicitly, this sequence is

$$\mathbf{x} \to \frac{\mathbf{x}}{\eta(\mathbf{x})} \to \frac{\mathbf{x}+\mathbf{c}}{\eta(\mathbf{x}+\mathbf{c})} \to \frac{\frac{\mathbf{x}}{\eta(\mathbf{x})}+\mathbf{c}}{\eta\left(\frac{\mathbf{x}}{\eta(\mathbf{x})}+\mathbf{c}\right)} = \frac{\mathbf{x}+\eta(\mathbf{x})\mathbf{c}}{w(\mathbf{c},\mathbf{x})} \tag{11}$$

with

$$w(\mathbf{c},\mathbf{x}) \equiv 1 + 2\eta(\mathbf{c},\mathbf{x}) + \eta(\mathbf{c})\eta(\mathbf{x}).$$

The special conformal transformation is undefined at points where $w(\mathbf{c},\mathbf{x}) = 0$, but again, this defect can be repaired by extending the manifold (section 7). Where it's defined, it is a conformal isometry (6) with scale function

$$\Omega(\mathbf{x}) = \frac{1}{w(\mathbf{c},\mathbf{x})}.$$

6 Weyl transformations

As defined in section 3, a *conformal isometry* re-arranges the points of spacetime in such a way that the line element is unaffected except through an overall $\mathbf{x}$-dependent scale. In contrast, a **Weyl transformation** replaces the original metric $g_{ab}(\mathbf{x})$ with a new one of the form

$$\hat{g}_{ab}(\mathbf{x}) \equiv \Omega^2(\mathbf{x}) g_{ab}(\mathbf{x}),$$

without re-arranging the points of spacetime.[16]

Any smooth function $\Omega^2(\mathbf{x})$ that is positive for all $\mathbf{x}$ may be used to define a Weyl transformation, but the set of scale functions $\Omega(\mathbf{x})$ that can be produced by a diffeomorphism (equation (2)) is limited. Section 9 describes an example of a scale function that cannot be produced by any diffeomorphism. Weyl transformations and conformal isometries are distinct concepts.[17]

Two metric structures that are related to each other by a Weyl transformation are said to be **conformally equivalent**, because they define angles (but not necessarily distances) the same way, even if they cannot be obtained from each other by any conformal isometry. In particular, for any $\Omega(\mathbf{x})$, a metric with components

$$g_{ab}(\mathbf{x}) = \Omega^2(\mathbf{x})\, \eta_{ab}$$

is called **conformally flat**, even if it cannot be obtained from the flat metric η_{ab} by any conformal isometry.

In this article, Weyl transformations are important because, on any given smooth manifold, two metrics that are conformally equivalent to each other have the same conformal isometries.

[16] The name *conformal transformation* is often used for the thing that I'm calling a *conformal isometry*, as in Farnsworth *et al* (2017) and definition 1.2 in Schottenloher (2008). In contrast, appendix D in Wald (1984) uses the name *conformal transformation* for the thing that I'm calling a *Weyl transformation*. Both concepts are important in this article, so to avoid confusion, this article doesn't use the name *conformal transformation* at all.

[17] The distinction between these two concepts is emphasized in Farnsworth *et al* (2017).

7 Conformal completion

Consider a conformal isometry that is defined everywhere on an N-dimensional manifold $\overline{\mathcal{M}}$. If we define another N-dimensional manifold $\mathcal{M} \subset \overline{\mathcal{M}}$ by by omitting some of $\overline{\mathcal{M}}$'s points, then the given conformal isometry might not be defined everywhere on $\mathcal{M}$, simply because some of the points in $\overline{\mathcal{M}}$ that the conformal isometry tries to mix with each other are missing from $\mathcal{M}$.

Section 5 described examples of conformal isometries that are defined almost everywhere, but not everywhere, in N-dimensional Minkowski spacetime. We can think of N-dimensional Minkowski spacetime as a subset $\mathcal{M} \subset \overline{\mathcal{M}}$ of another N-dimensional manifold $\overline{\mathcal{M}}$ on which those conformal isometries are defined everywhere. They are undefined at some points of $\mathcal{M}$ only because some points in $\overline{\mathcal{M}}$ are missing from $\mathcal{M}$. More precisely, N-dimensional Minkowski spacetime is *conformally equivalent* to a dense[18] subset $\mathcal{M} \subset \overline{\mathcal{M}}$ of a closed[19] manifold $\overline{\mathcal{M}}$ on which those conformal isometries are defined everywhere. The manifold $\overline{\mathcal{M}}$ is called a **conformal completion** (or **conformal compactification**)[20] of $\mathcal{M}$. Conformally equivalent metrics have the same conformal isometries, so conformal completion can be viewed as a way of adding points to $\mathcal{M}$ – including points that would be infinitely far away in the original metric – so that its conformal isometries become well-defined everywhere.

For the smooth manifold $\mathbb{R}^N$ equipped with a flat metric of any signature (such as Minkowski spacetime), sections 10-18 will explain how to construct a closed manifold $\overline{\mathcal{M}}$ with a dense subset $\mathcal{M} \subset \overline{\mathcal{M}}$ that is conformally equivalent to the original flat manifold. When $N \geq 3$, the conformal isometries of $\overline{\mathcal{M}}$ include all of the conformal isometries of $\mathcal{M}$, so $\overline{\mathcal{M}}$ is a conformal completion of $\mathcal{M}$ when $N \geq 3$.

[18] Intuitively, this means that every point in $\overline{\mathcal{M}}$ "touches" points in $\mathcal{M}$.

[19] A manifold is called **closed** if it is compact and doesn't have any boundary. Intuitively, a closed manifold wraps back on itself in every direction. The sphere S^N is one example of a closed manifold.

[20] The conformal completion of Minkowski spacetime is compact, but this is not necessarily true for other geometries (`https://mathoverflow.net/questions/127473`).

8 Induced metric: concept

The construction in sections 10-18 starts with a higher-dimensional flat manifold from which the N-dimensional manifolds $\overline{\mathcal{M}}$ and $\mathcal{M}$ are distilled. In that construction, the flat metric on the higher-dimensional manifold is used to define metrics on $\overline{\mathcal{M}}$ and $\mathcal{M}$, using the concept of an *induced metric*. This section explains what that means, and section 9 shows an example.

Suppose that an N-dimensional manifold $\mathcal{M}$ is embedded in a higher-dimensional manifold $\mathcal{E}$. (For a familiar example with $N = 2$, we can take $\mathcal{E}$ to be three-dimensional flat euclidean space, and we can take $\mathcal{M}$ to be the set of points whose distance from the origin is 1.) A coordinate system on the lower-dimensional manifold $\mathcal{M}$ only needs N coordinates, which we can write collectively as $\mathbf{x}$. A coordinate system on the higher-dimensional manifold $\mathcal{E}$ needs a larger number of coordinates, which we can write collectively as $\mathbf{X}$. Within the submanifold $\mathcal{M} \subset \mathcal{E}$, the components of $\mathbf{X}$ can be be regarded as functions of $\mathbf{x}$. This is one way of defining $\mathcal{M}$ as a submanifold of $\mathcal{E}$.

Suppose that geometry is defined in $\mathcal{E}$ by the line element

$$G_{ab}(\mathbf{X})dX^a\,dX^b. \tag{12}$$

The line element implicitly defines a metric tensor, with components $G_{ab}(\mathbf{X})$. The line element defines geometry by assigning a length or duration to each path, depending on the sign of (12) along that path, as explained in article 48968. The **induced metric** on $\mathcal{M}$ is defined simply by considering only those paths in $\mathcal{E}$ that lie entirely within $\mathcal{M}$.

To express the induced line element on $\mathcal{M}$ in terms of the N coordinates $\mathbf{x}$, let $\mathbf{X}(\mathbf{x})$ be the functions that define $\mathcal{M}$ as a subset of $\mathcal{E}$, as explained in the previous paragraph. Substitute the functions $\mathbf{X}(\mathbf{x})$ into the line element (12) and use the identity

$$dX^c = \frac{\partial X^c}{\partial x^a}\,dx^b,$$

which is valid for any path that lies entirely within $\mathcal{M}$, to get the induced line element

$$G_{cd}\big(\mathbf{X}(\mathbf{x})\big)\,\frac{\partial X^c}{\partial x^a}\,\frac{\partial X^d}{\partial x^b}\,dx^a\,dx^b.$$

This can also be written

$$g_{ab}(\mathbf{x})dx^a\,dx^b \tag{13}$$

with

$$g_{ab}(\mathbf{x}) \equiv G_{cd}\big(\mathbf{X}(\mathbf{x})\big)\,\frac{\partial X^c}{\partial x^a}\,\frac{\partial X^d}{\partial x^b}. \tag{14}$$

This expresses the components $g_{ab}(\mathbf{x})$ of the induced metric on $\mathcal{M}$ in terms of the components $G_{ab}(\mathbf{X})$ of the ambient metric on $\mathcal{E}$, but now that we have the new line element (13), we can treat it as an intrinsic line element on the submanifold $\mathcal{M}$ itself, without regard for the embedding that motivated it. In fact, in this article, we won't need equation (14) at all, because handling the line element as a whole is sufficient (and easier). The next section uses an example to show how this works.

9 Induced metric: example

This section uses a simple example – the standard N-sphere – to illustrate the process of deriving an induced metric. For the rest of this article, each index will be written as a subscript, and the abbreviations

$$\mathbf{A}\cdot\mathbf{B}\equiv\sum_k A_k B_k \qquad\qquad \mathbf{A}^2\equiv\mathbf{A}\cdot\mathbf{A} \tag{15}$$

will be used, where the sum is over however many components the boldface quantities have.

Let

$$\mathbf{U}=(U_1,...,U_{N+1}) \tag{16}$$

denote the coordinates in the higher-dimensional space $\mathcal{E}$. In this example, $\mathcal{E}$ is an $(N+1)$-dimensional euclidean space.[21] Geometry is defined in $\mathcal{E}$ by the line element[22]

$$d\mathbf{U}\cdot d\mathbf{U}. \tag{17}$$

Given any $R\neq 0$, the N**-sphere** (or just **sphere**) S^N consists of the points that satisfy

$$\mathbf{U}^2=R^2. \tag{18}$$

In words, S^N is the set of points in $(N+1)$-dimensional euclidean space whose distance from the origin is R.

The sphere (18) is an N-dimensional manifold,[23] so any given point on the sphere has a neighborhood that can be covered using only N coordinates, even though equation (18) describes the sphere using $N+1$ coordinates (16). Conceptually, the induced metric on the sphere comes from expressing the $N+1$ coordinates

[21] In subsequent sections, the higher-dimensional space has *two* extra dimensions and has a non-euclidean signature. To distinguish it from the present example, a different symbol ($\mathcal{A}$ instead of $\mathcal{E}$) will be used for that space.

[22] Article 21808

[23] The zero-dimensional "sphere" S^0 is a pair of points, because 1-dimensional euclidean space has two points whose distance from the origin is R.

(16) in terms of N coordinates and substituting those expressions into the line element (17).

A coordinate system that uses only N coordinates cannot cover the whole sphere, but the whole sphere can be covered using two such coordinate systems, each of which omits a single point. We can take one of these coordinates systems to be

$$\mathbf{x} = (x_1,\, x_2,\, ...,\, x_N)$$

with

$$\mathbf{x} \equiv \frac{\tilde{\mathbf{U}}}{R + U_{N+1}} \qquad\qquad \tilde{\mathbf{U}} \equiv (U_1, ..., U_N). \tag{19}$$

The $\mathbf{x}$ coordinate system covers the whole sphere except for the single point with where the denominator is zero.[24]

We can use the $\mathbf{x}$ coordinate system to obtain an explicit expression for the line element on the sphere (18) induced by the line element (17) in $\mathcal{E}$. The resulting expression for the line element will be defined everywhere except at the one point where the coordinate system itself is undefined. To do this, use the abbreviations

$$W \equiv R + U_{N+1}.$$

Take the differential of (19) to get

$$d\mathbf{x} = \frac{d\tilde{\mathbf{U}}}{W} - \frac{\tilde{\mathbf{U}}\, dU_{N+1}}{W^2}. \tag{20}$$

Take the differential of (18) to get

$$\mathbf{U} \cdot d\mathbf{U} = 0,$$

which may be rewritten as

$$\tilde{\mathbf{U}} \cdot d\tilde{\mathbf{U}} + U_{N+1}\, dU_{N+1} = 0. \tag{21}$$

[24]To cover this point, we can use the coordinate system $\mathbf{x} = \tilde{\mathbf{U}}/(R - U_{N+1})$.

We also have the identities

$$1+\mathbf{x}^2=\frac{2R}{W}\qquad\qquad d\mathbf{U}\cdot d\mathbf{U}=d\tilde{\mathbf{U}}\cdot d\tilde{\mathbf{U}}+(dU_{N+1})^2. \tag{22}$$

Use equations (20)-(22) to deduce

$$d\mathbf{x}\cdot d\mathbf{x}=\frac{(1+\mathbf{x}^2)^2}{4R^2}\,d\mathbf{U}\cdot d\mathbf{U}.$$

The quantity $d\mathbf{U}\cdot d\mathbf{U}$ on the right-hand side is the metric (17) in $\mathcal{E}$, with the understanding that the coordinates $\mathbf{U}$ are now restricted to the sphere (18). Altogether, this shows that the induced metric structure on the sphere is

$$\boxed{\frac{4R^2}{(1+\mathbf{x}^2)^2}\,d\mathbf{x}\cdot d\mathbf{x}} \tag{23}$$

in the $\mathbf{x}$ coordinate system. This is the standard metric on S^N, and this derivation shows that it is conformally flat,[25] It also shows that N-dimensional euclidean space (with metric $d\mathbf{x}\cdot d\mathbf{x}$) is conformally equivalent to a sphere S^N with one point deleted.[26]

This is also an example of a Weyl transformation whose effect cannot be reproduced by any conformal isometry (section 6). This follows from the fact that the Ricci scalar is zero for flat N-dimensional euclidean space but is nonzero for the sphere with the standard metric (23). These two metrics are related to each other by a Weyl transformation with scale function $\Omega(\mathbf{x})\propto 1/(1+\mathbf{x}^2)$, but they cannot be related to each other by any diffeomorphism, because a diffeomorphism cannot change the Ricci scalar from zero to nonzero.

[25] According to page 380 in Goldberg and Kobayashi (1962), "[Every] compact, simply connected conformally flat Riemannian manifold is conformal to an ordinary sphere."

[26] Section 18 will show that when $N\geq 3$, the N-sphere is a conformal completion of N-dimensional euclidean space, as a special case of a more general result that holds for arbitrary signature.

10 The embedding space formalism

To describe a conformal completion $\overline{\mathcal{M}}$ of a flat N-dimensional space with arbitrary signature (p, q), the **embedding space formalism** uses an auxiliary space $\mathcal{A}$ with two extra dimensions, equipped with a flat metric with signature $(P, Q) = (p + 1, q + 1)$. Points of $\overline{\mathcal{M}}$ correspond to null lines through the origin in the auxiliary space $\mathcal{A}$. Ordinary origin-preserving isometries in $\mathcal{A}$ permute these null lines with each other, and the corresponding permutation of the points of $\overline{\mathcal{M}}$ turns out to be a conformal isometry of $\overline{\mathcal{M}}$. If $p + q \geq 3$, then all conformal isometries of the original N-dimensional flat manifold can be described this way.[27,28]

Here's a summary of the notation:

- $\mathcal{M}$ is conformally equivalent to the original N-dimensional space, with topology $\mathbb{R}^N$. It has a conformally flat metric with signature (p, q) with $p + q = N$.
- $\overline{\mathcal{M}}$ is an N-dimensional closed manifold. Like $\mathcal{M}$, has a conformally flat metric with signature (p, q). It is a conformal completion of $\mathcal{M}$.
- $\mathcal{A}$ is the auxiliary $N + 2$-dimensional space. It has a flat metric with signature $(P, Q) = (p+1, q+1)$. Each null line through the origin in $\mathcal{A}$ corresponds to an individual point in $\overline{\mathcal{M}}$, and conversely.

Here's an outline:

- Sections 11-14 describe $\mathcal{M}$ as a submanifold of $\mathcal{A}$ and show that ordinary origin-preserving isometries of $\mathcal{A}$ are conformal isometries of $\mathcal{M}$.
- Sections 15-17 describe $\overline{\mathcal{M}}$ as a quotient of a submanifold of $\mathcal{A}$ and show that the induced metric on $\overline{\mathcal{M}}$ is conformally flat.
- Sections 18-19 show that $\overline{\mathcal{M}}$ is a conformal completion of $\mathcal{M}$ when $N \geq 3$.
- Sections 20-21 explain how $\mathcal{M} \cong \mathbb{R}^N$ "wraps back on itself" to fit within the closed manifold $\overline{\mathcal{M}}$.

[27] Schottenloher (2008), theorem 2.9

[28] A related insight is described in Fefferman and Graham (2007) and Cap (2009).

11 From $\mathcal{A}$ to $\mathcal{M}$

The auxiliary space $\mathcal{A}$ has a flat metric with signature (P, Q) with $P \geq 1$, $Q \geq 1$, and $P + Q = N + 2$. To make the minus signs explicit, denote the coordinates of a point in $\mathcal{A}$ by $(\mathbf{U}, \mathbf{V})$, with

$$\mathbf{U} = (U_1, ..., U_P) \qquad\qquad \mathbf{V} = (V_1, ..., V_Q),$$

and write the line element for $\mathbf{A}$ as

$$d\mathbf{U} \cdot d\mathbf{U} - d\mathbf{V} \cdot d\mathbf{V}, \tag{24}$$

using the dot-product notation that was defined in equation (15). With this notation, the line element (24) doesn't have any hidden minus signs. The only minus sign is the one that's written explicitly in (24).

Define the **cone** $\mathcal{C}$ to be the hypersurface

$$\mathbf{U}^2 - \mathbf{V}^2 = 0. \tag{25}$$

In words, $\mathcal{C}$ is the union of the null lines through the origin.[29] Let $\mathcal{P}$ be the hyperplane defined by

$$U_P + V_Q = 1. \tag{26}$$

Every null line through the origin in $\mathcal{A}$ intersects $\mathcal{P}$ exactly once, *except* the null lines with

$$U_P + V_Q = 0. \tag{27}$$

The intersection $\mathcal{C} \cap \mathcal{P}$ represents the manifold $\mathcal{M}$.

The manifold $\mathcal{M}$ is topologically equivalent to $\mathbb{R}^N$. To deduce this, write $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ for the first $p = P - 1$ and $q = Q - 1$ components of $\mathbf{U}$ and $\mathbf{V}$, respectively:

$$\tilde{\mathbf{U}} \equiv (U_1, ..., U_{P-1}) \qquad\qquad \tilde{\mathbf{V}} \equiv (V_1, ..., V_{Q-1}). \tag{28}$$

[29] When the signature is lorentzian, null lines are often called **lightlike** lines.

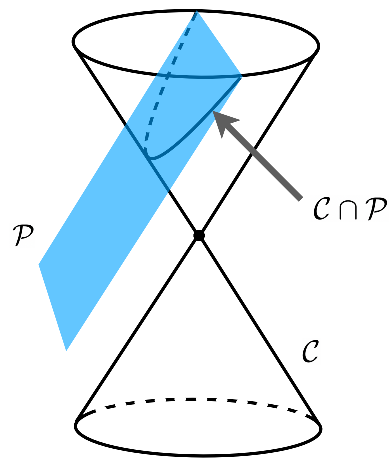
Using (26) to express V_Q in terms of U_P gives this equation for the hypersurface $\mathcal{M} = \mathcal{C} \cap \mathcal{P}$:

$$\tilde{\mathbf{U}}^2 - \tilde{\mathbf{V}}^2 = (1 - U_P)^2 - U_P^2 = 1 - 2U_P \qquad \Rightarrow \qquad U_P = \frac{1 + \tilde{\mathbf{V}}^2 - \tilde{\mathbf{U}}^2}{2}.$$

This shows that every $(\tilde{\mathbf{U}}, \tilde{\mathbf{V}})$ corresponds to a unique point in $\mathcal{M}$, and conversely, so $\mathcal{M}$ is topologically equivalent to $\mathbb{R}^{p+q} = \mathbb{R}^N$.

12 Graphic example: $N = 1$

To illustrate the construction described in section 11, consider simplest case $N = 1$. In this case, the auxiliary space $\mathcal{A}$ is three-dimensional, and the construction of $\mathcal{M}$ described in section (11) can be depicted like this:



The cone $\mathcal{C}$ defined by equation (25) is shown in the picture as the black outline, and the hyperplane (26) is shown in the picture as a blue rectangle. The hyperplane is parallel to one of the lines that makes the surface of the cone (namely the line described by equation (27)), so its intersection with the cone is a parabola, as indicated in the picture. This parabola corresponds to $\mathcal{M}$.

Section 10 defined the manifold $\overline{\mathcal{M}}$ as the set of null lines through the origin in $\mathcal{A}$: each null line through the origin corresponds to a single point in $\overline{\mathcal{M}}$, and conversely. When extended indefinitely, the parabola drawn above eventually intersects every line on the surface of the cone except the one to which the plane is parallel, so $\mathcal{M}$ is obtained from $\overline{\mathcal{M}}$ by deleting a single point.

When both p and q are nonzero, $\mathcal{M}$ is obtained from $\overline{\mathcal{M}}$ by deleting more than one point, but the construction is hard to draw in that case because the auxiliary space $\mathcal{A}$ is has $N + 2 \geq 4$ dimensions.

13 Induced metric on $\mathcal{M}$

To derive the induced metric on $\mathcal{M}$, we can use an approach like the one that was used in section 9. Define

$$\tilde{\mathbf{U}} \equiv (U_1, ..., U_{P-1}) \qquad\qquad \tilde{\mathbf{V}} \equiv (V_1, ..., V_{Q-1}) \tag{29}$$

and

$$\mathbf{u} \equiv \frac{\tilde{\mathbf{U}}}{W} \qquad\qquad \mathbf{v} \equiv \frac{\tilde{\mathbf{V}}}{W} \qquad\qquad W \equiv U_P + V_Q. \tag{30}$$

We can use $(\mathbf{u}, \mathbf{v})$ as coordinates on $\mathcal{C} \cap \mathcal{P}$ everywhere except points with $U_P + V_Q = 0$. The differentials of $\mathbf{u}$ and $\mathbf{v}$ are

$$d\mathbf{u} = \frac{d\tilde{\mathbf{U}}}{W} - \frac{\tilde{\mathbf{U}}\, dW}{W^2} \qquad\qquad d\mathbf{v} = \frac{d\tilde{\mathbf{V}}}{W} - \frac{\tilde{\mathbf{V}}\, dW}{W^2}.$$

The intersection $\mathcal{C} \cap \mathcal{P}$ is described the equations (36), which imply

$$\tilde{\mathbf{U}}^2 - \tilde{\mathbf{V}}^2 = V_Q^2 - U_P^2 \qquad\qquad \tilde{\mathbf{U}} \cdot d\tilde{\mathbf{U}} - \tilde{\mathbf{V}} \cdot d\tilde{\mathbf{V}} = V_Q\, dV_Q - U_P\, dU_P.$$

Use those results to get

$$\begin{aligned} d\mathbf{u} \cdot d\mathbf{u} - d\mathbf{v} \cdot d\mathbf{v} &= \frac{d\tilde{\mathbf{U}} \cdot d\tilde{\mathbf{U}} - d\tilde{\mathbf{V}} \cdot d\tilde{\mathbf{V}}}{W^2} + \frac{2\, dW}{W^3}(U_P\, dU_P - V_Q\, dV_Q) \\ &\qquad + \frac{(dW)^2}{W^4}(V_Q^2 - U_P^2), \\ &= \frac{d\tilde{\mathbf{U}} \cdot d\tilde{\mathbf{U}} - d\tilde{\mathbf{V}} \cdot d\tilde{\mathbf{V}}}{W^2} + \frac{dW}{W^2}(dU_P - dV_Q) \\ &= \frac{d\mathbf{U} \cdot d\mathbf{U} - d\mathbf{V} \cdot d\mathbf{V}}{W^2}. \end{aligned}$$

Altogether, this shows that the induced metric on $\overline{\mathcal{M}}$ is conformally equivalent to the flat metric $d\mathbf{u} \cdot d\mathbf{u} - d\mathbf{v} \cdot d\mathbf{v}$ with signature $(p, q) = (P-1, Q-1)$.

14 Specific conformal isometries

This section shows how each of the conformal isometries of $\mathcal{M}$ that were described in section 5 can be explicitly implemented as ordinary origin-preserving isometries in $\mathcal{A}$. Define $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ as in equations (28), and use the abbreviation

$$\tilde{\mathbf{X}} \equiv (\tilde{\mathbf{U}}, \tilde{\mathbf{V}}).$$

This is N of the coordinates on $\mathcal{A}$. The other two coordinates on $\mathcal{A}$ are U_P and V_Q. Define $\mathbf{u}$ and $\mathbf{v}$ as in equations (30), and use the abbreviation

$$\mathbf{x} \equiv (\mathbf{u}, \mathbf{v}) = \frac{\mathbf{X}}{U_P + V_Q}.$$

These will be used as the N coordinates on $\mathcal{M}$.

First consider a Lorentz boost in the U_P-V_Q plane in $\mathcal{A}$:

$$\begin{aligned} \tilde{\mathbf{X}} &\to \tilde{\mathbf{X}} \\ U_P &\to U_P \cosh\theta + V_Q \sinh\theta \\ V_Q &\to V_Q \cosh\theta + U_P \sinh\theta. \end{aligned}$$

To see what this does in $\mathcal{M}$, use the fact that the same transformation may be expressed like this:

$$\begin{aligned} \tilde{\mathbf{X}} &\to \tilde{\mathbf{X}} \\ U_P + V_Q &\to e^{-\theta}(U_P + V_Q) \\ U_P - V_Q &\to e^{\theta}(U_P - V_Q). \end{aligned}$$

According to equation (30), the effect of this on the point $\mathbf{x} \in \mathcal{M}$ is

$$\mathbf{x} \to e^{\theta}\mathbf{x}.$$

This is a dilation in $\mathcal{M}$, as defined in section 5.

Next, consider this reflection in $\mathcal{A}$:

$$\begin{aligned} \tilde{\mathbf{X}} &\to \tilde{\mathbf{X}} \\ U_P &\to -U_P \\ V_Q &\to V_Q. \end{aligned} \tag{31}$$

The effect of (31) on $\mathbf{x} \in \mathcal{M}$ is

$$\mathbf{x} \equiv \frac{\mathbf{X}}{U_P + V_Q} \to \frac{\mathbf{X}}{V_Q - U_P}. \tag{32}$$

Compare this to the effect of the inversion

$$\mathbf{x} \to \frac{1}{\eta(\mathbf{x})}\mathbf{x} = \frac{V_Q + U_P}{\eta(\tilde{\mathbf{X}})}\tilde{\mathbf{X}} \tag{33}$$

with $\eta(\cdots)$ defined as in section 5. For points on the cone (25), we have

$$\eta(\tilde{\mathbf{X}}) \equiv \tilde{\mathbf{U}}^2 - \tilde{\mathbf{V}}^2 = (V_Q + U_P)(V_Q - U_P),$$

so the right-hand sides of (32) and (33) are the same. This shows that the inversion in $\mathcal{M}$ permutes the points of $\mathcal{M}$ the same way that the reflection (31) permutes the corresponding null lines in $\mathcal{A}$.

Finally, choose any $\tilde{\mathbf{X}}_0$ with $\eta(\tilde{\mathbf{X}}_0) \neq 0$, and consider consider the effect of a reflection in $\mathcal{A}$ along $(\tilde{\mathbf{X}}, U_P, V_Q) = (\tilde{\mathbf{X}}_0, 1, -1)$ followed by a reflection along $(\tilde{\mathbf{X}}_0, 0, 0)$.[30] The effect of the first reflection is

$$(\tilde{\mathbf{X}}, U_P, V_Q) \to (\tilde{\mathbf{X}}, U_P, V_Q) - 2\frac{\eta(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}_0) + U_P + V_Q}{\eta(\tilde{\mathbf{X}}_0)}(\tilde{\mathbf{X}}_0, 1, -1). \tag{34}$$

[30] This combination of reflections gives a **null rotation** in $\mathcal{A}$, because the (two-dimensional) plane defined by the two given directions contains exactly one null direction (article 39430). A null rotation is a borderline case between an ordinary rotation (in a plane that contains no null directions) and a Lorentz boost (in a plane that contains two null directions).

When combined with the effect of the second reflection, the net effect is

$$\begin{aligned}
\tilde{\mathbf{X}} &\to \tilde{\mathbf{X}} - 2\frac{U_P + V_Q}{\eta(\tilde{\mathbf{X}}_0)}\tilde{\mathbf{X}}_0 \\
U_P &\to U_P - 2\frac{U_P + V_Q - \eta(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}_0)}{\eta(\tilde{\mathbf{X}}_0)} \\
V_Q &\to V_Q + 2\frac{U_P + V_Q - \eta(\tilde{\mathbf{X}}, \tilde{\mathbf{X}}_0)}{\eta(\tilde{\mathbf{X}}_0)}.
\end{aligned}$$

Altogether, this says that the given transformation preserves the hyperplane (26) (because it doesn't change the value of $U_P + V_Q$), and its effect on $\mathbf{x}$ is just a translation

$$\mathbf{x} \to \mathbf{x} - 2\frac{\tilde{\mathbf{X}}_0}{\eta(\tilde{\mathbf{X}}_0)}.$$

The direction of the translation can be either timelike ($\eta(\tilde{\mathbf{X}}_0) > 0$) or spacelike ($\eta(\tilde{\mathbf{X}}_0) < 0$). A translation in a null direction may be achieved by a sequence of two translations, one timelike and one spacelike. This shows how ordinary translations in $\mathcal{M}$ are implemented by isometries in the auxiliary space.

Altogether, this shows how each of the conformal isometries of $\mathcal{M}$ that were described in section 5 – including special conformal transformations – can be implemented as ordinary origin-preserving isometries in $\mathcal{A}$.

15 From $\mathcal{A}$ to $\overline{\mathcal{M}}$

Section 10 defined the manifold $\overline{\mathcal{M}}$ as the set of null lines through the origin in $\mathcal{A}$: each null line through the origin corresponds to a single point in $\overline{\mathcal{M}}$, and conversely. This section introduces an alternate description of $\overline{\mathcal{M}}$ as the intersection of $\mathcal{C}$ with another hypersurface $\mathcal{H}$ in $\mathcal{A}$, modulo the isometry that exchanges opposite points. The metric structure of the auxiliary manifold $\mathcal{A}$ is invariant under that isometry, so the induced metric on $\mathcal{C}\cap\mathcal{H}$ defines a metric on $\overline{\mathcal{M}}$. That's why this alternative description of $\overline{\mathcal{M}}$ is useful. Section 17 will show that this metric structure on $\overline{\mathcal{M}}$ is conformally flat, like the one on $\mathcal{M}$.

Consider the intersection of $\mathcal{C}$ with the hypersurface $\mathcal{H}$ defined by

$$\mathbf{U}^2+\mathbf{V}^2=2. \tag{35}$$

The intersection $\mathcal{C}\cap\mathcal{H}$ is the set of points satisfying

$$\mathbf{U}^2=1 \qquad\qquad \mathbf{V}^2=1. \tag{36}$$

This shows that $\mathcal{C}\cap\mathcal{H}$ is topologically the Cartesian product $S^p\times S^q$ of two spheres S^p and S^q with $(p,q)=(P-1,Q-1)$, because the conditions (36) can both be satisfied by choosing any point $\mathbf{U}$ satisfying $\mathbf{U}^2=1$ and any point $\mathbf{V}$ satisfying $\mathbf{V}^2=1$ independently of each other.

The intersection $\mathcal{C}\cap\mathcal{H}$ is a smooth manifold with $p+q=N$ dimensions. Every null line through the origin in $\mathcal{A}$ intersects this manifold at two points: if a null line through the origin intersects $\mathcal{H}$ at one point $(\mathbf{U},\mathbf{V})$, then the same null line also intersects $\mathcal{H}$ at the point $(-\mathbf{U},-\mathbf{V})$. Therefore, each point of $\overline{\mathcal{M}}$ corresponds to a pair of points $\pm(\mathbf{U},\mathbf{V})$ in $S^p\times S^q$. This shows that $\overline{\mathcal{M}}$ is homeomorphic (topologically equivalent)[31] to the quotient space

$$\frac{S^p\times S^q}{(\mathbf{U},\mathbf{V})\sim(-\mathbf{U},-\mathbf{V})}.$$

[31] Article 93875

More concisely:

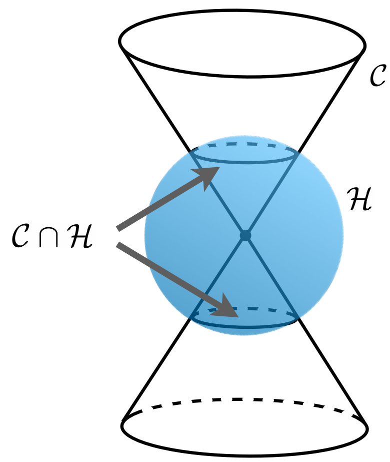
$$\mathcal{M} = \mathcal{C} \cap \mathcal{H}/\{\pm 1\} \cong S^p \times S^q/\{\pm 1\}.$$

In words, $\overline{\mathcal{M}}$ is obtained from the N dimensional manifold $\mathcal{C} \cap \mathcal{H} \cong S^p \times S^q$ by identifying opposite points with each other. The resulting manifold may or may not be homeomorphic to $S^p \times S^q$, depending on the signature (p, q),[32] but we can think of it intuitively as $S^p \times S^q$ as long as we remember that opposite points in $S^p \times S^q$ both represent the same point in $\overline{\mathcal{M}}$.

[32] https://math.stackexchange.com/questions/2316413

16 Graphic example: $N = 1$

To illustrate the construction described in section 15, consider simplest case $N = 1$. In this case, the auxiliary space $\mathcal{A}$ is three-dimensional, and the construction of $\overline{\mathcal{M}}$ described in section 15 can be depicted like this:



The cone $\mathcal{C}$ defined by equation (25) is shown in the picture as the black outline, and the hypersurface $\mathcal{H}$ defined by equation (35) is shown in the picture as the shaded green sphere. The intersection of the cone with that hypersurface is a pair of circles (equations (36)), indicated in the picture by the orange arrows. The pair of circles is topologically the same as $S^1 \times S^0$, because S^0 is just a pair of points. (In this case, the quantity $\mathbf{V}$ in equation (36) has only one component, so equation (36) says it has only two possible values, namely ± 1.) For this pair of circles, identifying opposite points amounts to equating the two circles with each other, so the manifold $\overline{\mathcal{M}}$ is a single circle S^1.

When both p and q are nonzero, the manifold $S^p \times S^q$ is connected, but the construction is hard to draw in that case because the auxiliary space $\mathcal{A}$ is has $N + 2 \geq 4$ dimensions.

17 Induced metric on $\overline{\mathcal{M}}$

Section 15 described $\overline{\mathcal{M}}$ using the intersection of the cone $\mathcal{C}$ (equation (25)) with a particular hypersurface (35). Topologically, we could also describe $\overline{\mathcal{M}}$ using any smooth deformation of the hypersurface (35) that still intersects each null line through the origin at two points that are each other's negatives. This section shows that the induced metric on $\overline{\mathcal{M}}$ is conformally flat for all such deformations. Section 18 will use this result to show that $\overline{\mathcal{M}}$ is a conformal completion of $\mathcal{M}$ when $N \geq 3$.

Let $\mathcal{H}_\rho$ be any $(P+Q-1)$-dimensional hypersurface described by the condition

$$\mathbf{U}^2 + \mathbf{V}^2 = 2\rho^2(\mathbf{U},\mathbf{V})$$

for any smooth function $\rho(\mathbf{U},\mathbf{V}) > 0$ satisfying $\rho(-\mathbf{U},-\mathbf{V}) = \rho(\mathbf{U},\mathbf{V})$. The case $\rho(\mathbf{U},\mathbf{V}) = 1$ is the hypersurface (35) that was used before. The intersection $\mathcal{C}\cap\mathcal{H}_\rho$ is the set of points satisfying

$$\mathbf{U}^2 = \rho^2(\mathbf{U},\mathbf{V}) \qquad\qquad \mathbf{V}^2 = \rho^2(\mathbf{U},\mathbf{V}). \tag{37}$$

The induced metric on $\mathcal{C}\cap\mathcal{H}_\rho$ is the metric defined by the line element (24) after using the constraints (37) to eliminate two of the coordinates.[33] The induced metric on $\mathcal{C}\cap\mathcal{H}_\rho$ is invariant under the isometry $(\mathbf{U},\mathbf{V}) \to (-\mathbf{U},-\mathbf{V})$, so this also defines a metric on $\overline{\mathcal{M}}$.

To derive the induced metric, we can choose a coordinate system on the N-dimensional manifold $\mathcal{C}\cap\mathcal{H}_\rho \subset \mathcal{A}$, express the original coordinates $\mathbf{U},\mathbf{V}$ in terms of these, and substitute the result into the line element (24) of $\mathcal{A}$. To choose a convenient coordinate system on $\mathcal{C}\cap\mathcal{H}_\rho$, first define rescaled coordinates by

$$\hat{\mathbf{U}} \equiv \frac{\mathbf{U}}{\rho(\mathbf{U},\mathbf{V})} \qquad\qquad \hat{\mathbf{V}} \equiv \frac{\mathbf{V}}{\rho(\mathbf{U},\mathbf{V})}. \tag{38}$$

Now the intersection $\mathcal{C}\cap\mathcal{H}_\rho$ is described by the conditions

$$\hat{\mathbf{U}}^2 = 1 \qquad\qquad \hat{\mathbf{V}}^2 = 1. \tag{39}$$

[33] This can always be done in a neighborhood of any given point, even though it can't be done everywhere at once.

Then define

$$\tilde{\mathbf{U}} \equiv (\hat{U}_1, ..., \hat{U}_{P-1}) \qquad\qquad \tilde{\mathbf{V}} \equiv (\hat{V}_1, ..., \hat{V}_{Q-1}) \tag{40}$$

and

$$\mathbf{u} \equiv \frac{\tilde{\mathbf{U}}}{W} \qquad\qquad \mathbf{v} \equiv \frac{\tilde{\mathbf{V}}}{W} \qquad\qquad W \equiv \hat{U}_P + \hat{V}_Q. \tag{41}$$

We can use $(\mathbf{u}, \mathbf{v})$ as coordinates on $\mathcal{C} \cap \mathcal{H}_\rho$ everywhere except points with $W = 0$. (To cover those points, we can use a similar coordinate system related to this one by an origin-preserving isometry of $\mathcal{A}$.) A calculation like the one in section 13 gives

$$d\mathbf{u} \cdot d\mathbf{u} - d\mathbf{v} \cdot d\mathbf{v} = \frac{d\hat{\mathbf{U}} \cdot d\hat{\mathbf{U}} - d\hat{\mathbf{V}} \cdot d\hat{\mathbf{V}}}{W^2}. \tag{42}$$

Rearrange equations (38) to get

$$\mathbf{U} = \rho\hat{\mathbf{U}} \qquad\qquad \mathbf{V} = \rho\hat{\mathbf{V}}, \tag{43}$$

and take the differentials of equations (39) and (43) to get

$$\begin{gathered}
\hat{\mathbf{U}} \cdot d\hat{\mathbf{U}} = 0 \qquad\qquad \hat{\mathbf{V}} \cdot d\hat{\mathbf{V}} = 0 \\
d\mathbf{U} = \rho\, d\hat{\mathbf{U}} + \hat{\mathbf{U}}\, d\rho \\
d\mathbf{V} = \rho\, d\hat{\mathbf{V}} + \hat{\mathbf{V}}\, d\rho.
\end{gathered} \tag{44}$$

Substitute these expressions for $d\mathbf{U}$ and $d\mathbf{V}$ into the metric (24) and use equations (39) and (44) to get

$$d\mathbf{U} \cdot d\mathbf{U} - d\mathbf{V} \cdot d\mathbf{V} = (d\hat{\mathbf{U}} \cdot d\hat{\mathbf{U}} - d\hat{\mathbf{V}} \cdot d\hat{\mathbf{V}})\rho^2. \tag{45}$$

Combine (42) and (45) to see that the induced metric on $\mathcal{C} \cap \mathcal{H}_\rho$ is conformally flat.

18 $\overline{\mathcal{M}}$ as the conformal completion of $\mathcal{M}$

If $\mathcal{H}_\rho$ is any hypersurface satisfying the conditions described at the beginning of section 17, then an ordinary origin-preserving isometry in $\mathcal{A}$ converts it to another hypersurface satisfying those conditions, so the result derived in section 17 implies that ordinary origin-preserving isometries in $\mathcal{A}$ act as conformal isometries in $\overline{\mathcal{M}}$. Those conformal isometries are defined everywhere in $\overline{\mathcal{M}}$.

Section 11 described $\mathcal{M}$ using the hyperplane $\mathcal{P}$ (equation (26)) instead of one of the hypersurfaces $\mathcal{H}_\rho$, but any finite piece of $\mathcal{P}$ can be treated as part of a hypersurface $\mathcal{H}_\rho$ of the type that was used in section 17, so the preceding paragraph applies to $\mathcal{M}$, too: the induced metric on $\mathcal{M}$ is the same as the one on $\overline{\mathcal{M}}$, up to conformal equivalence, and ordinary origin-preserving isometries in $\mathcal{A}$ act as conformal isometries in $\mathcal{M}$ – except at points that are mixed with points that $\mathcal{M}$ excludes.

When $N \geq 3$, the group of origin-preserving ordinary isometries in $\mathcal{A}$ gives the full group of conformal isometries of $\overline{\mathcal{M}}$ (section 19). The manifold $\mathcal{M}$ is dense in $\overline{\mathcal{M}}$ (section 7), so this shows that $\overline{\mathcal{M}}$ is a conformal completion of $\mathcal{M}$.

19 The group of conformal isometries

When $N \geq 3$, the group of origin-preserving ordinary isometries in $\mathcal{A}$ gives the full group G of conformal isometries of $\overline{\mathcal{M}}$,[34] and therefore also of $\mathcal{M}$.[35] In fact, when $N \geq 3$, the full group G of conformal isometries of $\mathcal{M}$ is generated by ordinary isometries, dilations, and inversions, all defined with respect to $\mathcal{M}$'s original flat metric. That can be inferred from this combination of observations:

- Section 4.1 in Di Francesco *et al* (1997) shows that when $N \geq 3$, ordinary isometries, dilations, and special conformal transformations generate the part of G that is continuously connected to the identity.

- Theorem 2.9 in Schottenloher (2008) shows that when $N \geq 3$, G is isomorphic to $O(p+1, q+1)/\{\pm 1\}$, where $O(p+1, q+1)$ is the orthogonal group of the $(N+2)$-dimensional auxiliary space $\mathcal{A}$.

- The group generated by ordinary isometries, dilations, and inversions in $\mathcal{M}$ has the same number of connected components as $O(p+1, q+1)/\{\pm 1\}$.[36,37]

This combination of observations implies that the whole group G of conformal isometries is generated by ordinary isometries, dilations, and inversions.

[34] Schottenloher (2008), theorem 2.9

[35] Some conformal isometries are defined only almost everywhere – not quite everywhere – in $\mathcal{M}$.

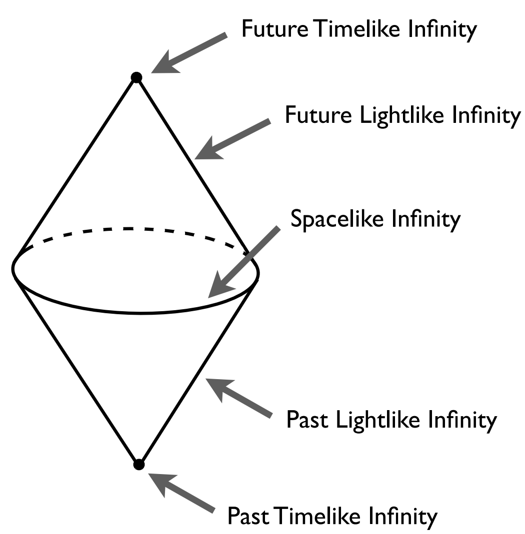
[36] The group $O(p+1, q+1)/\{\pm 1\}$, has four connected components when $p+1$ and $q+1$ are both even, and two otherwise. To deduce this, start with the fact that the group $O(p+1, q+1)$ has four connected components: one including transformations that reflect an odd number of timelike directions in the embedding space, one including transformations that reflect an odd number of spacelike directions in the embedding space, one that includes neither (this is the identity component), and one that includes both. If $p+1$ and $q+1$ are both even, the transformation -1 that reflects every direction belongs to the identity component (it reflects an even number of both timelike and spacelike directions), so the quotient doesn't merge any of the connected components. If at least one of $p+1$ or $q+1$ is odd, then -1 doesn't belong to the identity component, so the quotient merges the connected components in pairs (the details are shown in `https://physics.stackexchange.com/a/487288`).

[37] The fact that the group generated by ordinary isometries, dilations, and inversions has the same number of connected components is clear in the embedding space formalism, because every component of the group can be reached using reflections along the $N+2$ axes in the auxiliar space $\mathcal{A}$. N of these correspond to ordinary isometries in the original space, one of the others corresponds to the inversion in the original space, and the last one is redundant because of the quotient by $\{\pm 1\}$.

20 Conformal infinity: graphic preview

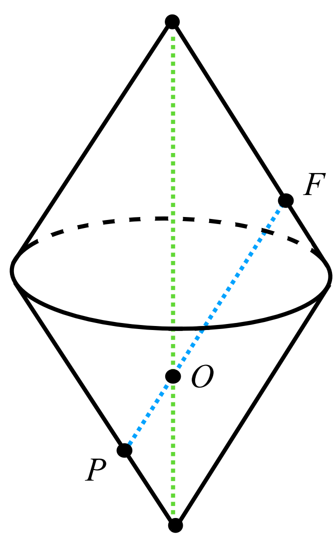
The manifold $\mathcal{M} \subset \overline{\mathcal{M}}$ is obtained by omitting some of the points in $\overline{\mathcal{M}}$. The omitted points correspond to limits in which one or more of the coordinates in $\mathcal{M}$ goes to infinity, in a coordinate system where the flat metric on $\mathcal{M}$ has the form shown in equation (4). For this reason, the omitted points are sometimes called **conformal infinity**. Each of those points can be approached from more than one direction in $\mathcal{M}$. Since $\overline{\mathcal{M}}$ is a closed manifold, each of these points can be approached from at least one pair of *opposite* directions within $\mathcal{M}$. Section 21 will describe the relationship between $\overline{\mathcal{M}}$ and $\mathcal{M}$ with that perspective in mind, specialized to the case of lorentzian signature so that $\mathcal{M}$ is (conformally equivalent to) Minkowski spacetime.

Here's a graphic preview of the results derived in section 21, for the case when $\mathcal{M}$ is three-dimensional Minkowski spacetime: $N = 3$ and $(p, q) = (2, 1)$. With the help of a Weyl transform, three-dimensional Minkowski spacetime can be depicted as the interior of a pair of back-to-back cones:[38]



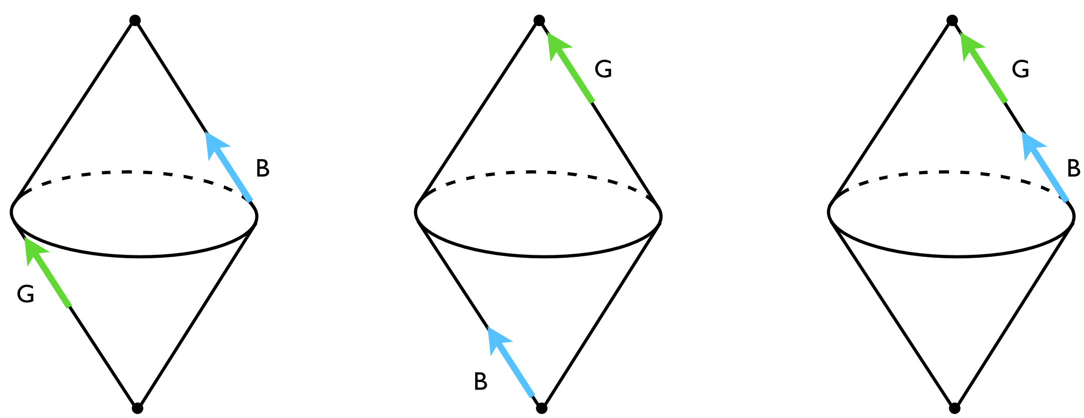
[38]These cones should not be confused with the "cone" $\mathcal{C}$ that was defined in section 15.

This is essentially a **Penrose diagram**, drawn with three dimensions instead of only two, as in Wald (1984), figure 11.2. Points on the boundary represent conformal infinity, with one caveat: they're not all distinct from each other in $\overline{\mathcal{M}}$, even though they are drawn as distinct points in the picture. Section 21 will show that the points labeled *future timelike infinity*, *past timelike infinity*, and *spacelike infinity* (the whole circle) are all the same single point in $\overline{\mathcal{M}}$. Section 21 will also show that any other point on the surface of the upper cone (future lightlike infinity) is the same (in $\overline{\mathcal{M}}$) as a corresponding point on the surface of the lower cone (past timelike infinity). The correspondence is simple: any lightlike line that passes through the origin of space at any time intersects the surfaces of the upper and lower cones, and those two intersections are the same single point in $\overline{\mathcal{M}}$. An example is depicted here:



In this picture, the vertical green dotted line is a timelike worldline at the origin of space (at all times), and the slanted blue dotted line is a lightlike worldline that passes through the origin at the event labeled O and approaches future and past lightlike infinity at the points labelled F and P, respectively. Both F and P correspond to the same single point in $\overline{\mathcal{M}}$.

Consider a path that starts somewhere on lightlike infinity (on the surface of one of the cones) and passes through spacelike infinity (the equator in these pictures). Spacelike infinity is a single point in $\overline{\mathcal{M}}$, so a continuous path that approaches the equator from past lightlike infinity could emerge from it into future lightlike infinity at any point on the equator. However, most of those continuous paths are not smooth: most of them have a kink at the point in $\overline{\mathcal{M}}$ that represents spacelike infinity in $\mathcal{M}$. Section 21 will show that in order for such a path to be *smooth*, it must emerge into future timelike infinity at the point on the equator that is opposite from where it entered, as illustrated here:



In these pictures, the path approaches spacelike infinity along the green arrow (G) and emerges from spacelike infinity along the blue arrow (B). All three pictures are different ways of drawing the same thing, thanks to the identifications described in the previous paragraph.[39]

[39] A lower-dimensional version of the picture on the left is shown in figure 5 in Strominger (2017), where it is used to illustrate the **antipodal matching condition** that characterizes the asymptotic properties of the electromagnetic field associated with a moving charge.

21 Conformal infinity: calculations

This section derives the results that were previewed in section 20. For the graphic preview in section 20, $\mathcal{M}$ was taken to be three-dimensional Minkowski spacetime, but the derivation here is for N-dimensional Minkowski spacetime with arbitrary N.

Work in an auxiliary space $\mathcal{A}$ with signature $(P, Q) = (N, 2)$, with coordinates

$$(\tilde{\mathbf{U}}, \tilde{V}, U_P, V_Q) \qquad\qquad \tilde{\mathbf{U}} \equiv (U_1, ..., U_{P-1}).$$

As before, let $\mathcal{C}$ be the cone defined by

$$\tilde{\mathbf{U}}^2 + U_P^2 = \tilde{V}^2 + V_Q^2.$$

To describe $\mathcal{M}$, let $\mathcal{P}$ be the hyperplane defined by

$$U_P + V_Q = 1. \tag{46}$$

The manifold $\mathcal{M}$ is $\mathcal{C} \cap \mathcal{P}$, as in section 11. Points on $\mathcal{C} \cap \mathcal{P}$ satisfy

$$\tilde{\mathbf{U}}^2 + U_P^2 = \tilde{V}^2 + (1 - U_P)^2,$$

which implies

$$U_P = \frac{1 - Z}{2} \qquad\qquad V_Q = \frac{1 + Z}{2}$$

with

$$Z \equiv \tilde{\mathbf{U}}^2 - \tilde{V}^2,$$

so points on $\mathcal{C} \cap \mathcal{P}$ have the form

$$(\tilde{\mathbf{U}}, \tilde{V}, U_P, V_Q) = \left(\tilde{\mathbf{U}},\, \tilde{V},\, \frac{1 - Z}{2},\, \frac{1 + Z}{2}\right). \tag{47}$$

These points are in one-to-one correspondence with $(\tilde{\mathbf{U}}, \tilde{V})$.

To describe the conformal completion $\overline{\mathcal{M}}$ of $\mathcal{M}$, let $\mathcal{H}$ be the hypersurface defined by the condition

$$\tilde{\mathbf{U}}^2 + U_P^2 + \tilde{V}^2 + V_Q^2 = 2. \tag{48}$$

The manifold $\overline{\mathcal{M}}$ is $\mathcal{C} \cap \mathcal{H}$ with opposite points identified, as explained in section 15. The intersection $\mathcal{C} \cap \mathcal{H}$ is given by the conditions

$$\tilde{\mathbf{U}}^2 + U_P^2 = 1 \qquad\qquad \tilde{V}^2 + V_Q^2 = 1, \tag{49}$$

so $\mathcal{C} \cap \mathcal{H}$ is topologically $S^{N-1} \times S^1$. A line through the origin and through the point (47) intersects $\mathcal{H}$ at the two points

$$\frac{\pm 1}{\sqrt{\tilde{V}^2 + (1+Z)^2/4}} \left(\tilde{\mathbf{U}},\, \tilde{V},\, \frac{1-Z}{2},\, \frac{1+Z}{2}\right). \tag{50}$$

This pair of points represents a single point in $\overline{\mathcal{M}}$.

Lines that don't intersect the hyperplane (46) represent points in $\overline{\mathcal{M}}$ that are not included in $\mathcal{M}$. To describe how these extra points in $\overline{\mathcal{M}}$ are approached from within $\mathcal{M}$, use the fact that line that don't intersect the hyperplane (46) have the form

$$(\tilde{\mathbf{U}}, \tilde{V}, U_P, V_Q) \propto (\tilde{\mathbf{U}}_0, \tilde{V}_0, 1, -1). \tag{51}$$

Lines of the form (51) correspond to taking $\tilde{\mathbf{U}}$ and/or $\tilde{V}$ to infinity in such a way that $Z \to \pm\infty$. If $\tilde{\mathbf{U}}$ is fixed and $|\tilde{V}| \to \infty$, then (50) approaches $\pm\,(\mathbf{0},\, 0,\, 1,\, -1)$. If $\tilde{V}$ is fixed and $|\tilde{\mathbf{U}}| \to \infty$, then (50) again approaches $\pm\,(\mathbf{0},\, 0,\, 1,\, -1)$. Opposite points in $\mathcal{C} \cap \mathcal{H}$ (differing from each other only by an overall sign) correspond to the same point in $\overline{\mathcal{M}}$, so this establishes the first result that was previewed in section 20:

> Spacelike infinity and future and past timelike infinity in $\mathcal{M}$ all correspond to the single point $\pm(\mathbf{0}, 0, 1, -1)$ in $\overline{\mathcal{M}}$.

To see what happens in lightlike directions, consider lines of the form

$$\tilde{\mathbf{U}} = (\tilde{V} - \tilde{V}_0)\mathbf{u} \tag{52}$$

with $\mathbf{u}^2 = 1$ so that $(d\tilde{\mathbf{U}})^2 - (d\tilde{V})^2 = 0$, which says that the line is lightlike. The slanted green dotted line in section 20, with endpoints labeled P and F, is an example of such a lightlike line. The constant $\tilde{V}_0$ is the value of the time coordinate $\tilde{V}$ at which this lightlike line crosses through the origin of space ($\tilde{\mathbf{U}} = 0$). In the picture shown in section 20, sliding the point labeled F upward from spacelike infinity to future timelike infinity (or sliding the point P upward from past timelike infinity toward spacelike infinity) corresponds to increasing $\tilde{V}_0$ from $-\infty$ to $+\infty$. Equation (52) implies

$$Z \equiv \tilde{\mathbf{U}}^2 - \tilde{V}^2 = (\tilde{V} - \tilde{V}_0)^2 - \tilde{V}^2 = \tilde{V}_0^2 - 2\tilde{V}_0\tilde{V},$$

so if $|\tilde{V}| \to \infty$ in (52), then the points (50) approach

$$\frac{\pm 1}{\sqrt{1 + \tilde{V}_0^2}}\Big(\mathbf{u},\, 1,\, \tilde{V}_0,\, -\tilde{V}_0\Big). \tag{53}$$

The signs correspond to $\tilde{V} \to \pm\infty$, which corresponds to appraoching F or P, respectively, along the slanted green dotted line in section 20. These two points correspond to the same point in $\overline{\mathcal{M}}$, because points in $\mathcal{C} \cap \mathcal{H}$ that differ from each other only by an overall sign represent the same point in $\overline{\mathcal{M}}$. This establishes the second result that was previewed in section 20:

> The past and future limits of any lightlike line in $\mathcal{M}$ that passes through the origin of space both correspond to the same point in $\overline{\mathcal{M}}$. This point in $\overline{\mathcal{M}}$ has the form (53).

Now consider this smooth path on $\mathcal{C}$, parameterized by λ:

$$(\lambda \mathbf{u}, \lambda, 1, -1) \tag{54}$$

with $\mathbf{u}^2 = 1$. These points don't exist in $\mathcal{M}$ because they have the form (51), but they define a smooth path in $\overline{\mathcal{M}}$. For $\lambda < 0$ and $\lambda > 0$, it has the form (53). At $\lambda = 0$, this path passes through spacelike infinity. According to equations (53) and (54), starting with $\lambda > 0$ and approaching $\lambda \to 0$ corresponds to starting with $\tilde{V}_0 > 0$ and approaching $\tilde{V}_0 \to +\infty$. This is illustrated, for a particular choice of $\mathbf{u}$, by the green arrow in the pictures at the end of section 20. If we continue along the path (54) with the same $\mathbf{u}$, then λ passes through zero and becomes increasingly negative ($\lambda < 0$), which corresponds to $\tilde{V}_0$ emerging from $-\infty$ and becoming less negative. This is illustrated by the blue arrow in the pictures. All three of the the pictures shown at the end of section 20 are different ways of drawing the same thing, because both signs in (53) represent the same point in $\overline{\mathcal{M}}$. This establishes the third result that was previewed in section 20:

> Suppose that Minkowski spacetime $\mathcal{M}$ is represented by a Penrose-like diagram as in section 20. Whenever a *smooth* curve on the lightlike-infinity surface passes through spacelike infinity, it "jumps" to the opposite side of space in the picture. This is necessary in order for the path to maintain a consistent direction within lightlike infinity, as required for the path to be *smooth*. The apparent discontinuity (the "jump") is an artifact of not drawing all of spacelike infinity as a single point in the picture, like it really is in $\overline{\mathcal{M}}$.

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