Free-Fall, Weightlessness, and Geodesics

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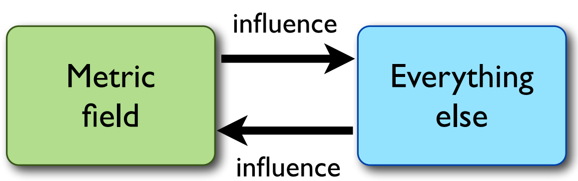
Abstract Article 48968 introduced the geometry of spacetime, which defines proper duration and proper length for timelike and spacelike worldlines, respectively. This article shows how to determine which worldlines are **geodesics**. A timelike geodesic describes the journey of an object in free-fall.

Contents

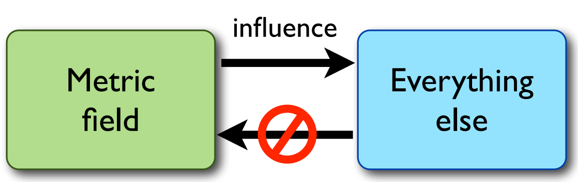
1 Context

In **general relativity**, gravity is mediated by the **metric field**. The **action principle** says that all physical influences must go both ways. In particular, influences between the metric field and everything else go both ways:



The metric field is part of the definition of **spacetime**. It is called the *metric* field (or just the *metric*) because it makes *geometry* possible (article 48968). In general relativity, all physical entities influence (and are influenced by) the metric field,[1] so physical entities modify geometry.

This article treats the metric field as a prescribed background field, exempt from the action principle. In such a model, the influence goes only one way:



Spacetime can still be curved in such a model (I like to call this **generalized special relativity**), but the geometry is prescribed instead of being influenced by other entities. The simplest example of such a model is **special relativity**, in which the spacetime geometry is prescribed to be flat (not curved). The **local flatness theorem** (article 48968) says that this is is a good approximation within any sufficiently small region of spacetime.

[1] Analogy: charges and currents influence (and are influenced by) the electromagnetic field.

2 Free-fall

Article 48968 introduces the concept of a worldline. A **geodesic** is a special type of worldline. Geometrically, geodesics generalize the concept of of "straight" lines to the context of curved space(time). Physically, a timelike or lightlike geodesic describes the motion of a test object in free-fall. A **test object** is an object with negligible size that is influenced by the metric field, but whose influence on the metric field is negligible, and a test object is in **free-fall** if its motion is not being influenced by anything other than the metric field.

Even though I said it's easy, everyday language tends to mix things up pretty badly. Words like "gravity" and "acceleration" are used inconsistently, with meanings that vary from one context to another, and such inconsistencies of language can obfuscate concepts that would otherwise be very simple. To help compensate for that, here are a few examples to illustrate the concept of free-fall:

- If air resistance is negligible, then a rock thrown upward is in free-fall even while it's still moving upward. Free-fall does *not* imply downward motion.

- A rock drifting alone in interstellar space is in free-fall (if influences like dust and light are negligible).

- An object in orbit is in free-fall (if atmospheric drag is negligible).

- A person standing on the surface of the earth is *not* in free-fall, because the ground is pushing upward on the person's feet.

- A person "floating" underwater is *not* in free-fall, because the water pressure below the person's body is greater than the water pressure above the person's body – a net influence that prevents the person from freely falling.

- A skydiver is *not* in free-fall if the skydiver's motion is being significantly influenced by air resistance, such as after reaching terminal velocity. The skydiver's direction of motion relative to the earth is irrelevant to the concept of free-fall. The word *free* is there to remind us of this.

3 Weightlessness

The feeling of being in free-fall is precisely the feeling of **weightlessness**. Don't confuse weightlessness with floating: when you're floating in water, you are *not* weightless, and you are *not* in free-fall. If you've ever been bunjee jumping, then you should appreciate that the feeling of being in free-fall is diffferent from the feeling of floating! You have your usual weight when standing on a scale inside a submarine floating underwater, but that same scale strapped lightly to the soles of your feet registers *zero* weight during the first second or so of a bunjee jump, before other influences like air resistance become significant.

Most importantly, weight has an intrinsic direction: when you're being pushed or pulled by something other than gravity (like the floor of the submarine), you feel the *direction* of that force. You don't weigh yourself by balancing a scale on your head while standing upright, you weigh yourself by standing upright with the scale under your feet. The direction matters. In contrast, free-fall does not have any intrinsic direction. When you're in free-fall, the scale registers zero no matter how it's positioned relative to your body.[2]

Like the previous section mentioned, the word "acceleration" is overloaded. Acceleration in the **relative** (or **extrinsic**) sense refers to the relative motion between two objects. When you drop a rock, the rock accelerates toward the earth in the *relative* sense. Equivalently, the earth accelerates toward the rock in the relative sense. Acceleration in the relative sense depends on what coordinate system we use to describe the motion: an object's *relative* acceleration (relative to the coordinate system) can be zero in one coordinate system and nonzero in another coordinate system. In contrast, acceleration in the **absolute** (or **intrinsic**) sense refers to a deviation from free-fall. An object accelerating in the absolute sense has a weight, and the direction and magnitude of the weight correspond to the direction and magnitude of the deviation from free-fall. An object's *absolute* acceleration does not depend on what coordinate system we use to describe it.

[2]This assumes that the body+scale system is small enough so that gravitational tidal effects are negligible, which is implied by the definition of *test object*.

4 Characterizing geodesics mathematically

Mathematically, a **geodesic** is a worldline whose tangent vector remains tangent to the worldline when parallel-transported along the worldline. Article 03519 explains what this means, but it requires more mathematical background than I want to assume here.[3]

As a substitute, I'll describe an alternative way to characterize geodesics.[4] This alternative characterization is not as satisfying conceptually, but it is more convenient for many purposes. In this alternative approach,

- A timelike geodesic is a worldline that extremizes the proper duration between its endpoints.
- A spacelike geodesic is a worldline that extremizes the proper length between its endpoints.

Recall that the **extrema** of an ordinary function of one variable are the points where its derivative is zero. Similarly, the extrema of the proper duration (or length), which are "functions" of timelike (or spacelike) worldlines with the given endpoints, are worldlines for which the first variational derivatives are all zero. This characterization leads to a system of **Euler-Lagrange equations**[5] (article 46044), whose solutions are precisely the geodesics. Lightlike geodesics are not so directly characterized in terms of an extremization principle,[6] but the general Euler-Lagrange equations that we obtain for timelike and spacelike geodesics also turn out to be valid for lightlike geodesics.[7]

[3]Article 03519 introduces the required background and shows that the definition based on parallel transport is consistent with the alternative approach used here.

[4]This alternative approach is also advocated in Martin (1988).

[5]Don't worry – this is easier than it might sound!

[6]That's is one reason why this alternative approach is not as satisfying conceptually.

[7]This can be proven using the parallel-transport definition, which I've chosen not to review here.

5 The Euler-Lagrange equations for a geodesic

Let A and B be two events connected by a timelike worldline $x^a(\lambda)$ parameterized by λ. We want to determine whether the given worldline is a geodesic. It's a geodesic if and only if it extremizes the (proper) duration.[8] The duration is[9]

$$\text{duration} = \int_{\lambda_A}^{\lambda_B} d\lambda\ L^{1/2} \tag{1}$$

with[10]

$$\boxed{L \equiv g_{ab}(x)\dot{x}^a\dot{x}^b.} \tag{2}$$

The extremization condition means that the variation of (1) is zero to first order in $\delta x^a(\lambda)$, for all variations $\delta x^a(\lambda)$ about the given worldline, keeping the endpoints fixed. To first order, the variation of (1) is

$$\delta(\text{duration}) = \frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda\ L^{-1/2}\,\delta L \tag{3}$$

Now suppose that λ is an **affine parameter**, which (for a timelike worldline) means $\lambda = \alpha\tau + \beta$ where τ is the proper time running along the given worldline and α, β are constants. This implies that $L =$ constant for the given worldline. The variation of L can still be nonzero, because the variation explores worldlines that differ from the given worldline, so equation (3) becomes

$$\delta(\text{duration}) \propto \int_{\lambda_A}^{\lambda_B} d\lambda\ \delta L. \tag{4}$$

[8] Witten (2019), section 2, page 7.

[9] I'm assuming that you're familiar with the material in article 48968.

[10] I'm using the mostly-minus convention for g_{ab}, so $L > 0$ for timelike worldlines, I'm also using the **summation convention** for indices repeated in the same term.

The **lagrangian** L depends only on $x^a(\lambda)$ and its first derivatives $\dot{x}^a$, so we can use the technique explained in article 46044 to deduce that the extremization condition $\delta(\text{duration}) = 0$ is equivalent to the **Euler-Lagrange** equations

$$\boxed{\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{x}^c} = \frac{\delta L}{\delta x^c}} \tag{5}$$

when λ is an affine parameter.

That was for timelike worldlines. For spacelike worldlines, equation (1) is replaced by

$$\text{length} = \int_{\lambda_A}^{\lambda_B} d\lambda\ (-L)^{1/2} \tag{6}$$

with L defined as before.[11] The extremization condition $\delta(\text{length}) = 0$ again leads to equation (5) when λ is an affine parameter, which now means $\lambda = \alpha s + \beta$ where s is the proper length running along the worldline.

As explained in section 4, the extremization conditions $\delta(\text{duration}) = 0$ and $\delta(\text{length}) = 0$ characterize timelike and spacelike geodesics, respectively. For lightlike geodesics, the approach used above doesn't make sense,[12] but the result – equation (5) – still holds even for lightlike geodesics, again when a suitable parameterization is used. Altogether, equation (5) characterizes geodesics of all kinds (timelike, spacelike, lightlike) when L is defined by (2) and an affine parameter is used.

[11] I'm using the mostly-minus convention for g_{ab}, so $L < 0$ for spacelike worldlines.

[12] It doesn't make sense because $L = 0$ for a lightlike geodesic, so variations about such a geodesic can make L have both signs, one of which makes $L^{1/2}$ undefined.

6 Geodesics for a general metric

This section uses the Euler-Lagrange equation (5) to derive a more explicit version of the geodesic equation for a generic metric. Use (2) to get

$$\frac{\delta L}{\delta x^c} = (\partial_c g_{ab})\dot{x}^a\dot{x}^b \qquad\qquad \frac{\delta L}{\delta \dot{x}^c} = g_{cb}\dot{x}^b + g_{ac}\dot{x}^a = 2g_{cb}\dot{x}^b. \tag{7}$$

The last step used the the fact that the metric is symmetric: $g_{ab} = g_{ba}$. The second of equations (7) gives

$$\frac{d}{d\lambda}\frac{\delta L}{\delta \dot{x}^c} = 2g_{cb}\ddot{x}^b + 2(\partial_a g_{cb})\dot{x}^a\dot{x}^b,$$

so the Euler-Lagrange equation (5) becomes

$$2g_{cb}\ddot{x}^b + 2(\partial_a g_{cb})\dot{x}^a\dot{x}^b = (\partial_c g_{ab})\dot{x}^a\dot{x}^b.$$

We can write this in a more convenient form by using the fact that the metric is invertible (article 48968). The components g^{ac} of the inverse metric satisfy

$$g^{ac}g_{cb} = \delta^a_b.$$

Using this, we can re-arrange the preceding equation to get the final result

$$\boxed{\ddot{x}^c + \Gamma^c_{ab}(x)\dot{x}^a\dot{x}^b = 0} \tag{8}$$

with coefficients

$$\boxed{\Gamma^c_{ab}(x) \equiv \frac{1}{2}g^{cd}(x)\big(\partial_a g_{bd}(x) + \partial_b g_{ad}(x) - \partial_d g_{ab}(x)\big).} \tag{9}$$

The coefficients are constructed to be symmetric ($\Gamma^c_{ab} = \Gamma^c_{ba}$), because an antisymmetric part wouldn't contribute to equation (8) anyway.

Altogether, a worldline is a geodesic if and only if it can be parameterized so that it satisfies equation (8). The next section explains how the equation is modified when an arbitrary parameterization is used.

7 Other parameterizations

Equation (8) is the geodesic equation when the geodesic is parameterized in a special way, called an **affine parameterization**. To see what happens when an arbitrary parameterization is used, start with a worldline $x^a(\lambda)$ that satisfies equation (8), and then reparameterize the worldline by expressing λ as a monotonically-increasing function of some other parameter μ:

$$\frac{d\lambda}{d\mu} > 0.$$

The new functions

$$X^a(\mu) \equiv x^a\big(\lambda(\mu)\big)$$

describe the same worldline (the same smooth sequence of events), but now it's parameterized by μ instead of by λ. Use the identity

$$\frac{d}{d\lambda} = \left(\frac{d\lambda}{d\mu}\right)^{-1} \frac{d}{d\mu}$$

in equation (8) to get an equation of the form

$$\ddot{X}^c + \Gamma^c_{ab}(X)\dot{X}^a\dot{X}^b = f(\mu)\dot{X}^c,$$

where now the dots denote derivatives with respect to the new parameter μ. This is the geodesic equation for an arbitrary parameterization. The term $f(\mu)\dot{X}^c$ in this equation comes from the term $\ddot{x}^c$ in equation (8).

Physically-meaningful results don't depend on how a worldline is parameterized, so we might as well assume an affine parameterization so that the simpler equation (8) can be used.[13]

[13] Some authors, like Wald (1984), don't even call it a geodesic unless it's affinely parameterized.

8 Example: flat spacetime

In flat spacetime, we can choose the coordinate system so that the metric for flat spacetime has components

$$g_{ab} = \text{diag}(1, -1, -1, -1). \tag{10}$$

These components are independent of the coordinates,[14] so using (10) in (9) gives $\Gamma^c_{ab} = 0$, and then equation (8) reduces to

$$\ddot{x}^c = 0. \tag{11}$$

The general solution of (11) is

$$x^c(\lambda) = \alpha^c\lambda + \beta^c \tag{12}$$

with constant (λ-independent) coefficients α^c and β^c. This says that in flat spacetime, geodesics are those worldlines that are *straight* in this special coordinate system.

The same worldline can also be parameterized in other ways. Here's one example:[15]

$$x^c(\lambda) = \alpha^c\lambda + \alpha^c\lambda^3 + \beta^c.$$

These are different functions than (12), but they still describe the same worldline – the same smooth sequence of events in spacetime. It is still a *straight* line in this coordinate system. However, when parameterized this way, the worldline no longer satisfies (11). It satisfies an equation of the form

$$\ddot{x}^c = f(\lambda)\dot{x}^c$$

instead. This illustrates the fact that equation (8) describes geodesics only when a special parameterization is used.

[14] The components are different in different coordinate *systems*, but in this particular coordinate system, the components are independent of the coordinates.

[15] The superscript c on α^c and β^c is an index, but the superscript 3 on λ^3 is an exponent.

9 Radial geodesics in Schwarzschild spacetime

For many metrics of interest, starting with the Euler-Lagrange equation (5) is easier than starting with (8). We can specialize the definition of L (equation (2)) to the metric of interest and use this in equation (5), instead of deriving (8) first.

As an example, choose a constant $R > 0$ and consider the line element

$$\dot{\tau}^2 = \dot{w}^2 - \dot{\mathbf{x}}^2 - \frac{R}{r}\left(\dot{w} + \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{r}\right)^2. \tag{13}$$

This is the **Schwarzschild metric** in **Kerr-Schild coordinates** (article 24902). The independent coordinates are w and $\mathbf{x} = (x, y, z)$, with the dependent quantity r defined by $r \equiv \sqrt{x^2 + y^2 + z^2}$. This line element has spherical symmetry: the right-hand side of (13) is invariant under rigid rotations of the x, y, z coordinates. Thanks to spherical symmetry, the geodesic equation (5) has solutions with

$$x(\lambda) > 0 \qquad y(\lambda) = 0 \qquad z(\lambda) = 0. \tag{14}$$

If such a solution is timelike, then it describes an object in free-fall along the x-axis. Let's work this out explicitly. For the metric (13), the quantity L defined in (2) is

$$L = \dot{w}^2 - \dot{\mathbf{x}}^2 - \frac{R}{r}\left(\dot{w} + \frac{\mathbf{x}\cdot\dot{\mathbf{x}}}{r}\right)^2. \tag{15}$$

Calculate the variational derivatives of L and then apply the conditions (14) to get

$$\begin{aligned}
\frac{\delta L}{\delta w} &= 0 & \qquad \frac{\delta L}{\delta \dot{w}} &\propto \dot{w} - \frac{R}{x}(\dot{w} + \dot{x}) \\
\frac{\delta L}{\delta x} &= \frac{R}{x^2}(\dot{w} + \dot{x})^2 & \qquad \frac{\delta L}{\delta \dot{x}} &- 2\left(\dot{x} + \frac{R}{x}(\dot{w} + \dot{x})\right) \\
\frac{\delta L}{\delta y} &= 0 & \qquad \frac{\delta L}{\delta \dot{y}} &= 0 \\
\frac{\delta L}{\delta z} &= 0 & \qquad \frac{\delta L}{\delta \dot{z}} &= 0.
\end{aligned}$$

This gives the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\dot{w}-\frac{R}{x}(\dot{w}+\dot{x})\right)=0 \tag{16}$$

and

$$\frac{d}{d\lambda}\left(\dot{x}+\frac{R}{x}(\dot{w}+\dot{x})\right)=-\frac{R/2}{x^2}(\dot{w}+\dot{x})^2. \tag{17}$$

To reduce this to something more recognizable, add equations (16) and (17) to get

$$\frac{d}{d\lambda}\,(\dot{x}+\dot{w})=-\frac{R/2}{x^2}(\dot{w}+\dot{x})^2. \tag{18}$$

Now define

$$C\equiv\dot{w}-\frac{R}{x}(\dot{w}+\dot{x}),$$

which equation (16) says is constant. Use this together with (14) to reduce (13) to

$$\dot{\tau}^2=(\dot{w}+\dot{x})(C-\dot{x}). \tag{19}$$

For a timelike worldline, we can use its proper time τ as the parameter λ, which is a special case of an affine parameterization. This gives $\dot{\tau}=1$, so we can use equation (19) to rewrite equation (18) like this:

$$\frac{d}{d\lambda}\frac{1}{C-\dot{x}}=-\frac{R/2}{x^2}\frac{1}{(C-\dot{x})^2}.$$

After evaluating the derivative on the left-hand side, the factors of $(C-\dot{x})^2$ cancel, leaving the simple result

$$\boxed{\ddot{x}=-\frac{R/2}{x^2}.} \tag{20}$$

This describes the motion of an object in free-fall along the x-axis, for all $x>0$, parameterized by the object's own proper time.

10 Circular orbits in Schwarzschild spacetime

Rewrite the line element (13) in a new coordinate system (w,ρ,ϕ,z) with ρ,ϕ defined implicitly by

$$x=\rho\cos\phi \qquad\qquad y=\rho\sin\phi.$$

This gives

$$\dot{\tau}^2=L=\dot{w}^2-\dot{\rho}^2-\rho^2\dot{\phi}^2-\dot{z}^2-\frac{R}{r}\left(\dot{w}+\frac{\rho\dot{\rho}+z\dot{z}}{r}\right)^2 \tag{21}$$

with $r\equiv\sqrt{\rho^2+z^2}$. Let's determine if any timelike geodesics exist with

$$\rho(\lambda)=\rho_0 \qquad\qquad z(\lambda)=0, \tag{22}$$

where ρ_0 is a given positive constant. Calculate the variational derivatives of L and then apply the conditions (22) to get

$$\begin{aligned}
&\frac{\delta L}{\delta w}=0 && \frac{\delta L}{\delta\dot{w}}\propto\left(1-\frac{R}{\rho_0}\right)\dot{w}\\
&\frac{\delta L}{\delta r}=-2\rho_0\dot{\phi}^2+\frac{R}{\rho_0^2}\dot{w}^2 && \frac{\delta L}{\delta\dot{\rho}}-2\frac{R}{\rho_0}\dot{w}\\
&\frac{\delta L}{\delta\phi}=0 && \frac{\delta L}{\delta\dot{\phi}}\propto\rho_0^2\dot{\phi}\\
&\frac{\delta L}{\delta z}=0 && \frac{\delta L}{\delta\dot{z}}=0.
\end{aligned}$$

Use these in the Euler-Lagrange equations (5) to get

$$\dot{w}=\text{const} \qquad\qquad \dot{\phi}=\text{const} \qquad\qquad \rho_0^3\dot{\phi}^2=\frac{R}{2}\dot{w}^2. \tag{23}$$

If we parameterize the worldline using its own proper time, $\lambda=\tau$, then equations (21) and (22) give

$$1=\dot{w}^2-\rho_0^2\dot{\phi}^2-\frac{R}{\rho_0}\dot{w}^2.$$

Combine this with the third equation in (23) to deduce

$$1 = \left(\frac{2\rho_0^3}{R} - 3\rho_0^3\right)\dot{\phi}^2. \tag{24}$$

This gives us an explicit expression for the constant $\dot{\phi}$, the angular speed of the object's orbit according to its own proper time. Equation (24) also implies

$$\boxed{\rho_0 > \frac{3R}{2}.}$$

This says that timelike circular orbits are possible only if the given constant ρ_0 satisfies this inequality, even though this is outside the event horizon (article 24902).

If we also consider lightlike circular orbits, then the lightlike condition $\dot{\tau} = 0$ leads to

$$0 = \left(\frac{2\rho_0^3}{R} - 3\rho_0^3\right)\dot{\phi}^2$$

instead of equation (24). This shows that a lightlike circular orbit must have

$$\rho_0 = \frac{3R}{2}.$$

This is called the **innermost circular orbit**.

11 The local flatness theorem

The geodesic equation (8) explains the significance of the local flatness theorem that was introduced in article 48968 and illustrated in article 24902. The local flatness theorem says that for any given event p where the metric is well-defined, a coordinate system always exists in which the Taylor expansion of the metric about the event p doesn't have any terms that are linear in the coordinates. According to equation (9), this means $\Gamma^c_{ab} = 0$ at the event p in that coordinate system,[16] which in turn means that the relative accelerations of objects in free-fall at p are all zero.

For a general metric, even though we can always choose the coordinate system so that $\Gamma^c_{ab} = 0$ at the given event, we cannot always choose the coordinate system so that $\Gamma^c_{ab} = 0$ exactly throughout a finite neighborhood of that event. If we can, then we say that spacetime is (exactly) flat within that neighborhood. Otherwise, the spacetime is curved – even though it is always *locally* flat in the sense explained above.

[16] The quantities Γ^c_{ab} are *not* the components of a tensor (article 09894). They may all be zero at p in one coordinate system even if they're not all zero in another coordinate system.

12 References

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13 References in this series

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