Free Massless Scalar Quantum Fields

Randy S

Abstract Articles 00980 and 30983 studied the model of a single free scalar quantum field with a nonzero single-particle mass ($m > 0$). When $m = 0$, the model acquires a new symmetry called a **shift symmetry**. This article introduces two versions of the model with $m = 0$: one in which the shift symmetry is spontaneously broken, and one in which observables are required to be invariant under the shift symmetry.

Contents

1 Introduction

Articles 00980 and 30983 studied the model of a single free scalar quantum field with a nonzero single-particle mass ($m > 0$). Sometimes the field itself is called *massive*, meaning that the spectrum of the hamiltonian (the total energy) has a nonzero gap separating the lowest-energy state from all orthogonal states, even in the infinite-volume limit.[1] Such a model is also described as having a nonzero **mass gap**, and sometimes we simply say that the model is **gapped**.

Setting $m = 0$ gives the **massless free scalar field**, which is an example of a **gapless** model – one whose infinite-volume limit has an energy spectrum that extends continuously down to the lower bound. This is only a toy model, but it's worth studying because it's a simple example of a **scale-invariant** model (article 09193). Models with scale invariance are important to the general study of quantum field theory, because scale-invariant models are those to/from which all others flow when we zoom in or out.[2]

The model's definition is straightforward when space is treated as a lattice of finite size, but it becomes scale-invariant only in the continuum and infinite-volume limits. This article describes two different ways to define the infinite-volume limit, resulting in two slightly different variants of the model.

[1]This article defines the model initially by treating space as a finite lattice. This automatically makes the gap nonzero as long as the lattice has finite spatial size (space has finite volume). When we say that a model has a nonzero energy gap, we mean that the gap remains nonzero even in the infinite-volume limit.

[2]Simmons-Duffin (2016)

2 Review: free scalar quantum fields

Let $\mathbf{x} = (x_1, ..., x_D)$ denote a point in D-dimensional space, and let t denote the time coordinate. A model of a single free scalar quantum field can be constructed in a relatively striaightforward way by treating space as a lattice,[3] as explained in article 52890. The number of sites along a canonical axis will be denoted K, and ϵ will denote the distance between neighboring points. The abbreviation

$$L \equiv K\epsilon \tag{1}$$

will also be used. The equation of motion for the free scalar field $\phi(\mathbf{x}, t)$ has the form

$$\ddot{\phi} - \nabla^2\phi + m^2\phi = 0. \tag{2}$$

where each overhead dot denotes a derivative with respect to t, ∇ denotes the (lattice) gradient with respect to $\mathbf{x}$, and m is the mass of one particle. For any m, the field operators satisfy the equal-time commutation relations

$$\big[\phi(\mathbf{x},t), \phi(\mathbf{y},t)\big] = 0 \qquad \big[\dot{\phi}(\mathbf{x},t), \dot{\phi}(\mathbf{y},t)\big] = 0 \tag{3}$$
$$\big[\phi(\mathbf{x},t), \dot{\phi}(\mathbf{y},t)\big] = i\delta(\mathbf{x}-\mathbf{y}) \qquad \text{with } \delta(\mathbf{x}-\mathbf{y}) \equiv \begin{cases} 1/\epsilon^D & \text{if } \mathbf{x}=\mathbf{y}, \\ 0 & \text{otherwise.} \end{cases}$$

Observables localized in a spacetime region R are constructed from the field operators $\phi(\mathbf{x}, t)$ with $(\mathbf{x}, t) \in R$. The hamiltonian, the operator that generates translations in time, is

$$H = \epsilon^D \sum_{\mathbf{x}} \left(\frac{\dot{\phi}^2(\mathbf{x},t) + \big(\nabla\phi(\mathbf{x},t)\big)^2}{2} + m^2\phi(\mathbf{x},t) \right) + \text{constant}. \tag{4}$$

The hamiltonian is independent of time,[4] even though the integrand is not.

[3] This helps clarify what happens when $m = 0$.

[4] Article 52890

3 The massless free scalar field

Everything in section 2 is valid for any $m \geq 0$. This article studies the massless case $m = 0$.[5] Compared to the massive case, the massless case has two new features. First, for any real number c, the transformation

$$\phi(\mathbf{x}, t) \to \phi(\mathbf{x}, t) + c, \tag{5}$$

is a symmetry of the model, called the **shift symmetry**. Second, on a finite lattice, the $m = 0$ model does not have a vacuum state, even though its energy spectrum has a finite lower bound as required by general principles.[6]

This article describes two different ways to define an infinite-volume limit of the massless model, with different consequences:

- One way of defining the infinite-volume limit gives what this article calls the **frozen** variant of the model.[7] In this variant, the zero-momentum part of the field operator is independent of time ("frozen"). This approach works for any $D \geq 2$, where D is the number of dimensions of space, but not for $D = 1$. When it works, this variant of the model has a vacuum state. In fact, it has lots of vacuum states, because the shift symmetry is spontaneously broken.

- Another way of defining the infinite-volume limit gives what this article calls the **trimmed** variant of the model.[7] In this variant, observables are required to be invariant under the shift symmetry (5), so the set of observables is "trimmed." In particular, the zero-momentum part of the field operator no longer qualifies as an observable, not even on a finite lattice. This makes defining an infinite-volume limit relatively straightforward, and it works for any $D \geq 1$. This variant of the model again has a vacuum state, but the shift symmetry is no longer described as being spontaneously broken, because the transformation (5) doesn't affect observables at all.

[5] Articles 00980 and 30983 studied the massive case $m > 0$.
[6] Article 21916
[7] This name is not standard.

4 The untrimmed model for $D = 0$

In finite volume, the untrimmed version of the $m = 0$ model does not have a vacuum state. This section, together with section 5, demonstrates this in the simplest case $D = 0$, where space consists of a single point. Sections 6-7 demonstrate it for arbitrary D.

When $D = 0$, the field operators don't depend on $\mathbf{x}$, so we can write the field simply as $\phi(t)$. The equation of motion (2) reduces to

$$\ddot{\phi} + m^2\phi = 0. \tag{6}$$

When $m > 0$, the commutation relations (2) and the equation of motion (2) are both satisfied by

$$\phi(t) = \phi(0)\cos(mt) + \dot{\phi}(0)\frac{\sin(mt)}{m} \tag{7}$$

with

$$[\phi(0), \dot{\phi}(0)] = i. \tag{8}$$

When $m = 0$, equations (2) and (2) are satisfied instead by

$$\phi(t) = \phi(0) + \dot{\phi}(0)t, \tag{9}$$

which can also be obtained from (7) by taking $m \to 0$ with t fixed.[8]

The $m \to 0$ limit seems innocuous when described this way, but it has an interesting consequence: a lowest-energy state (vacuum state) does not exist when $m = 0$. The next section uses a Hilbert-space representation of the field operators to explain how this works.

[8]Another perspective: the limit $m \to 0$ is effectively the same as m being negligibly small compared to all other scales. Since t is the only other scale in this case, the limit $m \to 0$ is effectively the same as $m \ll 1/t$.

5 The untrimmed model for $D = 0$: Hilbert space

This section shows that a lowest-energy state (vacuum state) does not exist when $m = 0$ and $D = 0$.

As in the previous section, consider zero-dimensional space ($D = 0$). When $m \neq 0$, we can define an operator a by

$$a \equiv \frac{m\phi(0) + i\dot{\phi}(0)}{\sqrt{2m}}. \tag{10}$$

The field operators may be written

$$\phi(t) = \frac{ae^{-imt} + a^\dagger e^{imt}}{\sqrt{2m}}, \tag{11}$$

which clearly satisfies the equation of motion (2) and also satisfies the commutation relations (2) because the operator a and its adjoint $a^\dagger$ satisfy

$$[a, a^\dagger] = 1. \tag{12}$$

In terms of these operators, the hamiltonian

$$H = \frac{\dot{\phi}^2(0) + m^2\phi^2(0)}{2} + \text{constant} \tag{13}$$

may also be written

$$H = ma^\dagger a + \text{constant},$$

and the commutation relation (12) implies that a state $|0\rangle$ satisfying $a|0\rangle = 0$ has the lowest possible energy, and that the states $(a^\dagger)^n|0\rangle$ with $n \in \{0, 1, 2, ...\}$ form a basis for a complete Hilbert-space representation of the algebra of field operators.

When $m = 0$, the definition (10) does not make sense. To appreciate the consequence of taking $m \to 0$, we can construct the Hilbert space in a different way that doesn't refer to the value of m. We can take the Hilbert space to consist of complex-valued functions $f(s)$ of a single real variable s, modulo functions with zero norm,[9] with the inner product defined by $\langle f|g\rangle = \int ds\; f^*(s)g(s)$. Then the

[9] Article 90771

operators $\phi(0)$ and $\dot{\phi}(0)$ can be represented as

$$\phi(0)f(s) = s\,f(s) \qquad\qquad \dot{\phi}(0)f(s) = -i\frac{d}{ds}f(s). \tag{14}$$

The hamiltonian (13) with $m = 0$ is then represented by

$$H = \frac{\dot{\phi}^2(0)}{2} + \text{constant} = \left(-i\frac{d}{ds}\right)^2 + \text{constant}. \tag{15}$$

The operators $\phi(0)$ and $\dot{\phi}(0)$ are self-adjoint, so the hamiltonian (13) satisfies the spectrum condition: the energy spectrum is bounded from below.[10] This is true for any m, including $m = 0$, but when $m = 0$, but that lower bound is not attainable: a lowest-energy state does not exist.[11,12] To see why, let $|f\rangle$ be the state-vector represented by the function $f(s)$, and consider the quantity

$$\langle f|H|f\rangle = -\int ds\;\, f^*(s)\frac{d^2}{ds^2}f(s) = \int ds\;\, \left|\frac{df}{ds}\right|^2.$$

The last step used integration-by-parts, which doesn't produce a boundary term because f is normalizable. The right-hand side can be arbitrarily close to zero, but not equal to zero: a function with $df/ds = 0$ cannot be normalized, so it doesn't correspond to any state-vector in the Hilbert space.

[10]The lower bound can be chosen arbitrarily by choosing the value of the constant term.

[11]In this situation, we say that the energy spectrum has an **infimum** instead of a **minimum**.

[12]Mathematically, the case $m = 0$ and $D = 0$ is the same as the nonrelativistic single-particle model described in article 20554, but the interpretation is different. In that article, the observable $\phi(t)$ represented the coordinate of a particle in one-dimensional space. Here, it represents the amplitude of a field in zero-dimensional space. Same math, different interpretation.

6 The untrimmed model for any D

This section shows that the model with $m = 0$ on a finite lattice does not have a vacuum state for any D.

Define ϵ, K, and L as in section 2. In D-dimensional space, we can take the Fourier transform of the equation of motion (2) with respect to $\mathbf{x}$ to get

$$\ddot{\phi}_{\text{FT}}(\mathbf{p},t) + \omega^2(\mathbf{p})\phi_{\text{FT}}(\mathbf{p},t) = 0, \tag{16}$$

where the subscript FT indicates the Fourier-transformed field

$$\phi_{\text{FT}}(\mathbf{p},t) \equiv \epsilon^D \sum_{\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{x},t). \tag{17}$$

When space is treated as a lattice of finite size, the components of $\mathbf{p}$ are integers times $2\pi/L$. For $m \to 0$, the quantity $\omega(\mathbf{p})$ is

$$\omega(\mathbf{p}) \equiv \sqrt{\hat{\mathbf{p}}^2}$$

and the components of $\hat{\mathbf{p}}$ are[13]

$$\hat{p}_n = \frac{2\sin(p_n\epsilon/2)}{\epsilon},$$

which becomes $\hat{p}_n \to p_n$ in the limit $\epsilon \to 0$ with $K\epsilon$ held fixed.[14] The commutation relations (2) imply

$$\big[\phi_{\text{FT}}(\mathbf{p},t),\, \dot{\phi}^{\dagger}_{\text{FT}}(\mathbf{p}',t)\big] = \begin{cases} (2\pi)^D L^D & \text{if } \mathbf{p} = \mathbf{p}' \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

For $\mathbf{p} \neq \mathbf{0}$, we can solve equation (16) just like in article 00980: we can write

$$\phi_{\text{FT}}(\mathbf{p},t) = \frac{e^{-i\omega(\mathbf{p})t}a(\mathbf{p}) + e^{i\omega(\mathbf{p})t}a^\dagger(-\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} \qquad \text{for } \mathbf{p} \neq \mathbf{0}, \tag{19}$$

[13] Article 71852

[14] Notice that this implies $K \to \infty$.

where the operators $a(\mathbf{p})$ and their adjoints $a^\dagger(\mathbf{p})$ satisfy

$$[a(\mathbf{p}), a(\mathbf{p}')] = 0 \tag{20}$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^D\, \delta(\mathbf{p}' - \mathbf{p}) \equiv \begin{cases} L^D & \text{if } \mathbf{p} = \mathbf{p}' \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

For $\mathbf{p} = \mathbf{0}$, the $m = 0$ version of equation (16) reduces to the $m = 0$ version of equation (6), so

$$\phi_{\text{FT}}(\mathbf{0}, t) = \phi_{\text{FT}}(\mathbf{0}, 0) + \dot{\phi}_{\text{FT}}(\mathbf{0}, 0)t, \tag{22}$$

as in equation (9). Equations (19) and (22) show that the original field operator is

$$\begin{aligned} \phi(\mathbf{x}, t) = \frac{1}{L^D}\sum_{\mathbf{p}} \; e^{i\mathbf{p}\cdot\mathbf{x}}\phi_{\text{FT}}(\mathbf{p}, t) &= \frac{1}{L^D}\left(\sum_{\mathbf{p}\neq\mathbf{0}} \frac{a(\mathbf{p})e^{-i\omega(\mathbf{p})t + i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega(\mathbf{p})}} + \text{adjoint}\right) \\ &\quad + \frac{1}{L^D}\big(\phi_{\text{FT}}(\mathbf{0}, 0) + \dot{\phi}_{\text{FT}}(\mathbf{0}, 0)t\big). \end{aligned} \tag{23}$$

The hamiltonian is[15]

$$H = \frac{1}{L^D}\sum_{\mathbf{p}} \frac{\dot{\phi}^\dagger_{\text{FT}}(\mathbf{p}, 0)\dot{\phi}_{\text{FT}}(\mathbf{p}, 0) + \omega^2(\mathbf{p})\phi^\dagger_{\text{FT}}(\mathbf{p}, 0)\phi_{\text{FT}}(\mathbf{p}, 0)}{2} + \text{constant} \tag{24}$$

$$= \frac{1}{L^D}\sum_{\mathbf{p}\neq\mathbf{0}} \omega(\mathbf{p})a^\dagger(\mathbf{p})a(\mathbf{p}) + \frac{1}{L^D}\frac{\dot{\phi}^2_{\text{FT}}(\mathbf{0}, 0)}{2} + \text{constant.} \tag{25}$$

As in section 4, the spectrum of H has a lower bound, because each non-constant term is manifestly a positive operator, but the next section shows that the lower bound cannot be attained by any state in the Hilbert space.

[15] This satisfies $\dot{\phi}(\mathbf{x}, t) = -i[\phi(\mathbf{x}, t), H]$ and $\ddot{\phi}(\mathbf{x}, t) = -i[\dot{\phi}(\mathbf{x}, t), H]$.

7 The untrimmed model for any D: Hilbert space

The construction described in section 5 can be generalized to arbitrary D. A state-vector $|f\rangle$ is represented by a function $f[s]$ of a collection of real variables $s(\mathbf{p})$, one per wavenumber $\mathbf{p}$. The inner product is defined by

$$\langle f|g\rangle \equiv \int [ds]\; f^*[s]g[s],$$

where the integral is over $-\infty < s(\mathbf{p}) < \infty$ for each variable $s(\mathbf{p})$. Only normalizable functions – functions for which $\langle f|f\rangle$ is finite – represent elements of the Hilbert space. The field operators $\phi_{\text{FT}}(\mathbf{p},t)$ and $\dot{\phi}_{\text{FT}}(\mathbf{p},t)$ are represented at $t=0$ by

$$\phi_{\text{FT}}(\mathbf{p},0)f[s] = \beta\, s(\mathbf{p})f[s] \qquad\qquad \dot{\phi}_{\text{FT}}(\mathbf{p},0)f[s] = -i\,\frac{L^D}{\beta}\,\frac{\partial}{\partial s(\mathbf{p})}f[s] \tag{26}$$

with an arbitrary-but-fixed positive value of β, and the field operators for arbitrary t are given by

$$\phi_{\text{FT}}(\mathbf{p},t) = U^{-1}(t)\phi_{\text{FT}}(\mathbf{p},0)U(t)$$

with $U(t) = e^{-iHt}$. This is consistent with the equation of motion (16) and the commutation relations (18).

We can minimize the contribution of the $\mathbf{p}\neq\mathbf{0}$ terms in (25) by choosing a state $f[s]$ that satisfies $a(\mathbf{p})f[s]=0$, but, just like in section 5, the Hilbert space doesn't have any state that minimizes the contribution of the $\mathbf{p}=\mathbf{0}$ term, which is

$$H_{\mathbf{p}=\mathbf{0}} \equiv \frac{1}{L^D}\,\frac{\dot{\phi}^2_{\text{FT}}(\mathbf{0},0)}{2} = \frac{-(K\epsilon)^D}{2\beta^2}\left(\frac{\partial}{\partial s(\mathbf{0})}\right)^2 \tag{27}$$

in this representation. This shows that the $m\to 0$ limit of the free scalar model does not have a lowest-energy state (vacuum state), at least not on a finite lattice, even though the energy spectrum has a finite lower bound. Sections 8-9 explore what happens in the infinite-volume limit $L\to\infty$.

8 The infinite-volume limit when $m > 0$

Superficially, the infinite-volume limit is $L \to \infty$, with the understanding that K also goes to ∞ so that sums $\frac{1}{L^D}\sum_{\mathbf{p}}\cdots$ are promoted to integrals $\int \frac{d^Dp}{(2\pi)^D}\cdots$.[16] Just sending the parameters L and K to ∞ is not enough, though. When the summand in $\frac{1}{L^D}\sum_{\mathbf{p}}\cdots$ is made of operators, we need to specify a context in which the sum converges to another operator. We can do this by specifying a class of observables and states that will be retained in the limit, chosen so that the resulting model can be represented on a separable Hilbert space.[17] This section explains (roughly, without trying to be precise) how that can be done when $m > 0$, and the next section addresses the case $m = 0$.

When $L \to \infty$ equations (18) imply that the operators $\phi_{\text{FT}}(\mathbf{p}, t)$ cannot remain defined as ordinary operators on a Hilbert space, because the right-hand side of (18) does not remain an ordinary function.[18] This is true no matter how we try to represent them as operators on a Hilbert space, even if $m > 0$. On the other hand, the **smeared** operators

$$\phi_{\text{FT}}(g, t) = \frac{1}{L^D}\sum_{\mathbf{p}} g(\mathbf{p})\phi_{\text{FT}}(\mathbf{p}, t) \tag{28}$$

behave better, if we restrict the set of smearing functions g so that they become smooth and normalizable when $L \to \infty$. The equal-time commutators of such smeared field operators remain ordinary functions in that limit. We can choose a Hilbert space on which the smeared field operators and their time-derivatives at a given time, say $t = 0$, remain ordinary operators in the infinite-lattice limit. The key question is whether they can remain ordinary operators at all times, using the hamiltonian to evolve them forward/backward in time.

[16] From now on, the condition $K \to \infty$ is understood whenever $L \to \infty$.

[17] Witten (2021), section 2.2

[18] They may be definable as "operator-valued distributions," but not as ordinary operators on a Hilbert space – just like the Dirac delta distribution is not a function.

When $m > 0$, the hamiltonian is

$$H = \frac{1}{L^D} \sum_{\mathbf{p}} \omega(\mathbf{p}) a^\dagger(\mathbf{p}) a(\mathbf{p}) + \text{constant.} \tag{29}$$

In contrast to the $m = 0$ case (25), here the term with $\mathbf{p} = \mathbf{0}$ is included in the sum, because the operator $a(\mathbf{p})$ is well-defined for $\mathbf{p} = \mathbf{0}$. With the hamiltonian (29), a state $|0\rangle$ satisfying $a(\mathbf{p})|0\rangle = 0$ for all $\mathbf{p}$ has the lowest possible energy, and we can construct a Hilbert space by acting on it with the algebra of smeared field operators. If we restrict the set of smearing functions so that they remain smooth and normalizable when $L \to \infty$, then the hamiltonian remains well-defined as an ordinary operator on this class of states. This approach can be used to define an infinite-volume limit for the free scalar model when $m > 0$.[19]

[19] Article 44563 explained how the free scalar model with $m > 0$ can be defined directly in continuous-and-infinite spacetime, using smeared field operators. (That article used symplectic smearing instead of the purely spatial smearing described in this section, but this is just a difference in the way the model is formulated, not a difference in the resulting model's content.) This article doesn't use that formulation because it relies on the distinction between the positive- and negative-frequency parts of a field, which becomes undefined when $m = 0$ and $\mathbf{p} = \mathbf{0}$. Starting with the finite-lattice formulation makes things more clear, in my opinion.

9 An infinite-volume limit with $m = 0$: frozen variant

When $m = 0$, the hamiltonian is given by equation (25) instead of (29). The difference is in the term with $\mathbf{p} = \mathbf{0}$, denoted $H_{\mathbf{p}=\mathbf{0}}$ in equation (27). This term prevents the model from having a lowest-energy state. One consequence of this is that the model does not have any one distinguished state from which to construct a Hilbert-space representation as described in the previous section.[20] That's okay. When K is finite, we can start with any state $|0\rangle$ that satisfies

$$a(\mathbf{p})|0\rangle = 0 \qquad\qquad \text{for all } \mathbf{p} \neq \mathbf{0}. \tag{30}$$

Applying $\phi_{\text{FT}}(\mathbf{0}, t)$ to any such state gives another such state, because $\phi_{\text{FT}}(\mathbf{0}, t)$ commutes with $a(\mathbf{p})$. We can build a Hilbert-space representation by acting on any such state $|0\rangle$ with the algebra of field operators.

We can define an infinite-volume limit by using only smeared field operators, but compared to the $m > 0$ case, the $m = 0$ case involves a new twist. As a warm-up, consider the quantity

$$\langle 0|\phi_{\text{FT}}(g, t)|0\rangle \tag{31}$$

with $\phi_{\text{FT}}(g, t)$ defined by (28). Use equations (19), (28), (30), and then (22) to get

$$\begin{aligned} \langle 0|\phi_{\text{FT}}(g, t)|0\rangle &= \frac{1}{L^D} g(\mathbf{0})\, \langle 0|\phi_{\text{FT}}(\mathbf{0}, t)|0\rangle \\ &= \frac{1}{L^D} g(\mathbf{0})\, \big\langle 0\big|\big(\phi_{\text{FT}}(\mathbf{0}, 0) + \dot{\phi}_{\text{FT}}(\mathbf{0}, 0)t\big)\big|0\big\rangle. \end{aligned} \tag{32}$$

Without the factor of L^D in the denominator (which comes from equation (28)), this would be undefined when $L \to \infty$, because the commutation relation (18) implies that either $\phi_{\text{FT}}(\mathbf{0}, 0)$ or $\dot{\phi}_{\text{FT}}(\mathbf{0}, 0)$ (or both) must diverge in that limit. This

[20] According to Witten (2021), section 2.1, the lack of a distinguished state is an unusual situation in flat spacetime, but it is the typical situation in spacetimes whose metric is not independent of time in any coordinate system.

is clear in the representation (26). The factor of L^D in the denominator comes to the rescue: if we set

$$\beta = L^D \tag{33}$$

in (26), then the $\dot{\phi}_{\text{FT}}$ term in (32) goes to zero when $L \to \infty$, leaving[21,22]

$$\lim_{L\to\infty} \langle 0|\phi_{\text{FT}}(g,t)|0\rangle = -g(\mathbf{0}) \int [ds]\ \left|f_{|0\rangle}[s]\right|^2 s(\mathbf{0}), \tag{34}$$

which is finite for a set of normalizable functions that is dense in the Hilbert space.

More generally, we can define an infinite-volume limit by making sure that all quantities of the form

$$\langle 0|\phi_{\text{FT}}(g_1,t)\cdots\phi_{\text{FT}}(g_n,t)|0\rangle \tag{35}$$

remain finite. The components of ϕ_{FT} with $\mathbf{p} \neq \mathbf{0}$ (equation (19)) commute with the $\mathbf{p} = \mathbf{0}$ components (equation (22)), so we can handle the $\mathbf{p} \neq \mathbf{0}$ components just like in section 8, and we can handle the $\mathbf{p} = \mathbf{0}$ components using the choice (33) as described above. Then $\dot{\phi}_{\text{FT}}(\mathbf{0},t)$ goes to zero when $L \to \infty$, so $\phi_{\text{FT}}(\mathbf{0},t)$ is independent of time in that limit – hence the nane *frozen* for this variant of the model.[23] This eliminates the $\mathbf{p} = \mathbf{0}$ term (27) from the hamiltonian (25), so now any state $|0\rangle$ satisfying (30) qualifies as a vacuum state: they all have the lowest possible energy. After taking this limit, the operator $\phi_{\text{FT}}(\mathbf{0},t)$ commutes with everything, so we can take the functions in the representation (26) to be eigenfunctions of $\phi_{\text{FT}}(\mathbf{0},t)$ to an arbitrarily good approximation, with any eigenvalue we want. This variant of the model doesn't have any observables left that can mix states with different eigenvalues of $\phi_{\text{FT}}(\mathbf{0},t)$, so the shift symmetry is spontaneously broken (footnote 23).

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[21] $f_{|0\rangle}$ denotes the function corresponding to $|0\rangle$ in the representation (26).

[22] We could define a different limit by choosing β to be some other power of L. Choosing how β depends on L amounts to choosing how the widths of the functions in (26) scale with L.

[23] This name is not standard, but regardless of the name, this variant does appear to be a standard choice. Example: the text below equation (4.12) in section 4.1 of Qualls (2015) asserts that the shift symmetry is spontaneously broken, which is a feature of this particular variant of the model.

10 No spontaneous symmetry breaking when $D \leq 1$

Section 9 sketched a way of defining an infinite-volume limit, but it was only a sketch, not a complete proof. This section shows that something goes wrong when space is only one-dimensional ($D = 1$): the frozen variant does not exist in that case.

To expose the problem, consider the quantity

$$\langle 0|\phi_{\text{FT}}(g_1,t)\phi_{\text{FT}}(g_2,t)|0\rangle.$$

According to equations (19), (21), (28), and (30), the part that comes from terms with $\mathbf{p} \neq \mathbf{0}$ is

$$\langle 0|\phi_{\text{FT}}(g_1,t)\phi_{\text{FT}}(g_2,t)|0\rangle\Big|_{\text{terms with } \mathbf{p}\neq\mathbf{0}} = \frac{1}{L^D}\sum_{\mathbf{p}\neq\mathbf{0}} \frac{g_1(\mathbf{p})g_2(-\mathbf{p})}{2\omega(\mathbf{p})}. \tag{36}$$

In the limit $L \to \infty$, this becomes[24]

$$\int \frac{d^Dp}{(2\pi)^D}\,\frac{g_1(\mathbf{p})g_2(-\mathbf{p})}{2\omega(\mathbf{p})} \tag{37}$$

with a finite domain of integration (because we're not taking the *continuum* limit here).[25] For $D \geq 2$, this integral is well-defined.[26] Even though the denominator $\omega(\mathbf{p}) = |\mathbf{p}|$ goes to zero as $\mathbf{p} \to 0$, the measure of integration can be written as $|\mathbf{p}|^{D-1}d|\mathbf{p}|$ times the angular part, so the ratio remains finite as $\mathbf{p} \to 0$ if $D \geq 2$. But if $D = 1$, then the integral is undefined whenever the smearing functions both have nonzero limits as $\mathbf{p} \to \mathbf{0}$, because nothing in the numerator compensates for the rate at which the denominator goes to zero as $\mathbf{p} \to \mathbf{0}$. As a result, this particular way of defining an infinite-volume limit – the *frozen* variant of the model – does not work when $D = 1$.

[24]In this limit, the fact that the $\mathbf{p} = \mathbf{0}$ term is absent in (36) doesn't matter. The problem comes from the terms with $\mathbf{p} \to \mathbf{0}$, not from the terms with $\mathbf{p} = \mathbf{0}$.

[25]Article 71852

[26]For the $m > 0$ model, it's well-defined for all D, because $\omega(\mathbf{p}) \to m$ as $\mathbf{p} \to \mathbf{0}$.

The **Mermin-Wagner theorem** says that, under some relatively general conditions, a compact continuous symmetry group cannot be spontaneously broken if $D \leq 1$.[27,28] The theorem is named after the authors of Mermin and Wagner (1966). Dobrushin and Shlosman (1975) derived a more general result that applies to any compact connected Lie group.[29] The shift symmetry group is noncompact, so the usual Mermin-Wagner theorem doesn't apply, but the outcome is the same: the shift symmetry isn't spontaneously broken when $D = 1$, because the model doesn't even have an infinite-volume limit when when $D = 1$.[30]

Section 11 describes a way to trim the set of observables so that an infinite-volume limit can be defined even when $D = 1$, but then the shift symmetry is no longer spontaneously broken for any D.

[27] Here, D is the number of dimensions of space, so spacetime has $D+1$ dimensions. The Mermin-Wagner theorem is usually stated in terms of the number of spacetime dimensions – which is equal to the number of *space* dimensions in the corresponding classical statistical mechanics setting, through the correspondence defined by the euclidean-action formulation of quantum field theory.

[28] A discrete symmetry group can be spontaneously broken when $D = 1$. Article 81040 analyzes one example in detail. Regarding the distinction between *continuous* and *discrete*, see the text near the end of section 1 in Harlow and Ooguri (2021).

[29] Sometimes physicists forget to include the adjective *compact* when citing these papers.

[30] This is emphasized in Coleman (1973), below equation (5).

11 The trimmed variant

In quantum field theory, observables are usually expressed in terms of field operators. The preceding sections treated the field operators $\phi(\mathbf{x},t)$ as observables, but we have another option: we can trim the set of observables down to include only those that are invariant under the shift symmetry (5).[31] Then the operators

$$\dot{\phi}(\mathbf{x},t) \qquad \nabla\phi(\mathbf{x},t) \qquad \phi(\mathbf{x},t)-\phi(\mathbf{y},t) \qquad e^{z\phi(\mathbf{x},t)}e^{-z\phi(\mathbf{y},t)} \tag{38}$$

all qualify as observables, but $\phi(\mathbf{x},t)$ itself does not. To describe the set of observables more systematically, we can use the Fourier-transformed field operators (17). The only operators that qualify as observables are those that can be expressed in terms of $\phi_{\text{FT}}(\mathbf{p},t)$ with $\mathbf{p}\neq\mathbf{0}$ and $\dot{\phi}_{\text{FT}}(\mathbf{0},t)$. The operator $\phi_{\text{FT}}(\mathbf{0},t)$ is excluded because it's not invariant under the shift symmetry (5).

This version of the model has a vacuum state even when the lattice is finite. To see why, remember how quantum theory works: predictions involve observables and states, and the states only need to be defined for operators in the algebra generated by observables. The Hilbert space is really just a tool for constructing states,[32] so only *observables* need to be represented as operators on the Hilbert space. Including other operators can be very convenient, but it's not strictly necessary. When the operator $\phi_{\text{FT}}(\mathbf{0},0)$ is excluded from the set of observables, the observable $\dot{\phi}_{\text{FT}}(\mathbf{0},t)=\dot{\phi}_{\text{FT}}(\mathbf{0},0)$ commutes with all other observables, so we can use a representation in which it is proportional to the identity operator, with an arbitrary proportionality factor,[33] which we might as well take to be zero. Then the hamiltonian (25) reduces to

$$H=\frac{1}{L^D}\sum_{\mathbf{p}\neq\mathbf{0}}\omega(\mathbf{p})a^\dagger(\mathbf{p})a(\mathbf{p})+\text{constant}. \tag{39}$$

[31] Unobservable symmetries are often associated with gauge fields. We could introduce a gauge field associated with the unobservable shift symmetry, but that wouldn't really change anything (Tong (2009), section 8.3.1).

[32] Article 03431

[33] Di Francesco *et al* (1997), section 6.3.3

We can use a Hilbert-space representation like the one in section 7, except that now a state-vector is represented by a function $f[s]$ that depends only on the variables $s(\mathbf{p})$ with $\mathbf{p} \neq \mathbf{0}$. This works because $\dot{\phi}_{\text{FT}}(\mathbf{0}, 0)$ is proportional to the identity operator and because $\phi_{\text{FT}}(\mathbf{0}, 0)$ doesn't need to be represented at all. With these changes, a nonzero state-vector $|0\rangle$ that satisfies $a(\mathbf{p})|0\rangle = 0$ for all $\mathbf{p} \neq \mathbf{0}$ has the lowest possible energy. This shows that the trimmed model has a vacuum state.

The infinite-volume limit can be defined like it was for $m > 0$ in section 8, but using smeared versions of $\dot{\phi}(\mathbf{x}, t)$ and $\nabla\phi(\mathbf{x}, t)$ instead of $\phi(\mathbf{x}, t)$. Each derivative inserts a factor of $\omega(\mathbf{p})$ or $\mathbf{p}$ into the numerator of (37), and then the integrand remains finite as $\mathbf{p} \to \mathbf{0}$ even if $D = 1$. As a result, the model has a good infinite-volume limit for any $D \geq 1$. The shift symmetry is no longer described as being spontaneously broken, because it doesn't affect observables at all.

12 Free fields with nonzero spin

The particles associated with the free scalar field have zero spin (no intrinsic angular momentum). Free fields whose associated particles have nonzero spin can also be defined, and for them, the difference between the massive and massless cases can be even more significant. Zakharov (1970) and van Dam and Veltman (1970) describe a discontinuity that occurs in the $m \to 0$ limit of free fields with spins 1 and 2.[34] The discontinuity occurs because particles in the $m \neq 0$ model have extra spin-states compared to those in the $m = 0$ model, and they don't decouple as $m \to 0$. In contrast, a scalar particle has the same number of spin-states (namely one) for both $m \neq 0$ and $m = 0$.

[34] The conclusion is summarized in Lüben *et al* (2019) and qualified in Babichev and Deffayet (2012).

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