Conservation Laws and a Preview of the Action Principle

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Abstract Conservation laws, including those for energy, momentum, and angular momentum, are consequences of the **action principle**. This article introduces the action principle and the resulting conservation laws in the context of newtonian physics, using a model of a system of objects that interact with each other through instantaneous forces. This includes Newton's model of gravity as a special case (article 50710).

Article 46044 introduces to the action principle using the lagrangian formalism, which is more general, and article 12342 uses that formalism to study conservation laws.

Contents

1 Introduction

Energy, momentum, and angular momentum are **conserved quantities**. They are properties of a system that do not change with time, even though the system itself is changing with time. Such **conservation laws** are useful because they tell us something simple about the allowed behaviors of a complicated system. They tell us *something* about the system's allowed behaviors, but they don't tell us everything. For that, we need the full set of **equations of motion**. The equations of motion define exactly which behaviors are allowed. Conservation laws are relatively simple consequences of the equations of motion.

The conserved quantities that we call energy, momentum, and angular momentum are each associated with a special **symmetry** of the equations of motion:

- **Energy** is the conserved quantity associated with symmetry under translations in time.

- **(Linear) momentum** is the conserved quantity associated with symmetry under translations in space.

- **Angular momentum** is the conserved quantity associated with symmetry under rotations.

This article explains the connection between symmetries and conservation laws, using a specific model that has those symmetries: Newton's model of gravity for a system of pointlike masses. For the rest of this article, I'll just call it **Newton's model of gravity**.

This model does have realistic applications, but those applications are not the focus of this article. This article uses the model only to illustrate some general principles.

2 Notation

This article assumes a model of K objects moving in D-dimensional space. The location of the kth object will be denoted $\mathbf{x}_k$. The index k takes values in $\{1, 2, ..., K\}$. The distance between the jth and kth object will be denoted $|\mathbf{x}_j - \mathbf{x}_k|$. A boldface symbol like $\mathbf{x}$ denotes a point in D-dimensional space with the usual Cartesian coordinate system. The real world has $D = 3$, but keeping D general doesn't take any extra work. To represent a point in D-dimensional space, $\mathbf{x}$ should have D components. The object-index will be written as a subscript, and the component-index will be written as a superscript, so the location of the kth object is

$$\mathbf{x}_k = (x_k^1,\, x_k^2,\, ...,\, x_k^D).$$

The distance between the jth and kth objects is given by the familiar Pythagorean formula[1]

$$|\mathbf{x}_j - \mathbf{x}_k| \equiv \sqrt{\sum_{n=1}^{D} (x_j^n - x_k^n)^2},$$

which is what **Cartesian coordinate system** means. In this equation, n is an index,[2] not an exponent.

Let $\mathbf{x}_k(t)$ be the location of the kth object at time t:

$$\mathbf{x}_k(t) = \big(x_k^1(t),\, x_k^2(t),\, ...,\, x_k^D(t)\big).$$

We specify the object's behavior by specifying these D functions of time. The object's velocity, denoted $\dot{\mathbf{x}}_k$, is the time-derivative of its location:

$$\dot{\mathbf{x}}_k = \left(\frac{dx_k^1}{dt},\, \frac{dx_k^1}{dt},\, ...,\, \frac{dx_k^D}{dt}\right).$$

The object's acceleration, denoted $\ddot{\mathbf{x}}_k$, is the time-derivative of its velocity.

[1] The symbol $\equiv$ expresses a *definition* as opposed to an identity.

[2] The plural form of index is indices, pronouned in-dih-sees, but the singular form is still *index*. Indice ("in-dih-see") is **not a word**.

3 The model

In a perfectly realistic model, the set of allowed behaviors would be precisely the behaviors that can occur in the real world. Newton's model of gravity is not perfectly realistic, but it's good enough for many applications. In this model, the world consists of a finite number of objects, each of which has two attributes:

- A mass, which does not change with time,
- A location in space, which may change with time.

The mass of the kth object will be denoted m_k, and its location in space at time t will be denoted $\mathbf{x}_k(t)$.

We can imagine many behaviors that can't actually happen in the real world. Part of defining a model is to specify which behaviors are allowed (according to the model). That's what the **equations of motion** do. In Newton's model of gravity, the equations of motion are[3]

$$\ddot{\mathbf{x}}_k = \sum_{j\neq k} m_j \, \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|^N}. \tag{1}$$

with one equation for each object k. The integer N in the exponent is equal to 3 in the real world, but the extra generality doesn't take any extra work. These equations relate the acceleration of each object to the masses and locations of all of the other objects. The objects' locations depend on time, so equation (1) is really an abbreviation for

$$\ddot{\mathbf{x}}_k(t) = \sum_{j\neq k} m_j \, \frac{\mathbf{x}_j(t) - \mathbf{x}_k(t)}{\big|\mathbf{x}_j(t) - \mathbf{x}_k(t)\big|^N}.$$

Given any behavior of the system of objects (their locations as functions of time), that behavior is allowed if and only if it satisfies all of equations (1).

[3] As in article 50710, I'm using natural units in which Newton's gravitational constant is $G = 1$. Unlike in article 50710, I'm allowing $N \neq D$ here.

4 The action principle

If we multiply both sides of equation (1) by m_k and then call the right-hand side $\mathbf{F}_k$, then the equation looks like this:

$$m_k\ddot{\mathbf{x}}_k(t) = \mathbf{F}_k(t). \tag{2}$$

This illustrates the familiar rule "$F = ma$" (force equals mass × acceleration). We could consider forces $\mathbf{F}_k$ that differ from those in equation (1), but many of the choices we might consider would not satisfy the action principle.

A loose translation of the action principle says that for every action, there is an equal and opposite reaction. But that's too vague to be useful. The real action principle is more specific. For models of the type (2), the action principle may be expressed like this:

> There is a single function V such that the force on each object is equal to the gradient of $-V$ with respect to that object's location.

This ensures that the jth object's contribution to the force on the kth object is related in a specific way to the kth object's contribution to the force on the jth object. The function V is often called the **potential energy**[4] or just the **potential**.

For $N \neq 2$, the model introduced in section 3 satisfies the action principle with

$$V\big(\mathbf{x}_1,\, ...,\, \mathbf{x}_K\big) = \frac{-1}{N-2}\sum_{1\leq j<k\leq K}\frac{m_j m_k}{|\mathbf{x}_j - \mathbf{x}_k|^{N-2}}. \tag{3}$$

With this V, setting

$$\mathbf{F}_k = -\nabla_k V$$

in equation (2) reproduces equation (1). The notation $\nabla_k V$ means the gradient of V with respect to $\mathbf{x}_k$.

[4] Section 12 explains why.

5 Symmetries and conservation laws

The action principle is more fundamental than conservation laws. Each conservation law studied in this article is a consequence of the action principle combined with a particular continuous symmetry:[5]

Action principle + Continuous symmetry = Conservation law.

Even though we call it a "law," a conservation law is not an extra constraint on the object's motion. The equations of motion already tell us which behaviors are allowed. Conservation laws are useful *consequences* of the full equations of motion. Section 15 explains why they're useful.

The next few sections study the conservation laws associated with the symmetries that were listed in section 1. Each one assumes that the equations of motion may be written

$$m_k\ddot{\mathbf{x}}_k = -\nabla_k V. \tag{4}$$

The special case (3) has all of the symmetries that were listed in section 1, but the results are more general: the conservation laws hold for any V that has these symmetries.

The word **symmetry** is overloaded, used with different shades of meaning in different contexts. In this article, it roughly means a continuous group of transformations, each of which replaces each allowed behavior (each solution of the equations of motion) with another allowed behavior. The examples in the following sections illustrate what this means.

[5] Here, "continuous" means that every transformation in the group of symmetry transformations can be continuously deformed to the identity transformation without leaving the group.

6 Conservation of momentum

Consider a model whose equations of motion have the form (4), and suppose that the potential V is invariant under translations in space. This means

$$V(\mathbf{x}_1+\mathbf{c},\mathbf{x}_2+\mathbf{c},...)=V(\mathbf{x}_1,\mathbf{x}_2,...) \tag{5}$$

for any time-independent $\mathbf{c}$. In words, V is unchanged when the locations of all K objects are shifted by the same amount $\mathbf{c}$. In this case, if we take any solution of (4) and shift all of the locations by the same time-independent $\mathbf{c}$, then the result is another solution of (4). In other words, these transformations are **symmetries** in this model. The condition (5) is equivalent to

$$\nabla_{\mathbf{c}}V(\mathbf{x}_1+\mathbf{c},\mathbf{x}_2+\mathbf{c},...)=0, \tag{6}$$

where $\nabla_{\mathbf{c}}$ is the gradient with respect to $\mathbf{c}$. This condition may also be written

$$\sum_k \nabla_k V(\mathbf{x}_1,\,\mathbf{x}_2,\,...)=0. \tag{7}$$

Equations (4) and (7) imply $\sum_k m_k\ddot{\mathbf{x}}_k=0$, which can also be written

$$\boxed{\frac{d}{dt}\sum_k m_k\dot{\mathbf{x}}_k=0.} \tag{8}$$

This is a conservation law: it says that the quantity

$$\sum_k m_k\dot{\mathbf{x}}_k(t) \tag{9}$$

is **conserved**, which means that it is constant in time for all behaviors that satisfy the equations of motion. The quantity (9) is called the system's total **momentum**.

7 Conservation of angular momentum, part 1

Consider a model whose equations of motion have the form (4), and suppose that V is invariant under rotations about the origin. In this case, if we take any solution of (4) and apply an overall rotation about the origin, then the result is another solution of (4), so these transformations are **symmetries** in this model.

In the special case $D = 3$, we are taught to think of a rotation as something defined by an angle and an *axis*, but that only makes sense for $D = 3$. If we replace the concept "rotation about the x^3-axis" with "rotation in the x^1-x^2 plane," then the concept generalizes immediately to arbitrary D: a rotation about the origin in the x^1-x^2 plane is a transformation of the form

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \to \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

with other components unchanged. Rotations in other planes are defined similarly.

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, use the abbreviation $\mathbf{a} \wedge \mathbf{b}$ for the collection of components $a^j b^k - a^k b^j$ for all $j, k \in \{1, 2, ..., D\}$. The quantity $\mathbf{a} \wedge \mathbf{b}$ is called the **wedge product** of the two vectors (article 81674).

The wedge product of two vectors is a natural way to represent an oriented element of area, just like a single vector is a natural way to represent a directed element of length. The orientation is the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, and the magnitude is the area of the parallelogram whose edges are $\mathbf{a}$ and $\mathbf{b}$. The wedge product of two vectors has $D(D-1)/2$ independent components, which is the number of ways of choosing two distinct indices from $\{1, ..., D\}$. The wedge product is zero when $\mathbf{a} \propto \mathbf{b}$, as it should be, because two parallel vectors don't define a plane.

For $D = 3$, the wedge product has the same list of components as the **cross product** $\mathbf{a} \times \mathbf{b}$, but the cross product is treated as a "vector" orthogonal to the plane defined by $\mathbf{a}$ and $\mathbf{b}$, which only makes sense for $D = 3$. For $D = 2$, a direction orthogonal to the plane does not exist, and for $D \geq 4$, the directions orthogonal to a given plane are not unique. The wedge product is more natural, because it is more general.

8 Conservation of angular momentum, part 2

Using the notation defined above, $\mathbf{x}\wedge\nabla$ is the set of differential operators[6]

$$x^j\frac{\partial}{\partial x^k}-x^k\frac{\partial}{\partial x^j}.$$

The assumption that V is invariant under rotations about the origin is expressed by this analogue of equation (7):

$$\sum_k \mathbf{x}_k\wedge\nabla_k V(\mathbf{x}_1,\,\mathbf{x}_2,\,...)=0. \tag{10}$$

Equations (4) and (10) imply

$$\sum_k \mathbf{x}_k\wedge m_k\ddot{\mathbf{x}}_k=0,$$

and the trivial identity $\dot{\mathbf{x}}_k\wedge\dot{\mathbf{x}}_k=0$ implies that this can also be written

$$\boxed{\frac{d}{dt}\sum_k \mathbf{x}_k\wedge m_k\dot{\mathbf{x}}_k=0.} \tag{11}$$

This is another conservation law: it says that the quantity

$$\sum_k \mathbf{x}_k\wedge m_k\dot{\mathbf{x}}_k \tag{12}$$

is conserved. The quantity (12) is called the system's total **angular momentum** about the origin. Just like momentum is actually a collection of D conserved quantities, one for each direction in space, angular momentum is a collection of $D(D-1)/2$ conserved quantities, one for each *pair* of directions in space.

[6] Recall (section 2) that in this context, a superscript is an index, not an exponent.

9 Conservation of angular momentum, part 3

More generally, suppose that V is invariant under rotations about the point $\mathbf{c}$. This is expressed by the condition

$$\sum_k (\mathbf{x}_k - \mathbf{c}) \wedge \nabla_k V(\mathbf{x}_1,\, \mathbf{x}_2,\, ...) = 0. \tag{13}$$

In this case, following the same steps as before leads to the conclusion that the quantity

$$\sum_k (\mathbf{x}_k - \mathbf{c}) \wedge m_k \dot{\mathbf{x}}_k \tag{14}$$

is conserved. This is the system's total angular momentum about the point $\mathbf{c}$.

If V is invariant under translations in space and under rotations about any one point, then it is automatically invariant under rotations about all points. Likewise, if momentum is conserved and if the quantity (14) is conserved for any one $\mathbf{c}$, then the quantity (14) is automatically conserved for all $\mathbf{c}$. This is clear by inspection of equations (9) and (14).

10 Conservation of energy, part 1

Section 6 introduced the conservation law for momentum as a consequence of the action principle combined with invariance under translations in space. This section introduces the conservation law for *energy* as a consequence of the action principle combined with invariance under translations in *time*.

Equation (4) could be generalized by replacing the function $V(\mathbf{x}_1, ..., \mathbf{x}_K)$ with a function that depends on time, like this:

$$V(\mathbf{x}_1,\, ...,\, \mathbf{x}_K,\, t). \tag{15}$$

The derivation of the conservation law for energy starts by assuming that V does *not* depend on time:

$$\frac{\partial}{\partial t} V(\mathbf{x}_1, ..., \mathbf{x}_K, t) = 0. \tag{16}$$

In this case, we can just write $V(\mathbf{x}_1, ..., \mathbf{x}_K)$, as in equation (4). The quantity $V\big(\mathbf{x}_1(t), ..., \mathbf{x}_K(t)\big)$ still depends on time, but only because the locations $\mathbf{x}_k(t)$ of the objects depend on time.

When the condition (16) is satisfied, the set of allowed behaviors is invariant under overall translations in time: if the functions $\mathbf{x}_k(t)$ satisfy the equations of motion, then so do the functions $\mathbf{x}'_k(t) \equiv \mathbf{x}_k(t + c)$ for any constant c.

The following sections study the corresponding conservation law.

11 Conservation of energy, part 2

This section summarizes the conservation law that will be derived in the next section.

Consider a model that satisfies the action principle and that is invariant under translations in time (equation (16)), so that the equations of motion have the form (4). Then the quantity

$$E(t) \equiv \sum_k \frac{m_k}{2}\big|\dot{\mathbf{x}}_k(t)\big|^2 + V\big(\mathbf{x}_1(t),\, ...,\, \mathbf{x}_K(t)\big) \tag{17}$$

is conserved. This means that its time-derivative is zero whenever the functions $\mathbf{x}_k(t)$ all satisfy the equations of motion (4):

$$\dot{E} = 0 \qquad \text{(consequence of (4)).} \tag{18}$$

Equation (17) may be abbreviated

$$E \equiv \sum_k \frac{m_k}{2}\big|\dot{\mathbf{x}}_k\big|^2 + V, \tag{19}$$

with the understanding that V is a function of the dynamic variables $\mathbf{x}_k$, which in turn are functions of t. The quantity E is called the system's (total) **energy**, and equation (18) says that energy is conserved.

12 Conservation of energy, part 3

To derive the result (18), first take the derivative of equation (19) with respect to t to get

$$\dot{E} = \sum_k m_k \dot{\mathbf{x}}_k \cdot \ddot{\mathbf{x}}_k + \dot{V}. \tag{20}$$

Now use the fact that the function

$$V\big(\mathbf{x}_1(t),\, ...,\, \mathbf{x}_K(t)\big)$$

depends on time only via the locations x_k, which implies[7]

$$\frac{d}{dt} V\big(\mathbf{x}_1(t),\, ...,\, \mathbf{x}_K(t)\big) = \sum_k \dot{\mathbf{x}}_k(t) \cdot \nabla_k V\big(\mathbf{x}_1(t),\, ...,\, \mathbf{x}_K(t)\big).$$

More concisely,

$$\dot{V} = \sum_k \dot{\mathbf{x}}_k \cdot \nabla_k V. \tag{21}$$

Use this in equation (20) to get

$$\dot{E} = \sum_k m_k \dot{\mathbf{x}}_k \cdot \ddot{\mathbf{x}}_k + \sum_k \dot{\mathbf{x}}_k \cdot \nabla_k V = \sum_k (m_k \ddot{\mathbf{x}}_k + \nabla_k V) \cdot \dot{\mathbf{x}}_k.$$

Finally, use the equations of motion (4) to see that the quantity in parentheses is zero for each k if the behavior is allowed, which gives the result (18). Altogether, this shows that the energy (19) is conserved (that is, it remains the same for all time) in any model that respects the action principle and that is also invariant under translations in time.

The two terms on the right-hand side of equation (19) each have names of their own. The first term (the one involving the velocities $\dot{x}_k$) is called the **kinetic energy**, and the second term V is called the **potential energy** (or just the **potential**). Having separate names for these two terms is often convenient, even though they are not separately conserved. Only their sum (19) is conserved.

[7] This is an application of the **chain rule** for the derivative of a composite function.

13 Perspective

The quantity $m\dot{\mathbf{x}}$ is often called momentum, even without the context of the conservation law that ultimately motivates it. However, we should not become attached to the idea that $m\dot{\mathbf{x}}$ always represents the momentum of a single object, or that the total momentum is simply the sum of single-object momenta.[8] These things are true in some models (like the ones used in this article), but not in others. The *general* definition of momentum is this: momentum is the quantity which is conserved as a result of combining the action principle with symmetry under translations in space.

Similarly, we should not become attached to the idea that $m\dot{\mathbf{x}}^2/2$ always represents the energy of a single object. It does in some models (like the ones used in this article), but not in others. The *general* definition of energy is this: energy is the quantity which is conserved as a result of combining the action principle with symmetry under translations in time.

"Energy" is actually an overloaded word: it has different meanings that are not equivalent to each other. When energy is defined as the conserved quantity associated with time-translation symmetry, as it is defined in this article, we can add an arbitrary constant term to the system's total energy without changing the fact that it's conserved. In the context of general relativity, a different definition of "energy" is more useful, and then the constant term is *not* arbitrary. The two definitions are interchangeable in some cases, but not in general, and much confusion can be avoided simply by remembering that the same word doesn't always mean the same thing.

[8] The plural of "momentum" is "momenta."

14 Example: Newton's model of gravity

The conservation laws studied above hold in any model whose the equations of motion have the form (4) and which has the appropriate symmetries. In particular, they hold in Newton's model of gravity, where V is given by equation (3):

- That V depends only on the differences $\mathbf{x}_k - \mathbf{x}_j$ (rather than depending on each $\mathbf{x}_k$ independently), so (5) is automatically satisfied.

- That V depends only on the combinations $|\mathbf{x}_k - \mathbf{x}_j|$, so the condition (13) is automatically satisfied for all $\mathbf{c}$.

- That V is independent of t (except through the t-dependence of the objects' locations $\mathbf{x}_k$), so the condition (16) is automatically satisfied.

Beware that symmetry alone is not always enough: the action principle is important, too. To illustrate this, consider a single-object model with equation of motion

$$m\ddot{\mathbf{x}}(t) = \mathbf{k} \tag{22}$$

for some constant $\mathbf{k}$. In this model, the set of allowed behaviors (solutions of (22)) is invariant under translations in space: if $\mathbf{x}(t)$ is one solution and $\mathbf{c}$ is any constant shift, then $\mathbf{x}(t)+\mathbf{c}$ is another solution. However, the momentum $m\dot{\mathbf{x}}$ is clearly not conserved in this model unless $\mathbf{k}=0$. We could write (22) in the form $m\ddot{\mathbf{x}} = -\nabla V$ with $V = \mathbf{k}\cdot\mathbf{x}$, but then V does not have translation symmetry unless $\mathbf{k}=0$, so momentum is not conserved unless $\mathbf{k}=0$.

15 Why conservation laws are useful

Conservation laws are special consequences of the equations of motion, namely consequences that can be expressed in the form

$$\dot{\Omega} = 0 \text{ for all allowed behaviors,}$$

where Ω is a quantity constructed from the dynamic variables. (In the type of model considered in this article, the dynamic variables are the objects' locations $\mathbf{x}_k(t)$.) Conservation laws are useful because they may also be written like this:

$$\Omega(\text{at any time}) = \Omega(\text{at any other time}).$$

This can be used to draw important conclusions about the system's behavior without solving the equations of motion completely, which is usually much harder.

16 Velocity is relative

Consider a model of the form (4) in which V depends only on the distances $|\mathbf{x}_j - \mathbf{x}_k|$ between the objects. If the quantities $\mathbf{x}_k(t)$ satisfy the equations of motion for all k and all t, then so do the quantities

$$\mathbf{x}_k(t) \to \mathbf{x}_k(t) + \mathbf{v}t \tag{23}$$

for any $\mathbf{v}$ which is the same for all k and all t. This is another example of a symmetry, one that shifts the velocities of all of the objects by the same amount $\mathbf{v}$. The corresponding conserved quantity turns out to be (article 12342)

$$\sum_k m_k \mathbf{x}_k - t \sum_k m_k \dot{\mathbf{x}}_k. \tag{24}$$

Using the assumed symmetry of V, you can check directly that the time-derivative of this quantity is zero whenever the $\mathbf{x}_k$s satisfy the equation of motion (4). This is true even though the quantity (24) has an explicit factor of t. This conservation law doesn't have a special name, but it is familiar: $\sum_k m_k \dot{\mathbf{x}}_k$ is the system's momentum (which is conserved by itself), and $\sum_k m_k \mathbf{x}_k$ is the system's **center of mass**, so this conservation law says that the velocity of the center of mass is constant.

In a world with the symmetry (23), changing the velocities of all objects by the same amount does not have any observable effect. When physicists say that **velocity is relative** in Newton's model, this is what they mean. In special relativity, (23) is not a symmetry, but a different kind of transformation called a **Lorentz boost** is a symmetry, and a similar principle still holds: a Lorentz boost changes the velocities of all of the objects, and applying the same symmetry transformation to all objects has no observable effect. In this sense, velocity is relative in special relativity, too. That's where the name *relativity* comes from. When we say that a model is **relativistic**, we mean that it has Lorentz symmetry. A model like (1) is called **nonrelativistic**, even though velocity is relative in this model, too – but in the sense defined by (23) instead of Lorentz symmetry.

17 References in this series

Article **12342** (https://cphysics.org/article/12342):
"Conservation Laws from Noether's Theorem" (version 2022-02-05)

Article **46044** (https://cphysics.org/article/46044):
"The Action Principle in Newtonian Physics" (version 2022-02-05)

Article **50710** (https://cphysics.org/article/50710):
"Newton's Model of Gravity" (version 2022-02-05)

Article **81674** (https://cphysics.org/article/81674):
"Can the Cross Product be Generalized to Higher-Dimensional Space?" (version 2022-02-06)