How to Think About Units

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Abstract This article explains how to think about units. This is an elementary subject, but it's still important: the more clearly we understand the easy things, the less confusion we will have when learning more advanced things.

Contents

1 The idea of a unit

The idea is simple:

A **unit** is an amount to which other amounts are compared.

In this article, saying that two quantities can be "compared" means that one can be expressed as a pure number times the other.

Here's an example. The standard international (SI) unit of length is the **meter**. If we know how long one meter is, then the comparison

$$\text{height of the tree} = 7.3 \times (\text{one meter})$$

tells us the height of the tree, and the comparison

$$\text{width of the trunk} = 0.8 \times (\text{one meter})$$

tells us the width of the trunk.

Here's another example. Suppose we agree to use the word "dozen" as a synonym for the number 12. Then three dozen chickens is the same as 36 chickens:

$$3 \times \text{dozen} = 36.$$

When we express the number of chickens this way, we are using "one dozen" as a unit. Again, a unit is simply an amount to which other amounts are compared.

Units can be manipulated just like unknown nonzero numbers can be manipulated. Here's an example: If we know how long one meter is, then the first comparison shown above can also be written

$$\frac{\text{height of the tree}}{\text{one meter}} = 7.3,$$

because the equations $y = 7.3x$ and $y/x = 7.3$ both mean the same thing if $x \neq 0$. Units can be manipulated just like unknown nonzero numbers can be manipulated.

2 Comparable units, part 1

Different kinds of units are useful for different things:

- We know how to compare the height of a tree to one meter, but how would we compare the height of a tree to one dozen? For describing the height of a tree, "one meter" is a useful unit, but "one dozen" is not.

- We know how to compare the number of chickens to one dozen, but how would we compare the number of chickens to one meter? For describing a number of chickens, "one dozen" is a useful unit, but "one meter" is not.

Quantities that can be compared to each other are sometimes said to be **comparable** or **commensurate**. If two quantities A and B are comparable, then their ratio can be reduced to an ordinary number. The standard way of expressing this symbolically is $[A] = [B]$. Examples:

$$\begin{aligned} &[\text{height of a tree}] = [\text{one meter}] \qquad && [36] \neq [\text{one meter}] \\ &[\text{height of a tree}] \neq [\text{one dozen}] \qquad && [36] = [\text{one dozen}]. \end{aligned}$$

The relationship $[A] = [B]$ doesn't mean that A and B are equal, but it does mean that they are comparable: the ratio A/B can be reduced to an ordinary number.

3 Comparable units, part 2

Whether or not two units are comparable may depend on what standards we are using. Suppose for a moment that we lived in a culture where vertical distances were traditionally measured in inches and horizontal distances were traditionally measured in meters. One day, a tree falls down across a river, and suddenly we realize that vertical distances (like the height of a standing tree) can be usefully compared to horizontal distances (like the width of a river) by *rotating* one into the other. Using the rotation protocol, we determine that 100 inches is the same as 2.54 meters. However, prior to that Day of Enlightenment, the relationship between inches and meters was undefined. In particular, the slope of an incline (rise over run) would have been expressed in inches per meter, and the ratio

$$\frac{\text{one inch}}{\text{one meter}} \tag{1}$$

could not be simplified. Even after that Day of Enlightenment, using different units for vertical and horizontal distances may sometimes still be convenient. We may sometimes still prefer to leave ratio (1) like it is, without writing it explicitly as an ordinary real number.

As far as we know today, the expression

$$\cos\big(5 \text{ meters}\big)$$

is undefined, because the cosine function is defined only for ordinary real numbers, and a meter is not comparable to a ordinary real number. (The cosine of what length is equal to 1? It's undefined.) Someday we might discover some natural sense in which a meter is comparable to a natural number, but in the meantime, the cosine of a length is undefined. In contrast, if a quantity L is comparable to a meter, then the expression

$$\cos\left(\frac{5 \text{ meters}}{L}\right)$$

is well-defined, because the ratio is an ordinary real number.

4 Some standard but confusing language

The square-bracket notation $[A]$ is nice, but some of the associated language is not ideal.

- We call $[A]$ the **units** of A or the **dimensions** of A. This language is not ideal because one meter and one kilometer "have the same units" even though they are "not the same unit" (ugh), and because one meter and one kilometer "have the same dimensions" even though they "don't have the same size" ... and "dimension" is also used as a synonym for "size" (ugh again).

- If $[A] = [1]$, then we say that A is **unitless** or **dimensionless**. This language is not ideal because it suggests that "one dozen" is not a unit, which would be an unnatural restriction that makes units seem more mysterious than they really are. A unit is simply an amount to which other amounts are compared, and of course we can compare other ordinary numbers to one dozen.

5 Composite units

An object is said to have **constant speed** if the distance D it travels is proportional to the elapsed time T. The speed v is the proportionality factor:

$$D = vT.$$

If the object travels 6 meters in 2 seconds, then its speed v is the quantity that satisfies

$$6 \text{ meters} = v \times (2 \text{ seconds}).$$

Using the abbreviations

$$\text{m} = 1 \text{ meter} \qquad \text{s} = 1 \text{ second},$$

this can also be written

$$6 \text{ m} = v \times (2 \text{ s}).$$

Since units can be manipulated just like unknown numbers, we can solve this equation for v to get

$$v = \frac{6 \text{ m}}{2 \text{ s}} = \frac{6}{2} \times \frac{\text{m}}{\text{s}} = 3\,\frac{\text{m}}{\text{s}}.$$

The ratio m/s (pronounced "meter per second") is being used here as a unit of speed, an amount of speed to which other amounts of speed can be compared. The speed 3 m/s is three times greater than 1 m/s.

> **Question:** How can we divide one meter by one second? What does distance-divided-by-time *mean*?
>
> **Answer:** That's what you get when you solve the equation $D = vT$ for v, if D is a distance and T is a time. This is the *definition* of the speed v, so speed ends up with units of length divided by time. That's all it means. Don't overthink it!

6 Converting units

Converting from one unit to another comparable unit is easy, because units can be manipulated just like unknown numbers can be manipulated.

Here's an example. Suppose we want to compare two speeds, namely one meter per second (m/s) and one kilometer per hour (km/h). One kilometer is 1000 meters, and one hour is 3600 seconds:

$$\text{km} = 1000\ \text{m} \qquad\qquad \text{h} = 3600\ \text{s}.$$

This immediately gives

$$\frac{\text{km}}{\text{h}} = \frac{1000\ \text{m}}{3600\ \text{s}} = \frac{1}{3.6} \times \frac{\text{m}}{\text{s}},$$

which can be re-arranged to get

$$\text{m/s} = 3.6\ \text{km/h}.$$

In words, one meter per second is the same speed as 3.6 kilometers per hour.

7 Natural units: Example

To introduce the concept of natural units, consider Newton's model of the gravitational interaction between N pointlike objects. This model is defined by the equations of motion[1]

$$m_n\ddot{\mathbf{x}}_n(t) = \nabla_n \sum_{j<k} \frac{Gm_j m_k}{|\mathbf{x}_j - \mathbf{x}_k|}, \tag{2}$$

where $\mathbf{x}_n(t)$ is the location of the nth object at time t in a conventional Cartesian coordinate system, $\ddot{\mathbf{x}}_n$ is its second derivative with respect to t, m_n is the mass of the nth object, and ∇_n is the gradient with respect to $\mathbf{x}_n$. From equation (2), we can infer that Newton's gravitational constant G must have units

$$[G] = \frac{\text{distance}^3}{\text{time}^2 \times \text{mass}}.$$

In the context of this model by itself, the factor of G is superfluous, because we can eliminate it by writing

$$m_n = M_n/G. \tag{3}$$

When this expression for m_n is substituted into equation (2), the factors of G cancel. In the context of this model by itself, the quantities M_n are just as good a representation of the objects' masses as the original quantities m_n were, even though the quantities M_n have units

$$[M_n] = \frac{\text{distance}^3}{\text{time}^2}.$$

In the context of the model defined by equation (2), the constant G is nothing more than a units coversion factor that compensates for expressing the m_n in terms of a separate unit of mass. We might as well choose the unit of mass so that $G = 1$, so that $m_n = M_n$, because even if we don't do this, the simple substitution (3) would eliminate all factors of G anyway. Choosing the unit of mass so that $G = 1$ is **natural** in Newton's model of gravity.

[1] This model is introduced in article 50710

8 Natural units: Perspective

Natural units is any system that eliminates one or more quantities from all of the equations that define the model, effectively replacing those quantities with 1. This is "natural" in the sense that those quantities could have been eliminated anyway by simple substitutions, as illustrated in the previous section.

What if we want to go the other way? If we start with equations written in units that are natural in one context, can we convert them to a more general system of units without any ambiguity? As an example, consider the equations

$$E^2 - \mathbf{p}^2 = m^2 \qquad\qquad \frac{\mathbf{p}}{E} = \mathbf{v} \tag{4}$$

from special relativity (article 77597), where E is the total energy of an object, $\mathbf{p}$ is its momentum, m is its mass, and $\mathbf{v}$ is its velocity. Equations (4) are written in units where c (the speed of light in a vacuum) is equal to 1. Suppose we want to generalize equations (4) to a conventional system units in which c is not reduced to a pure number, where "conventional" means

$$[E] = [\mathbf{p}][c] \qquad\qquad [\mathbf{p}] = [m][c] \qquad\qquad [\mathbf{v}] = [c]. \tag{5}$$

The only generalization that is consistent with equations (5) and that reduces to (4) when $c = 1$ is

$$E^2 - \mathbf{p}^2c^2 = m^2c^4 \qquad\qquad \frac{\mathbf{p}}{E} = \frac{\mathbf{v}}{c^2}. \tag{6}$$

This illustrates how to generalize from natural units to conventional units. The culture described in section 3 could use this same technique to generalize even further, to accommodate their convention of expressing vertical distances in inches and horizontal distances in meters.

Whatever system of units we use, the basic message is always the same: a unit is an amount to which other amounts are compared. We can use any system of units if we understand how the comparisons are defined, so we might as well use whatever system is most convenient. Natural units are a good choice whenever "most convenient" means making the basic equations as simple as possible.

9 References in this series

Article **50710** (https://cphysics.org/article/50710):
"Newton's Model of Gravity" (version 2022-02-05)

Article **77597** (https://cphysics.org/article/77597):
"Energy and Momentum at All Speeds" (version 2022-02-18)