Universal Principal Bundles

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Abstract The concept of a principal G-bundle over a base space M is the mathematical foundation for the concept of a gauge field, where G is the gauged group and M is space or spacetime. This article reviews two ways of constructing principal G-bundles: one using quotients of groups, and one that starts with one principal bundle and constructs others over different base spaces using pullbacks. This is used to introduce the concept of a **universal G-bundle**, a principal G-bundle from which all other principal G-bundles may be constructed using pullbacks, at least in principle.

Contents

1 Introduction

Article 70621 introduced the concept of a principal G-bundle for the case where the total space and base space are finite-dimensional smooth manifolds and the group G is a Lie group. Some applications, including the concepts of *universal bundle* and *classifying space*, require a generalization in which the spaces are not necessarily finite-dimensional manifolds, smooth or otherwise. Section 3 will give a definition that allows the spaces to be arbitrary CW complexes.[1,2] The definition has the same structure as the one in article 70621, but with smooth maps and smooth manifolds generalized to continuous maps and topological spaces.

This article is mostly concerned with principal G-bundles for which the base space M is a CW complex.[3] That's sufficient for typical applications in physics, thanks to these relationships:[4]

- Every n-dimensional topological manifold with $n \neq 4$ is homeomorphic to a CW complex,[5] and every compact topological manifold (including $n = 4$) is homotopy equivalent to a CW complex.[6]
- Every smooth manifold is homeomorphic to a CW complex.[7]
- For a given G, we can always choose a classifying space BG to be a CW complex.[8] Section 17 will introduce the concept of a classifying space.

[1] Article 93875 defines **CW complex**.

[2] Section 12 in Mitchell (2011) mentions an alternative definition that works for a more general class of spaces. Remark 10.4 in Kolář *et al* (1993) mentions a definition that looks more efficient because it doesn't refer explicitly to local trivializations, but then it relies on having an implicit function theorem in order to construct local trivializations, so it might not be as general as the definition used here.

[3] One of the sources listed in footnote 14, namely McLean (2016), requires the spaces to be CW complexes. Another one, namely Cohen (2023), requires them to be Hausdorff and paracompact (introduction to chapter 2). Every CW complex satisfies those conditions (`https://ncatlab.org/nlab/show/CW-complexes+are+paracompact+Hausdorff+spaces`).

[4] Article 93875 lists more relationships between CW complexes and other special kinds of topological spaces.

[5] Manolescu (2016), section 2.2, page 4; Hatcher (2001), text below corollary A.12

[6] Hatcher (2001), corollary A.12

[7] Manolescu (2016), section 2.2, page 3; Mitchell (1997), example 8; McLean (2016), page 1

[8] Section 19

2 Conventions and notation

- *Map* means continuous map, and *function* means continuous function.
- Each topological space is assumed to be homeomorphic to a CW complex.[9,10]
- A fiber bundle with fiber F, total space E, base space M, and bundle projection p may be denoted as $p : E \to M$, or $F \to E \to M$, or just $E \to M$.
- S^n is an n-dimensional sphere, also called an ***n*-sphere**.
- T^n is an ***n*-torus**, the cartesian product of n circles.
- $\mathbb{Z}$ is the integers, $\mathbb{Z}_k$ is the integers modulo k, and $\mathbb{R}$ is the real numbers.
- If X and Y are topological spaces, then $[X, Y]$ is the set of homotopy classes[11] of maps from X to Y. The set of basepoint-preserving homotopy classes of maps is denoted $[X, Y]_0$. Article 69958 reviews the definitions.
- A group or homotopy set is called **trivial** if it has only one element.
- $\pi_j(X)$ is the jth homotopy group of a topological space X.[12]
- $H_j(X; \mathbb{Z})$ and $H^j(X; \mathbb{Z})$ are homology and cohomology groups, respectively.[13]
- A topological space X is called ***n*-connected** if $\pi_j(X)$ is trivial for all $j \leq n$. In particular, **1-connected** means $\pi_0(X)$ and $\pi_1(X)$ are both trivial. The word **connected** by itself is an abbreviation for 0-connected.
- If G and H are groups, then $G \simeq H$ means G and H are isomorphic to each other. If X and Y are topological spaces, then $X \simeq Y$ means X and Y are homeomorphic to each other.

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[9] Article 93875

[10] Every smooth manifold is homeomorphic to a CW complex (article 93875).

[11] A *homotopy class* is an equivalence class of maps, and a *homotopy type* is an equivalence class of manifolds (article 61813).

[12] Article 61813

[13] Article 28539

3 The concept of a principal bundle

A **principal G-bundle** consists of these ingredients:

- a topological space E called the **total space**,
- a connected topological space M called the **base space**,
- a continuous map $p: E \to M$ called the **(bundle) projection**,
- a topological group G,
- a continuous map $E \times G \to E$ called the **(right) action** of G. The image of (x, g) under this map will be denoted xg.

Those ingredients must satisfy these conditions:[14]

- For each $b \in M$, the **fiber** $p^{-1}(b) \subset E$ is homeomorphic to G.
- The right action of G on each fiber is free and transitive.[15]
- $p(xg) = p(x)$ for all $g \in G$ and all $x \in E$.
- Each point $b \in M$ has a neighborhood U for which a homeomorphism τ from $U \times G$ to $p^{-1}(U)$ exists with $p(\tau(u, f)) = u$ for all $(u, f) \in U \times G$. The inverse τ^{-1} is called a **local trivialization**.
- Define b, u, f as before. For each $g \in G$, the right action $p^{-1}(b) \to p^{-1}(b)g$ is a homeomorphism of the fiber to itself, and $\tau(u, f)g = \tau(u, fg)$.

If we only require G to be a topological space (instead of a group) and omit the items involving the action of G on E, then this becomes the more general concept of a **fiber bundle**.

[14] Cohen (2023), definitions 2.1 and 2.5; McLean (2016), definitions 1.1-1.4; Mitchell (2011), section 2

[15] Article 70621 explains what this means. Roughly, it means that the effect of each $g \in G$ on the fiber is just like its effect on G when each element of G is replaced by itself times g.

4 Isomorphism of fiber bundles

Two fiber bundles may be the same when regarded as abstract fiber bundles, even if they are implemented differently. This section makes that idea precise.

Let $p : E \to M$ and $\hat{p} : \hat{E} \to M$ be two fiber bundles over the same base space M, where $E, \hat{E}$ are their total spaces and $p, \hat{p}$ are their bundle projections. A **bundle morphism** from one to the other is a map $f : E \to \hat{E}$ with this property:[16]

$$p(u) = \hat{p}(f(u)) \qquad \text{for all } u \in E. \tag{1}$$

A bundle morphism is called an **isomorphism** if it has an inverse – a bundle morphism in the opposite direction, $\hat{f} : \hat{E} \to E$, for which[17]

$$f(\hat{f}(\hat{u})) = \hat{u} \qquad \hat{f}(f(u)) = u \tag{2}$$

for all $u \in E$ and $\hat{u} \in \hat{E}$. Intuitively, two bundles are isomorphic if they are the same when regarded as abstract fiber bundles, even if they are implemented differently.

For principal G-bundles, a morphism f is also required to be G-equivariant with respect to the right action of G on the total space:[18] $f(u)g = f(ug)$. A morphism of principal bundles is automatically an isomorphism.[19]

[16] Husemoller (1966), chapter 2, definition 3.2

[17] Husemoller (1966), chapter 2, definition 3.4; Davis and Kirk (2001), definition 4.11 and the text below it

[18] Article 70621

[19] Mitchell (2011), proposition 2.1

5 Isomorphism and subsets of the base space

Let M' be a subset of M. If two fiber bundles $p : E \to M$ and $\hat{p} : \hat{E} \to M$ with the same base space M are isomorphic to each other, then their restrictions to the subset M' are still isomorphic to each other. The proof is easy:

- The total space of the restriction of $p : E \to M$ to $M' \subset M$ is the largest subset $E' \subset E$ for which $p(E') = M'$.

- The total space of the restriction of $\hat{p} : \hat{E} \to M$ to $M' \subset M$ is the largest subset $\hat{E}' \subset \hat{E}$ for which $\hat{p}(\hat{E}') = M'$.

- If $f : E \to \hat{E}$ is a bundle morphism, then its restriction to $E' \subset E$ is a bundle morphism from E' to $\hat{E}'$, because equation (1) implies that u is in E' if and only if $\hat{p}(f(u))$ is in M', which implies that $f(u)$ is in $\hat{E}'$.

- If $\hat{f}$ satisfies (2) for all $u, \hat{u} \in E, \hat{E}$, then it satisfies (2) for all $u, \hat{u} \in E', \hat{E}'$.

6 Principal bundles from quotients, part 1

Let X be a topological space, let Y be any set, and let $f : X \to Y$ be a surjective map.[20] We can promote the set Y to a topological space by declaring that a subset $U \subset Y$ is open if and only if its preimage $f^{-1}(U) \subset X$ is open in X's given topology. This is called the **quotient topology** for Y. When endowed with the quotient topology, Y is called a **quotient space**, and f is the **quotient map**.[21] The definition of the quotient topology ensures that f is continuous.

Here's an example. Let Γ be a topological group, and let G be a subgroup of Γ. Each element $\gamma \in \Gamma$ defines a **coset** $\gamma G \subset \Gamma$. Let Γ/G denote the set of all cosets. If Γ is a topological group, then we can promote Γ/G to a topological space by giving it the quotient topology, using the quotient map $\Gamma \to \Gamma/G$ defined by $\gamma \mapsto \gamma G$.

The quotient space Γ/G admits a natural right-action $(\Gamma/G) \times G \to \Gamma/G$ that leaves every point of Γ/G invariant, namely

$$(\gamma G, g) \mapsto \gamma G g = \gamma G.$$

This doesn't always make $\Gamma \to \Gamma/G$ a principal G-bundle,[22] but sometimes it does. If it does, then the subgroup $G \subset \Gamma$ is called **admissible**.[23] If Γ is a Lie group and G is any closed subgroup of Γ, then G is always admissible.[23] In that case, $\Gamma \to \Gamma/G$ is a principal G-bundle,[24] and the quotient space Γ/G admits a unique smooth structure[25] for which the bundle projection $\Gamma \to \Gamma/G$ is smooth,[26] making it a *smooth* principal G-bundle.

[20]Surjective means $f(X) = Y$.

[21]Lee (2011), chapter 3, pages 65-66

[22]Pages 20-21 in Cohen (2023) describe an example of what can go wrong.

[23]Cohen (2023), text above proposition 2.1; Mitchell (2011), text above proposition 3.5

[24]Michor (2008), section 18.5

[25]The smooth manifold Γ/G is called a **homogeneous space** (Lee (2013), chapter 21, pages 550 and 553).

[26]Michor (2008), section 5.11; Lee (2013), theorem 21.17

7 Stiefel manifolds

Article 92035 defines the orthogonal and special orthogonal groups $O(k)$ and $SO(k)$, the unitary and special unitary groups $U(k)$ and $SU(k)$, and the symplectic groups $\text{Sp}(n)$.[27] The name **Stiefel manifold** refers to any of these quotient spaces:[28,29]

$$\frac{O(n+N)}{O(n)} \qquad \frac{U(n+N)}{U(n)} \qquad \frac{\text{Sp}(n+N)}{\text{Sp}(n)}.$$

This article uses only the first two. In those cases, the quotient is defined by thinking of the group in the numerator as a group of linear transformations of an $(n+N)$-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, respectively, and thinking of the group in the denominator as the subgroup that acts as the identity transformation on the last N dimensions of that vector space.[30] Both quotient spaces are unchanged when the groups are replaced by their unit-determinant subgroups:[31]

$$\frac{O(n+N)}{O(n)} = \frac{SO(n+N)}{SO(n)} \qquad \frac{U(n+N)}{U(n)} = \frac{SU(n+N)}{SU(n)}.$$

For each $G \in \{O, SO, U, SU, \text{Sp}\}$, the corresponding Stiefel manifold is the base space of a principal bundle

$$G(n) \to G(n+N) \to \frac{G(n+N)}{G(n)}.$$

This is an example of the setup described in section 6, with $\Gamma = G(n+N)$.

[27] Each element of the group $O(n)$, $U(n)$, or $\text{Sp}(n)$ may be represented as an $n \times n$ matrix over the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or quaternions $\mathbb{H}$, respectively (article 92035).

[28] Steenrod (1951), sections 7.7 (for O) and 25.7 (for U)

[29] n and N are both positive integers (not zero).

[30] Geometrically, $O(n)/O(n-k)$ is the manifold of orthonormal $(k-1)$-frames (sets of $k-1$ mutually orthogonal unit vectors) tangent to the unit sphere S^{n-1} in n-dimensional space (Steenrod (1951), section 7.7).

[31] Steenrod (1951), section 7.8 (for O); Mimura and Toda (1991), chapter 3, §3, pages 119 (for U) and 121 (for O)

8 Homotopy groups of Stiefel manifolds

A key property of Stiefel manifolds is that they are highly connected:[32]

$$\begin{aligned} \pi_k\big(O(n+N)/O(n)\big) &= 0 \qquad \text{if } k < n, \\ \pi_k\big(U(n+N)/U(n)\big) &= 0 \qquad \text{if } k < 2n+1. \end{aligned}$$

The first nonzero homotopy group in each case is[32]

$$\begin{aligned} \pi_n\big(O(n+N)/O(n)\big) &\simeq \begin{cases} \mathbb{Z} & \text{if } n \text{ is even or } N = 1, \\ \mathbb{Z}_2 & \text{if } n \text{ is odd and } N > 1 \end{cases} \\ \pi_{2n+1}\big(U(n+N)/U(n)\big) &\simeq \mathbb{Z}. \end{aligned}$$

These properties will be important in sections 15-16.

[32] Steenrod (1951), sections 25.6 (for O) and 25.7 (for U); Whitehead (1978), page 11 and chapter IV, theorem 10.12 (for O)

9 The concept of a pullback bundle

Given a fiber bundle $p : E \to M$ with fiber F and a map $f : M' \to M$ into M from another space M', the corresponding **pullback bundle** (also called the **induced bundle**) is a fiber bundle $p' : E' \to M'$ defined like this:[33,34]

- Its fiber is still F, but its base space is M' instead of M.
- Its total space E' is a subset of $M' \times E$, namely the subset consisting of all points (b', u) for which $f(b') = p(u)$.
- Its bundle projection is defined by $p'(b', u) = p(u)$.

Intuitively, the pullback bundle replaces the trivial single-fiber bundle over each point $b \in M$ with a trivial bundle over $f^{-1}(b) \in M'$, keeping these trivial bundles stitched together just like the trivial single-fiber bundles over individual points of M were stitched together in the original bundle over M. Section 10 will give a few examples.

If the original bundle is a principal G-bundle, then the pullback bundle is, too.[35]

[33] Davis and Kirk (2001), definition 4.12; Cohen (2023), section 2.2.1; Steenrod (1951), sections 10.2

[34] Beware that Steenrod (1951) uses the symbol B for the total space (which he calls the *bundle space*). Modern literature often uses the symbol B for the base space.

[35] Husemoller (1966), chapter 4, proposition 4.1

10 Examples

Examples of pullback bundles:

- If $M' = M$ and if f is the identity map on M', then the pullback bundle is the same as the original bundle.

- Suppose $M' = M \times \mathbb{R}$, and define $f : M' \to M$ by $f(b, r) = b$ for all $(b, r) \in M \times \mathbb{R}$. Then the pullback bundle defined by f may be described intuitively as a continuum of copies of the original bundle, one for each real number r. (For a more specific example, suppose that M is a circle, so that $M \times \mathbb{R}$ is a cylinder.)

- If f maps all of M' to a single point of M, then the pullback bundle is isomorphic to the trivial bundle with total space $E' = M' \times F$ and projection $p'(b', f) = f$ for all $(b', f) \in M' \times F$.

- Consider a Möbius strip ($\mathbb{R}$ bundle over S^1) pulled back by a 2-to-1 map $S^1 \to S^1$. The result is a trivial bundle. Intuitively, the pullback makes two copies of the original base space S^1, cuts each of them open and glues them together end to end to make a new S^1. The original bundle was nontrivial because it had a single twist, but the new bundle has two twists (one for each copy of the original base space), and these two twists cancel each other.

A pullback bundle cannot be topologically more complicated than the original bundle, but the last two examples show that a pullback bundle can be topologically simpler than the original bundle.

11 Fiber bundles over mutually homotopic bases

If $p : E \to M$ is a fiber bundle, M' is another space, and if two maps f and g from M' to M are homotopic to each other, then the pullback bundles defined by f and g are isomorphic to each other.[36] In other words, given a fiber bundle over M with fiber F, each element of $[M', M]$ gives an isomorphism class of fiber bundles over the new base space M' with the same fiber F as before.

Here's an important special case. Suppose $M' = M$, and let $f : M \to M$ be a map from M into itself that is homotopic to the identity map $\text{id}_M : M \to M$. Then the pullback bundle defined by f is isomorphic to the original bundle. Intuitively, even if the space $f(M) \subset M$ has fewer dimensions than M does, the part of the fiber bundle over $f(M)$ carries all of the essential information about the topology of the full fiber bundle over M.

That special case may also be expressed this way: if $r(M)$ is a deformation retract of M, then every bundle over M is isomorphic to the pullback of a bundle over $r(M)$.[37] Examples:

- Every bundle over a cylinder is isomorphic to a pullback of a bundle over a circle.
- Every bundle over a Möbius strip is isomorphic to a pullback of a bundle over a circle.
- Every bundle over a contractible base space is trivial (isomorphic to the pull-back of a bundle over a point).

[36]Cohen (2023), theorem 4.1; Kottke (2012), proposition 3.13; Nakahara (1990), theorem 9.4

[37]Proof: a deformation retraction $r : M \to r(M) \subset M$ is homotopic to the identity map on M, and every bundle over M is the pullback of a bundle over M by the identity map.

12 n-universal bundles

Consider a principal bundle $G \to E \to M$ whose total space E is $(n-1)$-connected, which means[38] that its homotopy groups $\pi_k(E)$ are zero for all $k < n$. Such a bundle is called an ***n*-universal** principal G-bundle.[39,40]

Any n-universal principal G-bundle has this property:[41,42] for any k-dimensional CW complex X with $k < n$, every principal G-bundle over X is isomorphic to the pullback of the n-universal bundle by a map $X \to M$. This gives a 1-to-1 correspondence between elements of $[X, M]$ and isomorphism classes of principal G-bundles over X.[43] In other words, $[X, M]$ **classifies** principal G-bundles over X. This is why n-universal bundles are important.

Here's a useful relationship between the topology of the base space M of an n-universal principal G-bundle and the topology of the group G:[44]

$$\pi_k(M) \simeq \pi_{k-1}(G) \qquad \text{for all } 1 \leq k \leq n-1. \tag{3}$$

An n-universal principal bundle exists for each compact Lie group G and for each n.[45,46] Sections 15 will construct such a bundle for each G and n, and section 16 will construct another one.

[38] Article 61813

[39] Toda (1986), text above theorem 1.1; Mimura and Toda (1991), chapter 2, lemma 6.5; Steenrod (1951), section 19.4

[40] Section 17 will consider the limit $n \to \infty$, which gives what is simply called a **universal** principal G-bundle.

[41] Steenrod (1951), sections 19.3-19.4; Toda (1986), theorem 1.1

[42] Husemoller (1966) uses a different naming convention, calling a principal G-bundle n-universal if it has this property for all $k \leq n$ instead of $k < n$ (chapter 4, definition 10.7). The $k \leq n$ convention feels more natural to me, but the $k < n$ convention is used in most of the sources I consulted.

[43] Toda (1986), theorem 1.1

[44] Steenrod (1951), section 19.9

[45] Steenrod (1951), theorem 19.6, using the *if and only if* theorem in section 19.4

[46] More generally, every topological group G has an n-universal principal bundle for every n (Milnor (1956), section 1; Mitchell (2011), text below theorem 7.6). Toda (1986) reviews the construction of such a bundle in the text above theorem 1.3.

13 Example

The **Hopf bundle**[47] $S^3 \to S^2$ is a principal $U(1)$-bundle whose total space S^3 is 2-connected, so it's a 3-universal principal $U(1)$-bundle. As a result, principal $U(1)$-bundles over a 1- or 2-dimensional CW complex X are classified by $[X, S^2]$. In particular:

- $[S^1, S^2] \simeq \pi_1(S^2)$ has only one element,[48] so all principal $U(1)$-bundles over S^1 must be trivial.
- $[S^1 \times S^1, S^2] \simeq \mathbb{Z}$ has more than one element,[49] so nontrivial principal $U(1)$-bundles over $S^1 \times S^1$ must exist.

[47] Article 03838
[48] Article 61813
[49] Article 69958

14 Principal bundles from quotients, part 2

Section 7 described principal bundles whose base space is a Stiefel manifold. Sections 15-16 will describe examples whose total space is a Stiefel manifold, using a generalization of the quotient construction that was introduced in section 6. This section introduces the generalization.

Let G and H be compact subgroups of a compact Lie group Γ whose intersection $G \cap H$ contains only the identity element,[50] and suppose that every element of G commutes with every element of H, so $GH = HG$. Then the set GH of their products is also a subgroup of Γ, so the quotients Γ/H and $\Gamma/(GH)$ are both smooth manifolds.[51] The goal is to construct a principal G-bundle with total space Γ/H and base space $\Gamma/(GH)$:

$$G \to \frac{\Gamma}{H} \to \frac{\Gamma}{GH}. \tag{4}$$

For that, we need two more ingredients: a right-action of G on the total space, and a projection from the total space to the base space. Here are those ingredients, using γ to denote an element of Γ:

- The right-action of $g \in G$ on Γ/H is defined by $(\gamma H) \cdot g = \gamma H g = \gamma g H$.
- The projection $\Gamma/H \to \Gamma/(GH)$ is defined by $\gamma H \to \gamma HG$.

These ingredients satisfy the conditions required for a principal G-bundle.[52,53,54]

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[50] A compact subgroup of a Lie group is automatically also a Lie group (article 92035), so G and H are Lie groups.

[51] Section 6

[52] Theorem 6.46 in Davis and Kirk (2001) says that if a compact Lie group G acts freely on a compact manifold Γ, then $\Gamma \to \Gamma/G$ is a principal G-bundle. **Acts freely** means that every non-identity element of G moves every point of Γ. That condition is satisfied here, because if $\gamma g H = \gamma H$, then $\gamma g h = \gamma h'$ for some $h, h' \in H$. Multiplying by γ^{-1} on the left and by h^{-1} on the right gives $g \in H$, which implies $g = 1$ because we assumed $G \cap H = \{1\}$.

[53] This is a special case of a theorem in Steenrod (1951), section 7.4, page 30, where GH is generalized to any subgroup of Γ that contains H as a normal subgroup.

[54] Another approach is to start with the principal bundle $GH \to \Gamma \to \Gamma/(GH)$ whose existence and smoothness was established in section 6. If we define equivalence relation on $\Gamma \times G$ by $(\gamma, gH) \sim (\gamma a, a^{-1} gH)$ for all $(\gamma, gH) \in$

15 n-universal bundles using orthogonal groups

Here is an important example of the construction that was described in section 14. Take the groups Γ, G, and H to be these:

- Take Γ to be the group $O(n+N)$ of rotations and reflections about the origin in $(n+N)$-dimensional euclidean space $\mathbb{R}^{n+N}$.
- Take H to be the group $O(n)$ of rotations and reflections about the origin in an n-dimensional subspace of $\mathbb{R}^{n+N}$.
- Take G to be any closed subgroup $O(N)$, the group of rotations and reflections about the origin in the orthogonal N-dimensional subspace of $\mathbb{R}^{n+N}$.

The factors G and H commute with each other, and their only shared element is the identity element, so the result reviewed in section 14 gives a principal G-bundle

$$G \to \frac{O(n+N)}{O(n)} \to \frac{O(n+N)}{O(n)\times G}. \tag{5}$$

This is an n-universal G-bundle.[55,56] This construction works for any compact Lie group G, because if G is any given compact Lie group, then we can choose N large enough so that G is isomorphic to a closed subgroup of $O(N)$.[57]

$\Gamma \times G$ and $a \in GH$, then we can recover $G \to \Gamma/H \to \Gamma/(GH)$ as an **associated** bundle (Michor (2008), section 18.7; Figueroa-O'Farrill (2006), box in section 1.4). This is a special case of an approach described in an answer to `https://mathoverflow.net/questions/404478/`, where GH is generalized to any subgroup of Γ that contains H as a normal subgroup.

[55] Toda (1986), theorem 1.2; Steenrod (1951), section 19.7

[56] This follows from the definition in section 12 and properties of the Stiefel manifold $O(n+N)/O(n)$ that were reviewed in section 8.

[57] Mimura and Toda (1991), chapter 5, theorem 2.14; Steenrod (1951), text below theorem 19.6

16 n-universal bundles using unitary groups

Here's a variation of the construction that was described in section 15. Take the groups Γ, G, and H to be these:

- Take Γ to be the unitary group $U(n+N)$.
- Choose a subgroup $U(n)\times U(N)\subset U(n+N)$. Take H to be the $U(n)$ factor, and take G to be any closed subgroup of the $U(N)$ factor.

The factors G and H commute with each other, and their only shared element is the identity element, so the result reviewed in section 14 gives a principal G-bundle

$$G\to\frac{U(n+N)}{U(n)}\to\frac{U(n+N)}{U(n)\times G}. \tag{6}$$

This is an n-universal G-bundle.[58,59] This construction works for any compact Lie group G, because if G is any given compact Lie group, then we can choose N large enough so that G is isomorphic to a closed subgroup of $U(N)$.[60]

[58] Mimura and Toda (1991), chapter 2, lemma 6.8

[59] This follows from the definition in section 12 and properties of the Stiefel manifold $U(n+N)/U(n)$ that were reviewed in section 8.

[60] This follows from $O(N)\subset U(N)$ and footnote 57 in section 15.

17 Universal bundles

Sections 12-16 showed that for any finite integer $n > 1$ and compact Lie group sg, every principal G-bundle over a k-dimensional base space M with $k < n$ is a pullback of an n-universal principal G-bundle. This section reviews a version that is not limited to finite n.

A principal G-bundle $EG \to BG$ is called **universal** if it has these properties:[61,62]

- For any CW complex X, each principal G-bundle over X is isomorphic to a pullback bundle induced by a map $f : X \to BG$.
- If $f, g : X \to BG$ are maps whose corresponding pullback bundles are isomorphic to each other, then f and g are homotopic to each other.

Every topological group G has a corresponding universal principal G-bundle.[63,64]

Mutually homotopic maps define isomorphic pullback bundles,[65] so the two properties listed above may be summarized like this: for any CW complex X, the homotopy set $[X, BG]$ **classifies** principal G-bundles over X. The base space BG is called a **classifying space** for G.

For most choices of G and X, we don't have any explicit expression for the homotopy set $[X, BG]$, so we don't have any explicit classification of principal G-bundles over X,[66] but this relationship can be useful.

[61] Cohen (2023), definition 4.1 in section 4.1

[62] The total space and base space of a universal G-bundle are often denoted EG and BG, respectively.

[63] Davis and Kirk (2001), theorem 8.22; Mitchell (2011), theorem 7.6

[64] Theorem 2.5 in Calegari (2019) says that this holds for any topological group G with the homotopy type of a CW complex. Every Lie group G has this property, because every smooth manifold has this property (article 93875).

[65] Section 11

[66] Nash and Sen (1983), section 7.22, near the bottom of page 203

18 Classification using free vs based homotopy sets

Section 17 reviewed the fact that principal G-bundles over X are classified by the free homotopy set $[X, BG]$.[67] Some sources include basepoints in the definition of *principal bundle*, and then principal G-bundles over X are classified by a based homotopy set $[X, BG]_0$.[68]

For maps $f, g : X \to Y$, the condition $[f]_0 = [g]_0$ always implies $[f] = [g]$, but the condition $[f] = [g]$ doesn't always imply $[f]_0 = [g]_0$ (with respect to designated basepoints).[69] The condition $[f] = [g]$ does imply $[f]_0 = [g]_0$ for some spaces Y, so whenever BG is one of those spaces, the two classification statements are interchangeable. Here are two of those cases:

- If G is connected, then BG is 1-connected,[70] so $[f] = [g]$ does imply $[f]_0 = [g]_0$ in that case.[69]

- If G is discrete, then BG is a topological group,[71] so $[f] = [g]$ does imply $[f]_0 = [g]_0$ in this case, too.[69]

[67] Sources that use this convention include Oliveira (2022), Davis and Kirk (2001), and Cohen (2023). Beware that Cohen (2023) uses the notation $[X, Y]$ both for free homotopy sets (section 4) and for based homotopy sets (beginning of section 7.1), depending on the context.

[68] Selick (1997), definitions 9.1.1 and 9.1.2, section 9.2; Mimura and Toda (1991), chapter 2, theorem 6.10 on page 88, using the convention established on page 162; Putman, "Classifying spaces and Brown representability" (https://www3.nd.edu/~andyp/notes/BrownRepresentability.pdf)

[69] Article 69958

[70] Section 19

[71] https://math.stackexchange.com/questions/1729402/

19 Topology of universal bundles

Any principal G-bundle $p : E \to M$ whose total space E is n-connected for all n is automatically a universal principal G-bundle.[72,73] This is the $n \to \infty$ version of the property that section 12 used in the definition of *n-universal bundle.*

For a CW complex, being n-connected for all n is equivalent to being contractible.[74] Every topological group G has a universal principal G-bundle whose total space is contractible.[75]

A classifying space is typically not a finite-dimensional manifold,[76] but every topological group has a classifying space that is a CW complex.[77] All classifying spaces for G are homotopy equivalent to each other.[78]

If BG is a classifying space for a Lie group G, then $\pi_k(BG)$ and $\pi_{k-1}(G)$ are isomorphic to each other for all $k \geq 1$:[79]

$$\pi_k(BG) \simeq \pi_{k-1}(G) \qquad \text{for all } k \geq 1. \tag{7}$$

This is the $n \to \infty$ limit of equation (3).

[72] Cohen (2023), theorem 4.8; Mitchell (2011), theorem 7.4

[73] The cited source calls a space **aspherical** if it is n-connected for all n (Cohen (2023), definition 4.2), but other sources use the word **aspherical** more generally for any path-connected space X whose homotopy groups $\pi_n(X)$ are zero for all $n \geq 2$ but not necessarily for $n = 1$ (Lück (2012), definition 1.1; Whitehead (1978), chapter 5, beginning of section 4).

[74] Cohen (2023), the note below definition 4.2

[75] Davis and Kirk (2001), theorem 8.22

[76] Some examples are described in section 4.2 in Cohen (2023) and example 2.3 in Kuhrman (2019).

[77] Mitchell (2011), proposition 7.5 and theorem 7.6

[78] Cohen (2023), theorem 4.9; May (2007), chapter 23, section 8; Mitchell (2011), proposition 7.5

[79] Cohen (2023), corollary 4.10; May (2007), chapter 23, section 8; Mitchell (2011), corollary 11.2

20 Example

The principal $\mathbb{Z}$-bundle $\mathbb{R} \to S^1$ is a universal bundle for $\mathbb{Z}$.[80] This follows from the fact that the homotopy groups $\pi_k(\mathbb{R})$ are trivial for all k. More generally, if $G = \mathbb{Z} \times \cdots \times \mathbb{Z}$ with n factors, then the n-dimensional torus $T^n \equiv S^1 \times \cdots \times S^1$ is a classifying space for G.[81] The total space of the universal bundle is $\mathbb{R} \times \cdots \times \mathbb{R}$.

In this example, the total space and base space of the universal bundle are finite-dimensional manifolds, which is unusual.

[80]Cohen (2023), section 4.2, page 83

[81]Kuhrman (2019), example 2.2

21 Universal bundles and Eilenberg-MacLane spaces

Choose a positive integer n and a group G. An **Eilenberg-MacLane space** $K(G, n)$ is a topological space X whose homotopy groups $\pi_k(G)$ are all trivial except $\pi_n(X) \simeq G$.[82] Such a space exists for any G if $n = 1$ and for any abelian G if $n \geq 2$.[82]

If G is a discrete group, then $K(G, 1)$ is a classifying space BG for principal G-bundles.[83] The circle S^1 is a $K(\mathbb{Z}, 1)$, so S^1 is a classifying space for principal $\mathbb{Z}$-bundles.[84] Equation (7) implies

$$\pi_k(BU(1)) \simeq \pi_{k-1}(U(1)) \simeq \pi_{k-1}(S^1).$$

Combine this with the fact that S^1 is a $K(\mathbb{Z}, 1)$ to infer that a $K(\mathbb{Z}, 2)$ is a classifying space for the group $U(1)$. More generally, the cartesian product of n copies of $K(\mathbb{Z}, 2)$ is a classifying space for the direct product of n copies of $U(1)$.[85]

[82] Article 69958

[83] Cohen (2023), corollary 4.11; Mitchell (2011), theorem 7.11; Kuhrman (2019), last paragraph in section 2

[84] Section 20

[85] Toda (1986), text above theorem 2.1

22 Principal $U(1)$-bundles and cohomology

The kth cohomology group of M with integer coefficients is denoted $H^k(M;\mathbb{Z})$.[86] Principal $U(1)$-bundles over M are classified by $H^2(M;\mathbb{Z})$. In symbols,

$$[M, BU(1)] \simeq H^2(M;\mathbb{Z}). \tag{8}$$

To derive (8), combine the relationship[87]

$$[M, K(\mathbb{Z}, k)] \simeq H^k(M;\mathbb{Z}) \tag{9}$$

with the fact that $K(\mathbb{Z}, 2)$ is a $BU(1)$.[88]

[86] Article 28539
[87] Article 69958
[88] Section 21

23 More about principal $U(1)$-bundles

The kth homology group of M with integer coefficients is denoted $H_k(M;\mathbb{Z})$.[89] If M is a manifold for which $H_2(M;\mathbb{Z})$ and $H_1(M;\mathbb{Z})$ don't have torsion,[90,91] then

$$[M \times S^1, BU(1)] \simeq H^2(M;\mathbb{Z}) \oplus H^1(M;\mathbb{Z}). \tag{10}$$

To derive (10), first use the Künneth formula to get this relationship among homology groups:[89]

$$\begin{aligned} H_2(M \times S^1;\mathbb{Z}) &= \big(H_2(M;\mathbb{Z}) \otimes H_0(S^1;\mathbb{Z})\big) \oplus \big(H_1(M;\mathbb{Z}) \otimes H_1(S^1;\mathbb{Z})\big) \\ &= \big(H_2(M;\mathbb{Z}) \otimes \mathbb{Z}\big) \oplus \big(H_1(M;\mathbb{Z}) \otimes \mathbb{Z}\big) \\ &= H_2(M;\mathbb{Z}) \oplus H_1(M;\mathbb{Z}). \end{aligned} \tag{11}$$

If torsion is absent, then the homology groups and cohomology groups are isomorphic to each other:[89]

$$H^k(M;\mathbb{Z}) \simeq H_k(M;\mathbb{Z}). \tag{12}$$

Use equations (9)-(12) and the fact that S^1 is a $K(\mathbb{Z},1)$ to get (10).

Taken together, equations (8) and (10) imply

$$[M \times S^1, BU(1)] \simeq [M, BU(1)] \oplus [M, U(1)], \tag{13}$$

which suggests a way of constructing all principal $U(1)$-bundles over $M \times S^1$ from principal $U(1)$-bundles over M and maps $M \to U(1)$. Article 33600 explains how to do this when the bundle over M is trivial.

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89 Article 28539

90 Article 28539 defines **torsion**.

91 Examples of manifolds that satisfy this condition include any n-dimensional torus and any closed, connected, orientable 2-dimensional manifold (article 28539). One example that doesn't satisfy this condition is $\mathbb{R}\mathrm{P}^3$, which is orientable but has 2-torsion (`https://mathoverflow.net/questions/195030/example-of-torsion-in-orientable-manifolds`).

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