Homology Groups

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Abstract Homology groups are examples of topological invariants: topologically equivalent spaces have the same homology groups. The idea behind homology groups is to consider a special family of topological spaces C for which the concept of a *boundary* makes sense, namely spaces made of simple polyhedra, and to use maps from those spaces into another topological space X as a way of exploring the topology of X. Roughly, the nth **homology group** of X describes continuous maps into X from those special n-dimensional spaces C that cannot be extended to a continuous map into X from any of the special $(n+1)$-dimensional spaces whose boundary is C. This article introduces homology groups. A brief overview of related topological invariants called **cohomology groups** and **cohomology rings** is also included.

Contents

1 Notation and conventions

In this article, the unqualified word *map* always means *continuous map*, and the unqualified word *manifold* means a finite-dimensional topological manifold with boundary.[1] The boundary may be empty, in which case it's a manifold without boundary. Some notation:

- $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ are the real numbers, rational numbers, and integers.
- $\mathbb{Z}_n$ is the integers modulo n. (Another common way to write $\mathbb{Z}_n$ is $\mathbb{Z}/n\mathbb{Z}$.)
- $\mathbb{R}$ is the field of real numbers.
- $\mathbb{R}^n$ is n-dimensional euclidean space.
- S^n is the n-dimensional sphere, the boundary of an $(n+1)$-dimensional ball.
- $\mathbb{R}\mathrm{P}^n$ is n-dimensional real projective space.
- If G and H are algebraic structures (like groups), then the notation $G \simeq H$ means that G and H are isomorphic to each other.[2,3]
- If X and Y are topological spaces, then $X \times Y$ is their cartesian product with the product topology.
- Sections 2-3 will define $\times$, $\oplus$, and $\otimes$ for abelian groups.
- $T(G)$ is the torsion part of an abelian group G (section 19).
- $\pi_k(X)$ is the kth **homotopy group**[4] of a topological space X.

Some references to Lee (2011) are paired with references to the earlier edition Lee (2000), because the earlier edition is freely accessible online.

[1] Many math texts – including some of the sources cited in this article – use a different convention in which the word *manifold* by itself implies *without boundary*.

[2] Article 29682 defines **isomorphism** of groups.

[3] Sometimes, distinguishing between isomorphism (equality as abstract groups) and other forms of equality is important. When this distinction is not important, isomorphism is sometimes written $G = H$.

[4] This is defined in article 61813.

2 The direct product and the direct sum

Let G and H be arbitrary groups. Their **direct product** $G \times H$ is the group consisting of pairs (g, h) with $g \in G$ and $h \in H$ and with the group operation defined by[5]

$$(g, h) \circ (g', h') \equiv (g \circ g', h \circ h').$$

This can be extended to an arbitrary number of factors, $G_1 \times G_2 \times \cdots$, in the obvious way.

The group operation $\circ$ is usually described as multiplication, but it is sometimes described instead as addition when the group is abelian.[6] The additive description is normally used for homology groups and their coefficient groups, which are always abelian. This article uses that convention. The **direct sum** of abelian groups, denoted $G_1 \oplus G_2 \oplus \cdots$, can be defined for any number of factors. When the number of factors is finite, which is the only case that will be needed in this article, the direct sum is the same as the direct product.[5,7] Only the notation is different (additive instead of multiplicative). In symbols:

$$A \oplus B = A \times B.$$

The composition rule for the direct sum $G \oplus H$ (and for the direct product $G \times H$ when additive notation is used) is[8]

$$(g, h) + (g', h') \equiv (g + g', h + h').$$

[5]Lee (2011), appendix C, page 402

[6]A group is called **abelian** if all of its elements commute with each other.

[7]Whitehead (1978), appendix B, theorem 1.4

[8]Sullivan (2020)

3 The tensor product

The **tensor product** $G \otimes H$ is another way of combining two groups to get a new group. When G and H are abelian, their tensor product is the group consisting of pairs (g, h) with $g \in G$ and $h \in H$ and with the group operation defined by[9]

$$(g, h) \circ (g', h) = (g \circ g', h) \qquad\qquad (g, h) \circ (g, h') = (g, h \circ h').$$

If G is any abelian group and $\{0\}$ is the trivial group, then $\{0\} \otimes G = G$ and $\mathbb{Z} \otimes G \simeq G$.[8]

In the context of homology (and cohomology), where the groups are abelian and the group operation is written as addition, the sum of n copies of $g \in G$ may be written ng, and the sum of n copies of the inverse of g may be written $-ng$. This defines a natural action of the ring $\mathbb{Z}$ of integers on the group G. The definition of the tensor product of abelian groups G and H implies that integer factors may be passed back and forth from one side of the tensor product to the other, and the tensor product is sometimes written $G \otimes_{\mathbb{Z}} H$ to indicate this. More generally, the ring $\mathbb{Z}$ of integers may be replaced by another commutative ring R that acts on the groups in a natural way.[10] The notation $G \otimes_R H$ indicates that coefficients in R may be passed back and forth from one side of the tensor product to the other.[11] In this article, $\otimes$ with no subscript means $\otimes_{\mathbb{Z}}$.

Remember[12] that the direct sum $\oplus$ of a finite number of abelian groups is the same as the direct product $\times$ of those groups, but the tensor product $\otimes$ is different. In symbols:

$$A \oplus B = A \times B \neq A \otimes B.$$

Starting in section 22, both $\oplus$ and $\otimes$ will appear together in some equations.

[9]Sullivan (2020) gives the precise definition. Unlike the direct product (or direct sum), the group operation $\circ$ in the tensor product is such that some combinations $(g_1, h_1) \circ (g_2, h_2)$ cannot be reduced to a single term (g, h). This is analogous to the situation called *entanglement* in the context of Hilbert spaces.

[10]Section 25

[11]Hatcher (2001), section 3.1, page 215

[12]Section 2

4 Homology groups: preview

Homotopy groups, which were defined in article 61813, are topological invariants: if two spaces are homeomorphic (topologically equivalent) to each other, then they have the same homotopy groups. Section 12 will introduce another collection of topological invariants called *homology groups*. One homology group $H_n(X;G)$ is defined for each topological space X, each positive integer n, and each abelian group G (called the **group of coefficients**).[13]

The concept of a *boundary* isn't defined for arbitrary topological spaces, but it is defined for manifolds[14] and for polyhedra. The idea behind homology groups is to consider a family of topological spaces for which the concept of a *boundary* makes sense, and to use maps from those spaces into another topological space X as a way of exploring the topology of X. Let M be a space with non-empty boundary ∂M, and let X be a space whose topology we want to explore. A subject called **bordism homology** explores the topology of X by asking questions like this: do any maps $\partial M \to X$ exist that cannot be reproduced by restricting the domain of a map $M \to X$ to the boundary ∂M?[15,16] The rest of this article is about **singular homology**, which uses a variation of that idea to make the math easier.

[13] Hatcher (2001), section 2.2, page 153

[14] Article 44113

[15] Freed (2013), lecture 12, page 103; also mentioned in Hatcher (2001), section 2.1

[16] Recall that in this article, *map* means *continuous map* (section 1).

5 An example for motivation

This section describes a pair of manifolds whose homology groups are different even though their homotopy groups are the same.[17,18,19] The two manifolds in this example are $X = \mathbb{R}\mathrm{P}^3 \times S^2$ and $Y = \mathbb{R}\mathrm{P}^2 \times S^3$. The manifolds X and Y are both five-dimensional, connected, closed, and smooth.

To show that they have different homology groups, start with the fact that X is orientable and Y is not. This follows from the fact that S^n is orientable for all n and the fact that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd.[20] Now invoke this general result about homology groups: if M is a closed connected n-dimensional manifold, then $H_n(M;\mathbb{Z}) \simeq \mathbb{Z}$ if M is orientable, and $H_n(M;\mathbb{Z}) = 0$ otherwise.[21] This shows that

$$H_5(X;\mathbb{Z}) \not\simeq H_5(Y;\mathbb{Z}),$$

so the homology groups of X are not all the same as those of Y.

On the other hand, their homotopy groups are the same. Results reviewed in article 61813 give

$$\begin{gathered}
\pi_k(\mathbb{R}\mathrm{P}^n \times S^m) \simeq \pi_k(\mathbb{R}\mathrm{P}^n) \times \pi_k(S^m) \text{ for all } k \geq 1 \\
\pi_k(\mathbb{R}\mathrm{P}^n) \simeq \pi_k(S^n) \text{ for all } k \geq 2 \\
\pi_1(\mathbb{R}\mathrm{P}^n) \simeq \mathbb{Z}_2 \text{ for all } n \geq 2 \\
\pi_1(S^n) = 0 \text{ for all } n \geq 2.
\end{gathered}$$

Combine these results to deduce that $\pi_k(X) \simeq \pi_k(Y)$ for all $k \geq 1$.

[17] This is example 1.19 in Maxim (2018). It's a special case of Whitehead (1978), section IV.7, example 1.

[18] If a map $X \to Y$ induces isomorphisms between $\pi_n(X)$ and $\pi_n(Y)$ for all n, then it also induces isomorphisms between $H_n(X;G)$ and $H_n(Y;G)$ (Maxim (2018), theorem 10.3), but the existence of isomorphisms between $\pi_n(X)$ and $\pi_n(Y)$ does not imply the existence of such a map.

[19] The opposite situation can also occur: two manifolds that have the same homology groups may have different homotopy groups. One example is the **Poincaré homology sphere**. The homology groups of this 3d manifold are the same as those of S^3 (that's why it's called a *homology sphere*), but its first homotopy group π_1 is different: zero for S^3, nonabelian for the Poincaré homology sphere. Article 61813 says more about this example.

[20] Intuition: $\mathbb{R}\mathrm{P}^n$ is $S^n \subset \mathbb{R}^{n+1}$ modulo $x \mapsto -x$, which preserves the orientation of S^n if and only if the number of reflected coordinates is even. Also see `https://ncatlab.org/nlab/show/real+projective+space`.

[21] Hatcher (2001), text below theorem 3.26, and the top of page 142 in chapter 2

6 Simplexes

An ***n*-simplex**, denoted Δ^n, is an n-dimensional polyhedron in $\mathbb{R}^n$ with $n+1$ vertexes,[22] with the vertexes listed in a particular order. Geometrically, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. This article uses a notation in which each vertex is represented by an integer. With this notation, $[0,1,2,3]$ denotes a 3-simplex, and $[0,1,2,4]$ denotes another 3-simplex that shares three of its vertexes with the first one.

Geometrically, the boundary of an n-simplex is a union of $(n-1)$-simplexes, each of whose vertex-lists is obtained by omitting one vertex from the list that defines the original n-simplex. As an example, consider the 3-simplex whose four vertexes are $[0,1,2,3]$. Its boundary is the union of these four 2-simplexes: $[1,2,3]$, $[0,2,3]$, $[0,1,3]$, and $[0,1,2]$. This is a set of four triangles, the faces of the original tetrahedron.

Many n-dimensional manifolds M may be constructed by gluing n-simplexes together face-to-face. Such manifolds may be used to probe the topology of another space X, as previewed in section 4. Homology does this in a clever way: instead of mapping the fully-assembled manifold M into X, it maps each constituent n-simplex into X, using an algebraic device to keep track of how the simplexes fit together to make M. The next few sections will explain how it works.

[22] This article uses *vertexes* as the plural form of *vertex*, and similarly for other nouns ending in *-ex*. The traditional rule for pluralizing these words is to replace *-ex* with *-ices*, maybe because that makes the plural form easier to pronounce, but that weird tradition has a negative side effect: newcomers who learn the plural form first often assume that the singular form must end in *-ice*. If we fix the language the right way by using *-exes* as the plural of *-ex*, then we can help well-meaning students avoid accidentally fixing it the wrong way.

7 Singular simplexes

Given a topological space X, a map $\sigma : \Delta^n \to X$ is called a **singular *n*-simplex**. The name *singular* refers to the fact that the map only needs to be continuous, so its image might not be a simplex geometrically. It might not even be n-dimensional. Homology uses these maps to explore the topology of X.

A singular n-simplex is a map $\sigma : \Delta^n \to X$, not just the subset $\sigma(\Delta^n) \subset X$, so the definition of *boundary* that was given in article 44113 cannot be applied to a singular n-simplex. We can apply it to the map's domain Δ^n, and we can even apply it to the map's image $\sigma(\Delta^n) \subset X$ if that image happens to be a manifold (which is not required), but we can't apply it to the map σ itself. Section 10 will introduce a concept of *boundary* that applies to such maps – actually to formal linear combinations of such maps – that accounts for the ordering of the vertexes and that has this key property: the boundary of a boundary is zero.

8 Which manifolds can be triangulated?

A **(euclidean) simplicial complex** is a set S of simplexes in $\mathbb{R}^n$, for some n, with these properties:[23]

- If a simplex is in S, then every face of that simplex is in S.
- The intersection of any two simplexes in S is either empty or is a face shared by both of them.
- Every point in S has a neighborhood that intersects at most finitely many simplexes in S.

An **abstract simplicial complex** is defined similarly, using only abstract vertex-sets without reference to any ambient euclidean space. Every finite abstract simplicial complex can be realized as a euclidean simplicial complex.[24]

A **polyhedron** is a topological space X that is homeomorphic to the union of the simplexes in a simplicial complex. Such a homeomorphism is called a **triangulation** of X, and a space that admits a triangulation is called **triangulable**.[25]

A manifold is called **triangulable** if it is homeomorphic to a polyhedron, and then the homeomorphism is called a **triangulation**.[26] For any triangulated manifold, every $(n-1)$-simplex is a face of no more than two n-simplexes.[27] Every 2-dimensional manifold is triangulable,[28] and so is every 3-dimensional manifold,[29] but some 4-dimensional manifolds are not,[30] and in 5 and more dimensions the situation is unknown.[30]

[23] Lee (2011), chapter 5, page 149 (also Lee (2000), chapter 5, page 93)
[24] Lee (2011), proposition 5.41 (also Lee (2000), exercise 5.5)
[25] Lee (2011), chapter 5, page 151 (also Lee (2000), chapter 5, page 100)
[26] Lee (2011), chapter 5, page 151 (also Lee (2000), chapter 5, page 91)
[27] Lee (2000), chapter 5, parenthetical remark on page 107
[28] Lee (2011), theorem 5.36 (also Lee (2000), theorem 5.12)
[29] Lee (2011), theorem 5.37 (also Lee (2000), theorem 5.13)
[30] Lee (2011), text after theorem 5.37 (also Lee (2000), text below theorem 5.13)

9 Singular n-chains

The boundary of an n-simplex Δ^n consists of $(n-1)$-simplexes called the **faces** of Δ^n.[31] Many spaces homeomorphic to n-dimensional topological manifolds may be constructed as a union of n-simplexes that share faces with each other. When two n-simplexes in $\mathbb{R}^n$ share a face, that face is not part of the boundary of their union.

We could build interesting topological spaces from n-simplexes and then use maps from those spaces into X as a way of exploring the topology of X. Homology uses a slightly different idea: instead of assembling the n-simplexes first and then mapping their union into X, we map each individual n-simplex into X in a way that matches some of their faces with each other inside X. Then, instead of considering the boundary of the resulting shape inside X, we use a new concept of *boundary* that applies to the collection of maps instead of only to their images. To make this work, the collection of maps is treated as more than just a collection: it's treated as a formal linear combination called a *singular n-chain*.

Recall that a *singular n-simplex* is a map from Δ^n to another topological space X. A **singular n-chain with coefficients in G** is a formal linear combination of such maps, with coefficients in a given abelian group G. Since G is abelian, we can express the group operation as addition. Then the inverse of g is $-g$, and the identity element is expressed as zero: $g + (-g) = 0$. Given two singular n-chains, we can add them to each other in the obvious way: terms involving the same singular n-simplex may be combined by adding their coefficients, and terms involving different singular n-simplexes (different maps from Δ^n to X) remain as separate terms. Any term whose coefficient is zero may be discarded, and if no terms remain, then the whole thing is zero. In this way, the set of singular n-chains forms an abelian group, denoted $C_n(X; G)$.

Section 10 will define the *boundary* of a singular n-chain, the key idea that makes homology work.

[31] Hatcher (2001), section 2.1, page 103

10 The boundary of a singular n-chain

To define the boundary of a singular n-chain, first consider a singular n-chain c with only one term, so $c = g\sigma$ for some map $\sigma : \Delta^n \to X$ and some $g \in G$. The **boundary** of c, denoted ∂c, is a singular $(n-1)$-chain. Instead of writing out the general definition, here's an example with $n = 3$. If c is the standard 3-simplex $\Delta^3 \equiv [0, 1, 2, 3]$, then the boundary of c is

$$\partial c = \partial(g\sigma) \equiv g\sigma|_{[1,2,3]} - g\sigma|_{[0,2,3]} + g\sigma|_{[0,1,3]} - g\sigma|_{[0,1,2]}, \tag{1}$$

where $\sigma|_s$ denotes the restriction of the map σ to the 2-simplex s. The general definition should be evident from this example. The relative sign of each term is determined by which vertex was omitted from the 3-simplex to get that 2-simplex. The definition of ∂ is extended to arbitrary singular n-chains by requiring ∂ to be a G-linear map from $C_n(X; G)$ to $C_{n-1}(X; G)$.

Calculating $\partial(\partial c)$ leads to a linear combination in which the restriction $\sigma|_{[v,v']}$ to each 1-simplex $[v, v']$ occurs twice, with opposite signs, so everything cancels. More generally if σ_1 and σ_2 are two maps from an n-simplex into X that are equal to each other when restricted to one face of the n-simplex, then that face does not contribute to the boundary of $g\sigma_1 - g\sigma_2$. As a result, the boundary of a boundary of every singular n-chain is zero, as promised in section 7:

$$\partial(\partial c) = 0. \tag{2}$$

To make this clear, consider an example using a 3-simplex $\Delta^3 = [0, 1, 2, 3]$. Let $\sigma_1 : \Delta^3 \to X$ and $\sigma_2 : \Delta^3 \to X$ be two maps for which $\sigma_1|_{[0,1,2]} = \sigma_2|_{[0,1,2]}$. Then equation (1) combined with the linearity of ∂ gives

$$\begin{aligned} \partial(g\sigma_1 - g\sigma_2) = \; & g\sigma_1|_{[1,2,3]} - g\sigma_1|_{[0,2,3]} + g\sigma_1|_{[0,1,3]} - g\sigma_1|_{[0,1,2]} \\ & - \big(g\sigma_2|_{[1,2,3]} - g\sigma_2|_{[0,2,3]} + g\sigma_2|_{[0,1,3]} - g\sigma_2|_{[0,1,2]}\big) . \end{aligned} \tag{3}$$

We have assumed that σ_1 and σ_2 are equal to each other when restricted to the face $[0, 1, 2]$, so the two terms involving that face cancel each other in the linear combination (3).

11 An example with non-spherical topology

An n-simplex is homeomorphic to (topologically equivalent to) an n-dimensional ball. This section describes a collection of 3-simplexes (tetrahedra) whose union is a 3-dimensional manifold homeomorphic to a solid ring. This will be used to construct a singular 3-chain c whose boundary ∂c does not involve any of the shared faces.

To begin, consider this sequence of 3-simplexes:[32]

$$\begin{aligned} \Delta_0 &= [0,1,2,3] \\ \Delta_1 &= [1,2,3,4] \\ \Delta_2 &= [2,3,4,5] \\ \Delta_3 &= [3,4,5,6] \\ &\vdots \end{aligned}$$

The union of these tetrahedra forms a **Boerdijk–Coxeter helix**. Each tetrahedron Δ_k with $k \geq 1$ shares exactly two of its faces with other tetrahedra in the sequence.[33] Now, truncate the sequence so that it has only a finite number N of tetrahedra, so that Δ_{N-1} is the last tetrahedron in the sequence. The faces $[0,1,2]$ and $[N, N+1, N+2]$ are not shared, and if $N \geq 4$, then the tetrahedra that own those faces don't don't share any faces with each other. Now, think of those two faces as opposite faces of a faceted "cylinder," and identify them with each other by identifying the points $N, N+1, N+2$ with the points $0, 1, 2$, in that order.[34] If $N \geq 4$, then the result is topologically equivalent to a solid ring.[35]

To do this in three-dimensional space, we would need to distort at least some of the tetrahedra so that at least some of their faces are no longer flat. That's fine,

[32] This section uses a subscript to distinguish different 3-simplexes and omits the superscript that previous sections used to indicate the number of dimensions.

[33] Example: Δ_1 shares the face $[1,2,3]$ with Δ_0, and it shares the face $[2,3,4]$ with Δ_2. Its other two faces, $[1,3,4]$ and $[1,2,4]$, are not shared.

[34] The order is important, because if we glued the faces together with the wrong orientation, then we would get something called a **solid Klein bottle** instead of a solid ring.

[35] You can check this by drawing a Boerdijk–Coxeter helix with $N = 4$ on paper, with the vertexes labelled.

because in this example, only the topology matters. Just for fun, though, here's an example of such a ring in which every tetrahedron is an undistorted regular tetrahedron, which means that its faces are all undistorted equilateral triangles. Such a ring would be impossible in three-dimensional euclidean space, but it is possible in four-dimensional euclidean space. This example uses eight tetrahedra ($N = 8$), and the eight vertexes have these coordinates:[36]

$$\begin{aligned} [0] &= (1,0,0,0) & [4] &= (-1,0,0,0) \\ [1] &= (0,1,0,0) & [5] &= (0,-1,0,0) \\ [2] &= (0,0,1,0) & [6] &= (0,0,-1,0) \\ [3] &= (0,0,0,1) & [7] &= (0,0,0,-1). \end{aligned}$$

With this sequence of vertexes, we can check by inspection that all eight of the tetrahedra defined above are regular (their faces are equilateral triangles), and we already know from the previous paragraph that their union is topologically equivalent to a solid ring.[37]

Now let's use this to construct a singular 3-chain c in which all of the shared faces cancel each other in the boundary ∂c. Let Δ^3 be an arbitrary 3-simplex, and let σ_k be a map from Δ^3 into $\mathbb{R}^n$ whose image is the kth 3-simplex Δ_k in the preceding solid-ring construction, where the number n of dimensions is large enough to allow identifying the vertexes $N, N+1, N+2$ with $0, 1, 2$. In symbols: $\sigma_k(\Delta^3) = \Delta_k \subset \mathbb{R}^n$. Let G be any abelian group, choose any nonzero element $g \in G$, and consider this singular 3-chain:

$$c = g\sigma_0 - g\sigma_1 + g\sigma_2 - g\sigma_3 + \cdots + g\sigma_{N-2} - g\sigma_{N-1}. \tag{4}$$

The alternating pattern of signs ensures that if N is even, then all of the shared faces cancel in the boundary ∂c,[38] so the union of the images of the surviving 2-simplexes is homeomorphic to the two-dimensional surface of a torus in $\mathbb{R}^n$.

[36] A vertex is a 0-simplex, so the notation $[k]$ for the kth vertex is a special case of the notation that section 6 introduced for any n-simplex.

[37] https://en.wikipedia.org/wiki/Boerdijk%E2%80%93Coxeter_helix gives additional information about this eight-vertex example.

[38] If $G = \mathbb{Z}_2$, the group with only two elements, then this also works when N is odd because $g = -g$.

12 Homology groups: definition

For each n, the boundary operator ∂ defined in section 10 is a homomorphism from the abelian group $C_n(X;G)$ of singular n-chains to the abelian group $C_{n-1}(X;G)$ of singular $(n-1)$-chains. In the sequence of homomorphisms

$$C_{n+1}(X;G) \xrightarrow{\partial} C_n(X;G) \xrightarrow{\partial} C_{n-1}(X;G),$$

equation (2) says that the image of the first map is contained within the kernel of the second map.[39] The image of the first map is an abelian subgroup of $C_n(X;G)$ denoted $B_n(X;G)$. Its elements are called **boundaries**, because they each have the form ∂c for some $c \in C_{n+1}(X;G)$. The kernel of the second map is an abelian subgroup of $C_n(X;G)$ denoted $Z_n(X;G)$. Its elements are called **cycles**, and it consists of the elements $c \in C_n(X;G)$ that satisfy $\partial c = 0$.

Equation (2) says that every boundary is a cycle, but some cycles might not be boundaries. As a result, the quotient group

$$H_n(X;G) \equiv Z_n(X;G)/B_n(X;G),$$

which consists of elements of $Z_n(X;G)$ modulo elements of $B_n(X;G)$, might be nontrivial. This is the ***n*th singular homology group** with coefficients in G.[40]

The case $G = \mathbb{Z}$ is especially important, so the more concise notation $H_n(X)$ is used as an abbreviation for $H_n(X;\mathbb{Z})$. The groups $H_n(X)$ are called **integral homology groups**. When the group G is not specified, $G = \mathbb{Z}$ is usually understood.

[39]This pattern is depicted graphically in article 29682.

[40]This article considers only singular homology, so the prefix *singular* will usually be omitted.

13 Some intuition from an example

This section uses the singular 3-chain that was defined in equation (4) to give some intuition about how homology groups can be sensitive to the topology of another manifold X. The conclusion will be that $H_2(S^1 \times S^1) \not\simeq H_2(\mathbb{R}^2)$.

The union of the images of the singular 3-chain c in equation (4) is a solid torus in $\mathbb{R}^n$. Let M denote this solid torus. The union of the images of the singular 2-chain ∂c is the boundary ∂M of M, which is homeomorphic to $S^1 \times S^1$. Let X be some other manifold whose topology we want to explore, and consider maps $\omega_3 : M \to X$ and $\omega_2 : \partial M \to X$. Here, the subscript on ω_k indicates the number of dimensions of the map's domain. By composing the maps in the singular 3-chain c with ω_3, we get a singular 3-chain c_3 whose target space is X. By composing the maps in the singular 2-chain c with ω_2, we get a singular 2-chain c_2 whose target space is X. We could choose the maps ω_3 and ω_2 so that $c_2 = \partial c_3$, but that's not required. In fact, to explore the topology of X, we really want to know if any choices of ω_2 exist for which c_2 is not equal to ∂c_3 for any ω_3 whatsoever.

If $X = S^1 \times S^1$, then such a choice for ω_2 does exist: just take ω_2 to be the obvious homeomorphism from ∂M to X. With that choice for ω_2, no matter how we choose ω_3, we cannot make $c_2 = \partial c_3$. Intuitively, this is clear because a continuous map $M \to \partial M$ that acts as the identity map on ∂M does not exist: a solid torus cannot be continuously retracted onto its boundary. The identity $\partial(\partial c) = 0$ implies $\partial c_2 = 0$, because composing ∂c with ω_2 can't separate any 2-simplexes that already coincide in the image of ∂c. This shows that c_2 is a cycle ($\partial c_2 = 0$), even though it's not a boundary ($c_2 \neq \partial c_3$ for any c_3). As a result, the homology group $H_2(X)$ nontrivial.

If $X = \mathbb{R}^2$ instead (or if X is any other two-dimensional contractible manifold), then no such choice for ω_2 would exist: we would always be able to choose a map ω_3 for which $c_2 = \partial c_3$. This is not obvious (to me), but it is a special case of the general fact that if X is contractible, then $H_k(X) = 0$ for all $k \geq 1$.[41]

[41] Lee (2011), corollary 13.11

14 Some properties of homology groups

Homology groups are topological invariants: if X and Y are homeomorphic to each other, then their homology groups $H_k(X;G)$ and $H_k(Y;G)$ are isomorphic to each other.[42] Homology groups are also invariant under a more inclusive equivalence relation: homotopy equivalent manifolds have isomorphic homology groups.[43] In particular, if X is contractible, then $H_k(X) = 0$ for all $k \geq 1$.[44] The fact that $H_k(\text{point}) = 0$ for all $k \geq 1$ has this generalization: if M is a triangulable compact n-dimensional manifold, then $H_k(M) = 0$ for $k \geq n+1$.[45]

[42] Lee (2011), corollary 13.3 (also Lee (2000), corollary 13.3)

[43] Lee (2011), corollary 13.9 (also Lee (2000), corollary 13.8); Hatcher (2001), corollary 2.11. Those results are stated for homology groups with integer coefficients, but the universal coefficient theorem (section 22) then implies that they also hold when other coefficient groups are used. Example: Eschrig (2011), section 5.5, page 136 (for coefficients in $\mathbb{R}$)

[44] Lee (2011), corollary 13.11 (also Lee (2000), corollary 13.9)

[45] Lee (2000), problem 13-7

15 Relating homology groups to homotopy groups

Section 5 mentions that two manifolds may have different homology groups even if their homotopy groups are identical, and conversely, but some relationships do exist between homology groups $H_k(X)$ and homotopy groups $\pi_k(X)$.

A relationship exists for $k = 1$, in spite of the fact that $\pi_1(X)$ can be nonabelian and $H_1(X)$ is always abelian. Let G be any group, not necessarily abelian.[46] The **commutator subgroup** of G, denoted $[G, G]$, is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$. (I'm using multiplicative notation here because G is not necessarily abelian.) The **abelianization** of a group G is the quotient group $G/[G, G]$.[47,48] The quotient group $G/[G, G]$ is abelian even if G is not. Now the relationship between $H_1(X)$ and $\pi_1(X)$ can be stated like this: if X is path-connected, then $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$.[49]

When $k \geq 2$, both $H_k(X)$ and $\pi_k(X)$ are always abelian, but they may still differ from each other. Here's one situation where at least some of them are equal to each other: if a manifold M is n-connected[50] with $n \geq 1$, then $H_k(M) = 0$ for $1 \leq k \leq n$, and $H_{n+1}(M) \simeq \pi_{n+1}(M)$.[51] That result is implied by this stronger result:[52,53] if X is a path-connected topological space, then the smallest value of k for which $H_k(X)$ is nontrivial is the same as the smallest value of k for which $\pi_k(X)$ is nontrivial, and $H_k(X) \simeq \pi_k(X)$ for that value of k. This is the **Hurewicz isomorphism theorem**.

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[46] I'm recycling the letter G here. This G is not related to the coefficient group G in $H_k(X; G)$.

[47] Lee (2011), text above theorem 10.19 (also Lee (2000), text above theorem 10.11)

[48] Article 29682 introduces the concept of a *quotient group*.

[49] Lee (2011), theorem 13.14 (also Lee (2000), theorem 13.11); Hatcher (2001), section 2.1, page 110

[50] n-connected means $\pi_k(M) = 0$ for $k \leq n$ (article 61813).

[51] Hatcher (2001), theorem 4.32; Maxim (2018), theorem 10.1

[52] Bott and Tu (1982), theorem 17.21; Whitehead (1978), chapter IV, corollaries 7.7 and 7.8

[53] Theorem 17.21 in Bott and Tu (1982) assumes that X is a CW complex, but the paragraph after remark 17.21.1 says that the theorem still holds when this condition is omitted.

16 Contrasting homology and homotopy groups

The intuition in section 13 illustrates an important difference between homology groups and homotopy groups. Roughly, the homotopy group $\pi_2(X)$ is defined using maps from S^2 into X. In contrast, the homology group $H_2(X)$ is defined using maps from a variety of topologically distinct spaces (including S^2 and $S^1 \times S^1$) into X. That's at least part of why $H_2(S^1 \times S^1)$ is nontrivial even though $\pi_2(S^1 \times S^1)$ is trivial.[54]

Homology groups and homotopy groups also differ in other ways: they differ in the way their group operations are defined, and they differ in the criterion they use for deciding whether a given map into X is trivial.[55] The message here is that they also differ in the set of spaces that they use to probe the space X: homotopy groups use only spheres, and homology groups use polyhedra, which are topologically more variable than spheres.

[54]Article 61813

[55]A homotopy group considers a map from S^k into X to be trivial if it can be continuously morphed into a map from S^k to a single point of X. A homology group considers a map from ∂M into X to be trivial if it's not the boundary of any map from M into X.

17 The zeroth homology group

Section 12 defined the nth homology group $H_n(X)$ as the group of n-chains c for which $\partial c = 0$ modulo the group of n-chains c for which $c = \partial c'$. Every 0-chain c satisfies $\partial c = 0$, so $H_0(X) = C_0(X)/B_0(X)$, where $B_0(X)$ is the kernel of the map $\partial : C_1(X) \to C_0(X)$.

If X is a single point, then only one singular 0-simplex exists (only one map from a single vertex to a single point), so every singular 0-chain is an integer multiple of this one singular 0-simplex. This gives $C_0(X) \simeq \mathbb{Z}$. The boundary of every singular 1-simplex is zero, so $H_0(X) \simeq \mathbb{Z}$ when X is a single point.[56] That implies $H_0(X) \simeq \mathbb{Z}$ for every contractible space X, because homotopy equivalent manifolds have isomorphic homology groups.[57] More generally, $H_0(X) \simeq \mathbb{Z}^k$ if X has k path-connected components.[58]

Some results can be stated more concisely in terms of the **reduced homology groups** $\tilde{H}_n(X)$. The definition won't be reviewed here, but the key properties are[59]

- $\tilde{H}_n(X) \simeq H_n(X)$ for all $n \geq 1$, for every space X.
- $\tilde{H}_0(X) = 0$ for every contractible space X.

For any space X, the last result generalizes to $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$.

[56] Hatcher (2001), proposition 2.8
[57] Section 14
[58] Hatcher (2001), proposition 2.7
[59] Hatcher (2001), section 2.1, page 110

18 Homology groups of S^n, $\mathbb{RP}^n$, and lens spaces

The homology groups of an n-sphere S^n with $n \geq 1$ are:[60,61]

$$H_k(S^n) \simeq \begin{cases} \mathbb{Z} & \text{if } k \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Another example related to spheres: using the convention $H_k(\cdot) \equiv 0$ for $k < 0$, the group $H_k(M \times S^n)$ is isomorphic to $H_k(M) \oplus H_{k-n}(M)$ for all k, n.[62]

The homology groups of n-dimensional real projective space $\mathbb{RP}^n$ are:[63]

$$H_k(\mathbb{RP}^n) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}_2 & \text{if } k \text{ is odd and } 1 \leq k < n, \\ \mathbb{Z} & \text{if } k \text{ is odd and } k = n, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Odd-dimensional real projective spaces are a special case of a more general pattern. To describe the generalization, think of S^{2n-1} as the unit sphere in $\mathbb{R}^{2n}$. Choose an integer $m \geq 2$ and a list of integers $k_{1,2}, k_{3,4}, ..., k_{2n-1,2n}$ that are relatively prime to m. Let G be the group generated by R, where R is the transformation that rotates through angle $2\pi k_{1,2}/m$ in the 1-2 plane, through angle $2\pi k_{3,4}/m$ in the 3-4 plane, and so on. The quotient space $M \equiv S^{2n-1}/G$ is called a **lens space**, and its homology groups $H_k(M)$ are[64] $\mathbb{Z}, \mathbb{Z}_m, 0, \mathbb{Z}_m, 0, ..., \mathbb{Z}_m, 0, \mathbb{Z}$ for $k = 0, 1, 2, ..., 2n-1$, respectively. This reduces to the previous result for $\mathbb{RP}^{2n-1}$ when $m = 2$ and $k_{\bullet,\bullet} = 1$ so that R has the same effect as reflecting every coordinate in $\mathbb{R}^{2n}$.

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[60] Lee (2011), proposition 13.23 (also Lee (2000), proposition 13.14)

[61] The restriction $n \geq 1$ is imposed here so that S^n is connected. The 0-sphere S^0 is a pair of points.

[62] Hatcher (2001), chapter 2, exercise 36

[63] Hatcher (2001), example 2.42; Miller (2016), proposition 17.1

[64] Hatcher (2001), example 2.43

19 Finitely generated abelian groups and torsion

Consider any abelian group of the form

$$G = G_1 \oplus G_2 \oplus G_3 \oplus \cdots \tag{7}$$

with a finite number of terms, where each term G_k is either $\mathbb{Z}$ or a finite cyclic group.[65] Then the **torsion subgroup** $T(G)$ is the group obtained by excluding all factors of $\mathbb{Z}$ from (7).[66] Examples:

$$T(\mathbb{Z}) = 0 \qquad T(\mathbb{Z}_n) \simeq \mathbb{Z}_n \qquad T(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2) \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_2.$$

A group is called **finitely generated** if it is generated by a finite number of elements.[67] Every finitely generated abelian group G may be written uniquely in the form[68]

$$G \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus T(G), \tag{8}$$

where the torsion $T(G)$ is a direct sum of cyclic groups of prime order, and the total number of summands is finite. The additive group of real numbers, $\mathbb{R}$, is one example of an abelian group that is not finitely generated.

[65] Every cyclic group with n elements is isomorphic to $\mathbb{Z}_n$ (Scott (1987), theorem 2.4.2).

[66] More generally, the *torsion subgroup* of an abelian group A is the subgroup consisting of all elements $g \in A$ with finite order (Scott (1987), text after theorem 5.1.2).

[67] Scott (1987), section 5.4

[68] Scott (1987), theorem 5.4.4

20 Homology groups of compact manifolds

When M is a compact manifold, the homology groups with coefficients in $\mathbb{Z}$ are finitely generated.[69,70] According to equation (8), this implies that the homology groups $H_k(M)$ of any compact manifold have the form

$$H_k(M) \simeq \mathbb{Z}^n \oplus T \tag{9}$$

for some n, where $\mathbb{Z}^n$ is the direct sum of n copies of $\mathbb{Z}$, the torsion part T is a finite abelian group. Examples:[71]

$$H_1(S^1 \times S^1) \simeq \mathbb{Z} \oplus \mathbb{Z} \qquad\qquad H_1(\text{Klein bottle}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \tag{10}$$

The number of $\mathbb{Z}$ summands in $H_k(M)$ is called the ***k*th Betti number** of M, and each integer n appearing in a summand $\mathbb{Z}_n$ is a **torsion coefficient**.[72] Informally, a finitely generated abelian group is sometimes said to **have *p*-torsion** if its torsion subgroup has $\mathbb{Z}_p$ as a direct summand.[73] Examples from equations (10):

- The first Betti number of the torus $S^1 \times S^1$ is 2.
- The first Betti number of the Klein bottle is 1.
- The first homology group of the torus does not have any torsion.
- The first homology group of the Klein bottle has 2-torsion.

If M is a closed and connected n-dimensional manifold, then $H_n(M;\mathbb{Z})$ is $\mathbb{Z}$ if M is orientable and is 0 otherwise.[74] For the same M, the torsion part of $H_{n-1}(M;\mathbb{Z})$ is 0 if M is orientable and is $\mathbb{Z}_2$ otherwise.[75]

[69] Hatcher (2001), by combining corollaries A.8 and A.9

[70] Section 19 defined *finitely generated*.

[71] Hatcher (2001), examples 2.3 and 2.47

[72] Hatcher (2001), section 2.1, page 130

[73] This language is common when the manifold M is a Lie group. Examples include Mimura and Toda (1991) and https://mathoverflow.net/questions/3700/.

[74] Hatcher (2001), text below theorem 3.26

[75] Hatcher (2001), corollary 3.28

21 Rings, principal ideal domains, and fields

This section briefly reviews a few of the algebraic structures that will appear in the remaining sections.

Article 29682 introduces the concept of a *group* G, one of the simplest mathematical structures with an operation that combines any two elements of G to get another element of G. This operation is usually called a *product* when it's not necessarily commutative. When it is commutative, it is often called a *sum*, as it is in this article because (co)homology groups are always commutative.

A **ring**[76] R is one of the simplest mathematical structures with two operations, each of which combines two elements of R to get another element of R. One is a commutative operation called a *sum* that makes R an abelian group. The other operation is called a *product*. The product is associative, and it distributes over addition, but it is not necessarily commutative, and elements of the ring don't necessarily have multiplicative inverses. A ring is called **commutative** if the product is commutative. Examples:

- The integers $\mathbb{Z}$ form a commutative ring.
- A matrix algebra forms a noncommutative ring.

A commutative ring is called a **field** if it has an identity element for multiplication and if every nonzero element has an inverse.[77] Examples include:[78]

- the field $\mathbb{R}$ of real numbers,
- the field $\mathbb{Q}$ of rational numbers,
- the field of integers modulo a prime number p, denoted $\mathbb{Z}_p$.

[76] Fraleigh (2014), definition 18.1; Pinter (1990), chapter 17, page 170

[77] Fraleigh (2014), definition 18.16; Pinter (1990), chapter 17, page 172

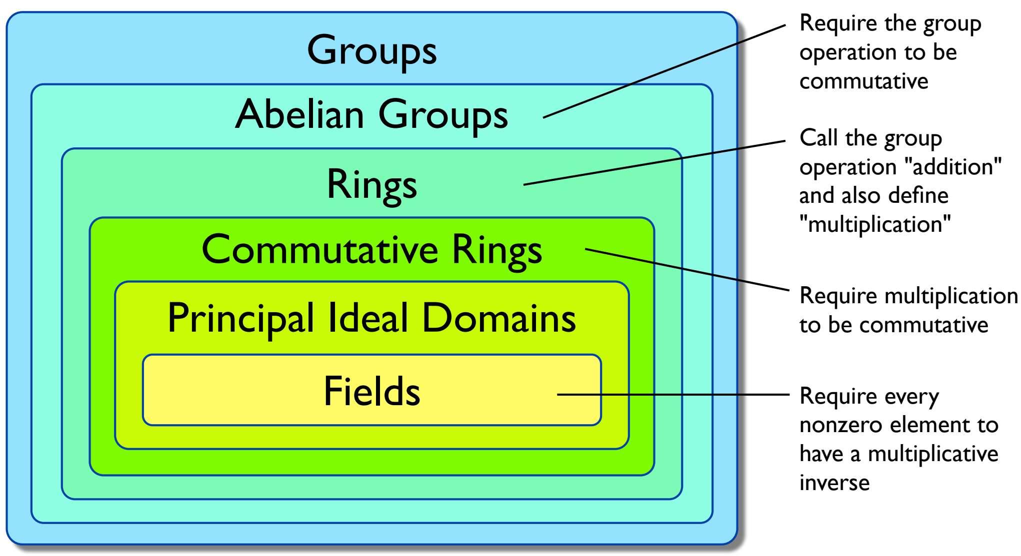
[78] Fraleigh (2014), example 18.18 and corollary 19.12

The integers $\mathbb{Z}$ do not form a field, because most nonzero integers do not have multiplicative inverses.

The concept of a **principal ideal domain** (often abbreviated **PID**) is more specific than the concept of a commutative ring[79] but more general than the concept of a field. Instead of reviewing the definition,[80] here are the examples that will be needed in this article:[81]

- The integers $\mathbb{Z}$ form a PID.
- Every field is also a PID.

This Venn diagram depicts the relationships:



[79] If n is not a prime number, then $\mathbb{Z}_n$ (the ring of integers modulo n) is a commutative ring but not a PID (Fraleigh (2014), example 18.17). It's not a PID because it has nonzero elements whose product is zero: if $j \neq 1$ and $k \neq 1$ are two integers for which $jk = n$, then the product of j and k is equivalent to zero in $\mathbb{Z}_n$. If n is a prime number, then $\mathbb{Z}_n$ is a field, and every field is a PID.

[80] Fraleigh (2014), definitions 19.6 and 45.7

[81] Fraleigh (2014), text below definition 45.7

22 The universal coefficient theorem

Each chain group with coefficients in G has the form[82]

$$C_n(X;G) \simeq C_n(X) \otimes G$$

where $C_n(X)$ is the chain group with coefficients in $\mathbb{Z}$. This leads to the **universal coefficient theorem** for homology groups, which says that if X is any topological space and G is any abelian group, then[83]

$$H_k(X;G) \simeq \big(H_k(X) \otimes G\big) \oplus \text{Tor}\big(H_{k-1}(X), G\big) \tag{11}$$

for all k. The general definition of $\text{Tor}(H,G)$ won't be reviewed here, but this is an important special case: if A and B are finitely generated abelian groups, then[84]

$$\text{Tor}(A,B) = T(A) \otimes T(B)$$

where $T(G)$ is the torsion subgroup of G as defined in section 19. In particular,

$$T(\mathbb{Z}) = 0 \qquad\qquad T(\mathbb{Z}_n) \simeq \mathbb{Z}_n.$$

Equation (11) says that the homology groups with coefficients in G don't convey any information about X beyond what the integral homology groups already convey. However, for any one value of k, $H_k(X;G)$ may convey information about X that $H_k(X)$ doesn't convey, because $H_k(X;G)$ depends on both $H_k(X)$ and $H_{k-1}(X)$.

[82] Section 1.4 in Maxim (2013) uses this to define $C_n(X;G)$ in terms of $C_n(X)$. This is equivalent to the definition in section (10), because $\mathbb{Z} \otimes G \simeq G$ (Sullivan (2020)).

[83] Bott and Tu (1982), theorem 15.14; Casacuberta (2015), the unnumbered equation after equation (7)

[84] Maxim (2013), equation (1.5.7)

23 The universal coefficient theorem: examples

This section uses the universal coefficient theorem to determine $H_n(M;G)$ for the cases $M = S^n$ and $M = \mathbb{R}\mathrm{P}^n$, whose integral homology groups were given in section 18.

First consider the case $M = S^n$ with $n \geq 1$. In this case, equation (5) implies that $T(H_k(S^n))$ is zero for all k, so equation (11) reduces to

$$H_k(S^n;G) \simeq H_k(S^n) \otimes G.$$

Combine this with equation (5) and the identity[85]

$$\mathbb{Z} \otimes G \simeq G$$

to get the final result[86]

$$H_k(S^n;G) \simeq \begin{cases} G & \text{if } k \in \{0,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Next, consider $M = \mathbb{R}\mathrm{P}^n$ and $G = \mathbb{Z}_2$. Use $T(\mathbb{Z}_2) \simeq \mathbb{Z}_2$ in (11) to get

$$H_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \simeq \big(H_k(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{Z}_2\big) \oplus \Big(T\big(H_{k-1}(\mathbb{R}\mathrm{P}^n)\big) \otimes \mathbb{Z}_2\Big).$$

Combine this with equation (6) and the identities[85]

$$\mathbb{Z} \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2 \qquad \mathbb{Z}_2 \otimes \mathbb{Z}_2 \simeq \mathbb{Z}_2 \qquad 0 \otimes \text{anything} = 0$$

to get the final result[87]

$$H_k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \simeq \mathbb{Z}_2 \qquad \text{for } 0 \leq k \leq n \text{ and } n \geq 1. \tag{12}$$

[85] Sullivan (2020)
[86] Maxim (2013), section 1.4, page 23
[87] Maxim (2013), example 1.4.1; Hatcher (2001), example 2.50; Miller (2016), section 19

24 Homology with coefficients in a field

When $\mathbb{F}$ is the field $\mathbb{R}$ of real numbers or the field $\mathbb{Q}$ of rational numbers,[88] the universal coefficient theorem gives[89]

$$H_k(M;\mathbb{F}) \simeq H_k(M;\mathbb{Z}) \otimes \mathbb{F} \qquad \text{for all } k, \tag{13}$$

and the relationships[90,91]

$$\mathbb{Z} \otimes \mathbb{F} \simeq \mathbb{F} \qquad T \otimes \mathbb{F} = 0 \qquad (T = \text{torsion part}) \tag{14}$$

imply that $H_k(M;\mathbb{F})$ doesn't know about the torsion part of $H_k(M;\mathbb{Z})$.

Equation (13) holds for a field of characteristic 0. If p is a prime number, then $\mathbb{Z}_p$ is a field with nonzero characteristic p. For this field, (11) implies[92]

$$H_k(M;\mathbb{Z}_p) \simeq H_k(M;\mathbb{Z}) \otimes \mathbb{Z}_p \qquad \text{if } H_{k-1}(M;\mathbb{Z}) \text{ doesn't have } p\text{-torsion.} \tag{15}$$

If the $(k-1)$th homology group of M does have p-torsion, then the equation on the left doesn't hold.[93]

[88] $\mathbb{Q}$ is sometimes denoted $\mathbb{Z}_0$. Example: Borel (1955), section 3, page 400.

[89] Hatcher (2001) states this for $\mathbb{F} = \mathbb{Q}$ in corollary 3A.6, and it can be deduced for $\mathbb{F} = \mathbb{R}$ by using proposition 3A.5(3) in the universal coefficient theorem (section 22).

[90] Derivation of the first relationship: If n is an integer and $r \in \mathbb{F}$, then $nr \in \mathbb{F}$. The definition of $\otimes$ allows integers to be passed from one side to the other, so $n \otimes r = 1 \otimes nr$. Every element of $\mathbb{Z} \otimes \mathbb{F}$ is a linear combination of elements of the form $n \otimes r$, so every element is equivalent to one of the form $1 \otimes$ (something). This gives $\mathbb{Z} \otimes \mathbb{F} \simeq \mathbb{F}$.

[91] Derivation of the second relationship: T is finite and abelian (using addition as the group operation), so for each element $t \in T$, a nonzero integer n exists for which $nt = 0$. If $r \in \mathbb{F}$, then $r/n \in \mathbb{F}$, so $t \otimes r = t \otimes (nr/n) = (nt) \otimes (r/n) = 0$ for all $t \otimes r$. Each element of $T \otimes \mathbb{F}$ is a linear combination of elements of the form $t \otimes r$, so $T \otimes \mathbb{F} = 0$.

[92] To derive this, use the fact that $\mathbb{Z}_p \otimes \mathbb{Z}_q = 0$ whenever p and q are distinct prime numbers (Sullivan (2020)).

[93] Example: $H_3(\mathbb{R}\mathrm{P}^5;\mathbb{Z})$ has 2-torsion (equation (6)), so the equation in (15) doesn't hold for $H_4(\mathbb{R}\mathrm{P}^5;\mathbb{Z}_2)$. Details: $H_4(\mathbb{R}\mathrm{P}^5;\mathbb{Z}) = 0$ (equation (6)) and $H_4(\mathbb{R}\mathrm{P}^5;\mathbb{Z}_2) \simeq \mathbb{Z}_2$ (equation (12)), so $H_4(\mathbb{R}\mathrm{P}^5;\mathbb{Z}_2)$ is not isomorphic to $H_4(\mathbb{R}\mathrm{P}^5;\mathbb{Z}) \otimes \mathbb{Z}_2$, even though $H_4(\mathbb{R}\mathrm{P}^5;\mathbb{Z})$ itself doesn't have 2-torsion.

25 Cartesian products of spheres

This section explains how to determine the homology groups of a cartesian product of any number of spheres with arbitrary dimensions. This is relevant to the topology of Lie groups, because when torsion is ignored, the homology groups of a compact connected Lie group are the same as the homology groups of a cartesian product of odd-dimensional spheres.[94]

If X and Y are CW complexes,[95] and if R is a principal ideal domain,[96] and if the homology groups $H_k(X;R)$ and $H_k(Y;R)$ don't have torsion,[97] then the homology groups $H_k(X\times Y;R)$ may be determined using[98]

$$\begin{aligned} H_n(X\times Y;R) = &\big(H_0(X;R)\otimes_R H_n(Y;R)\big)\\ &\oplus \big(H_1(X;R)\otimes_R H_{n-1}(Y;R)\big)\\ &\oplus\cdots\\ &\oplus \big(H_n(X;R)\otimes_R H_0(Y;R)\big) \qquad \text{(if no torsion)}, \end{aligned} \tag{16}$$

where the subscript on $\otimes_R$ means that elements of R may be passed from one side of $\otimes$ to the other. This is an example of a **Künneth formula**.

If we set $R=\mathbb{Z}$ and use the result shown in section 18 for the homology groups of a single sphere, then we can use (16) together with the identity[99] $\mathbb{Z}\otimes\mathbb{Z}=\mathbb{Z}$ to determine the integral homology groups for any cartesian product of spheres. Equation (16) holds in this case because the homology groups of an individual sphere don't have any torsion. Section 26 will show some examples.

[94]Article 92035

[95]Article 93875 reviews the definition of **CW complex**.

[96]Section 21

[97]Section 19

[98]This is theorem 3B.6 in Hatcher (2001), specialized to the case where $H_k(X;R)$ and $H_k(Y;R)$ don't have torsion. The result looks the same as corollary 3B.7 in Hatcher (2001), which assumes that R is a field (in which case the no-torsion condition is satisfied automatically), but here we will use it for $R=\mathbb{Z}$ (in which case the no-torsion condition is an additional condition).

[99]Section 23

26 Cartesian products of spheres: examples

In these examples, all spheres are assumed to have at least one dimension.[100] For the product of two spheres with different numbers of dimensions ($m \neq n$),[101]

$$H_k(S^m \times S^n) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0,\, m,\, n,\, m+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

For the product of two spheres with the same number of dimensions,[101]

$$H_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0,\, 2n\}, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

For the product of three spheres with different dimensions (ℓ, m, n all different),

$$H_k(S^\ell \times S^m \times S^n) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0,\, \ell,\, m,\, n,\, \ell+m,\, \ell+n,\, m+n,\, \ell+m+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that all homology groups of a single sphere are either $\mathbb{Z}$ or zero allows the general pattern to be expressed concisely using the **Poincaré polynomial**[102,103]

$$P(X,t) \equiv \sum_k t^k \, \text{Betti}\big(H_k(X;\mathbb{Z})\big).$$

Equation (16) implies that the Poincaré polynomial for a product of spheres is the product of the Poincaré polynomials of the individual spheres. Example:

$$P(S^\ell \times S^m \times S^n, t) = P(S^\ell, t)P(S^m, t)P(S^n, t) = (1+t^\ell)(1+t^m)(1+t^n),$$

with no restriction on ℓ, m, n.

[100] Cases involving a zero-dimensional sphere S^0, which is just a pair of points, are excluded.

[101] The results for a cartesian product of two spheres are also shown in Powell (2019), example 2.7.

[102] Mimura and Toda (1991), section 3.1, page 101

[103] The coefficients of the polynomial $P(X,t)$ are the Betti numbers of X that were defined in section 20.

27 From homology groups to cohomology groups

Cohomology groups are another set of topological invariants, closely related to homology groups. Like homology groups, cohomology groups are abelian groups expressed using addition as the group operation. The homology groups of a space determine its cohomology groups,[104] so the cohomology groups don't provide any new information, but cohomology groups can be promoted to *cohomology rings* that convey more information. Section 29 will introduce cohomology rings. This section highlights a relationship between cohomology groups and homology groups, as a substitute for reviewing their definition.[105]

The notation for cohomology groups is almost identical to the notation for homology groups:[106] homology groups are written with a subscript, as in $H_k(X;\mathbb{Z})$, and cohomology groups are written with a superscript, as in $H^k(X;\mathbb{Z})$.

If the homology groups are finitely generated,[107] as they are for any compact manifold M,[108] then the relationship between cohomology groups and homology groups is especially simple:[109]

$$\begin{aligned} H^k(M;\mathbb{Z}) \simeq\ &(\text{non-torsion part of } H_k(M;\mathbb{Z})) \\ &\oplus (\text{torsion part of } H_{k-1}(M;\mathbb{Z})). \end{aligned} \tag{17}$$

Example: if the manifold is $M = \mathbb{R}\mathrm{P}^2$, then[110]

$$\begin{array}{lll} H^0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \simeq \mathbb{Z} & H^1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = 0 & H^2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \simeq \mathbb{Z}_2 \\ H_0(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \simeq \mathbb{Z} & H_1(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \simeq \mathbb{Z}_2 & H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) = 0. \end{array}$$

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[104] Hatcher (2001), intro to chapter 3, page 185

[105] Section 3.1 in Hatcher (2001) introduces cohomology groups.

[106] For the rest of this article, the ring of coefficients will be indicated explicitly, even when it's $\mathbb{Z}$.

[107] Section 19 defined *finitely generated*.

[108] Section 20

[109] Hatcher (2001), section 3.3, second-to-last paragraph on page 231; Davis and Kirk (2001), section 2.6, text between exercise 31 and theorem 2.33; Mimura and Toda (1991), section 7.1, result 1.19 on page 372

[110] Davis and Kirk (2001), section 1.4, page 15

28 Cohomology with other coefficients

Cohomology has its own version of the universal coefficient theorem. It looks like this:[111]

$$H^k(X;G) \simeq \text{Hom}\big(H_k(X;R),G\big) \oplus \text{Ext}\big(H_{k-1}(X;R),G\big) \tag{18}$$

where R is a principal ideal domain (like $\mathbb{Z}$ or a field) that acts on G in a natural way.[112] The definitions of Hom and Ext won't be reviewed here, but special cases will be highlighted below. Notice the superscripts and subscripts: equation (18) involves both a cohomology group and homology groups.

One special case was already highlighted in section 27: if H is finitely generated, then $\text{Hom}(H,\mathbb{Z})$ and $\text{Ext}(H,\mathbb{Z})$ are isomorphic to the non-torsion and torsion parts of H, respectively,[113] so if M is a compact manifold, then (18) reduces to (17) when $R = G = \mathbb{Z}$.[114]

Another easy special case is when $R = G = \mathbb{F}$ for a field $\mathbb{F}$ of characteristic zero, like $\mathbb{Q}$ or $\mathbb{R}$. In that case, if we again suppose that the homology groups are finitely generated[115] (true for any compact manifold), then equation (18) gives[116]

$$H^k(M;\mathbb{F}) \simeq H_k(M;\mathbb{F}) \qquad \text{for all } k. \tag{19}$$

Equation (19) can also be inferred from equations (13) and (17), and so can

$$H^k(M;\mathbb{F}) \simeq H^k(M;\mathbb{Z}) \otimes \mathbb{F} \qquad \text{for all } k. \tag{20}$$

The cohomology groups over different fields are essentially interchangeable, but $H^k(M;\mathbb{Z})$ and $H^k(M;\mathbb{F})$ are not interchangeable when $\mathbb{F}$ is a field, because only

[111] Mimura and Toda (1991), chapter 3, equation (1.7); Casacuberta (2015), page 9 (for $R = \mathbb{Z}$); Hatcher (2001), section 3.1, page 198 (for $R = \mathbb{Z}$)

[112] More precisely: G is an R-module (Mimura and Toda (1991), chapter 3, text above equation (1.5)).

[113] Hatcher (2001), text above corollary 3.3

[114] https://ckottke.ncf.edu/docs/exttoruct.pdf, proposition 6.3

[115] https://math.stackexchange.com/questions/42581/ illustrates the importance of this condition.

[116] Mimura and Toda (1991), chapter 3, equation (1.8)

the first one knows about torsion. We can switch between the different coefficient-fields $\mathbb{Q}$ and $\mathbb{R}$ using[117,118]

$$H^k(M;\mathbb{R}) \simeq H^k(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \qquad\qquad \text{for all } k$$

and $\mathbb{Q}\times\mathbb{R} \simeq \mathbb{R}$. If $\mathbb{F}$ is $\mathbb{Q}$ or $\mathbb{R}$, then applying $\otimes\mathbb{F}$ to $H^k(M;\mathbb{Z})$ discards information because of the second equation in (14), but applying $\otimes_{\mathbb{Q}}\mathbb{R}$ to $H^k(M;\mathbb{Q})$ does not discard any information.

Even though they don't convey as much topological information as integral cohomology groups $H^k(M;\mathbb{Z})$ do, the real cohomology groups $H^k(M;\mathbb{R})$ are important partly because of this relationship: if M is a smooth manifold, then the **de Rham cohomology groups** $H^k_{\text{dR}}(M)$ can also be defined,[119] and the groups $H^k_{\text{dR}}(M)$ and $H^k(M;\mathbb{R})$ are isomorphic to each other.[120] This provides a calculus-based way of thinking about the non-torsion part of cohomology groups.[121]

[117]Mimura and Toda (1991), section 6.5, page 341

[118]The subscript on $\otimes$ indicates what kinds of factors can be passed back and forth from one side of $\otimes$ to the other.

[119]The kth de Rham cohomology group $H^k_{\text{dR}}(M)$ is the additive group of differential k-forms ω satisfying $d\omega = 0$ modulo terms of the form $d\lambda$, where λ is a $(k-1)$-form (Madsen and Tornehave (1997)).

[120]This is called **de Rham's theorem** (Davis and Kirk (2001), section 1.4, page 16).

[121]Madsen and Tornehave (1997)

29 Cohomology rings

For a given space X, the collection of cohomology groups may be promoted to a **cohomology ring** that has both a sum operation and a product operation. The sum operation is the one inherited from the cohomology groups, and the product operation is a new device called the **cup product** whose definition won't be reviewed here.[122] The goal in this section is to review just enough about the concept of a *cohomology ring* to explain how results about cohomology groups can be extracted from results that are expressed in terms of cohomology rings.

Homology and cohomology groups may both be defined using coefficients in an abelian group G.[123] Universal coefficient theorems, examples of which were reviewed in the preceding sections, relates those homology and cohomology groups to the ones with coefficients in $\mathbb{Z}$. In a cohomology *ring*, we need both a sum and a product,[124] so we use a *ring* R of coefficients instead of just an abelian group.[125] This is an easy step, because the typical choices for the group of coefficients – namely $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{R}$, and $\mathbb{Q}$ – are already rings. For a given space X, the cohomology ring with coefficients in R is denoted $H^*(X;R)$, with an asterisk instead of an index.

The ring $H^*(X;R)$ is generated by the groups $H^k(X;R)$. An extra bit of structure called a **grading** is used to keep track of the index k that labels the individual cohomology groups. This makes $H^*(X;R)$ a **graded ring**.[126] An element a of $H^*(X;R)$ is called **homogeneous** if it belongs to one of the subsets $H^k(X;R)$, and the subset to which it belongs is indicated by writing $|a| = k$. I'll call the integer $|a|$ the **grade**[127] of a. If two elements $a \in H^*(X;R)$ and $b \in H^*(X;R)$ have the same grade, then their sum $a+b$ is defined just like it is in that individual

[122] Hatcher (2001), chapter 3; Maxim (2013), chapter 3

[123] Hatcher (2001), section 2.2, page 153 (for homology groups) and section 3.1, page 197 (for cohomology groups)

[124] Section 21

[125] Hatcher (2001), section 3.2, page 206

[126] Hatcher (2001), section 3.2, page 212; https://math.stackexchange.com/questions/1581681/

[127] It's usually called the **degree** or the **dimension**, but those words are overloaded. The name **grade** seems like a more natural choice, because it relates naturally to the name *graded ring* (which is standard) and it is less overloaded.

cohomology group. If they have different grades, then we can still write their sum as $a+b$, subject to the usual axioms like $a+b=b+a$ and $a+0=a$. In this case, $a+b$ doesn't belong to any of the individual cohomology groups $H^k(M;R)$, but it still belongs to the ring $H^*(M;R)$.

The product operation in the cohomology ring $H^*(X;R)$ is a new structure that wasn't present in the cohomology groups $H^k(X;R)$. Its definition (not reviewed here) ensures that if a and b are homogeneous elements with grades j and k, respectively, then their product ab is homogeneous with grade $j+k$. The product is not necessarily commutative, and its "multiplication table" can hold information about the topology of X that is not conveyed by the cohomology groups alone. As an example, the cohomology groups of $SO(5)$ and $\mathbb{R}\mathrm{P}^7\times S^3$ with coefficients in $\mathbb{Z}$ are isomorphic to each other, but the cohomology rings are not.[128,129] This is possible because an isomorphism of cohomology rings must preserve the structures that make it a graded ring (the sum, the product, and the grading), but an isomorphism of cohomology groups only needs to preserve the structure that makes them groups (namely the sum).

[128] Hatcher (2001), section 3.E, page 309

[129] Another example is described in https://topospaces.subwiki.org/wiki/Cohomology_groups_need_not_determine_cohomology_ring/.

30 Cohomology rings: example

To illustrate the concept of a cohomology ring, consider the manifold[130]

$$X = S^3 \times S^3 \times S^5. \tag{21}$$

This section starts with an expression for the cohomology ring $H^*(X;\mathbb{Z})$ and then explains how to extract the cohomology groups $H^k(X;\mathbb{Z})$ from it.

The cohomology ring $H^*(X;\mathbb{Z})$ of the space (21) is an **exterior algebra** generated by three elements with grades 3, 3, and 5, respectively:[131]

$$H^*(X;\mathbb{Z}) \simeq \Lambda_{\mathbb{Z}}[a,b,c] \qquad |a| = |b| = 3,\ |c| = 5. \tag{22}$$

An exterior algebra Λ is a graded ring whose product is such that two homogeneous elements anticommute with each other if their grades are both odd and commute with each other otherwise. In this example, the generators all have odd grade, so they all anticommute with each other. This tells us that every element of $H^*(X;\mathbb{Z})$ is a linear combination of the elements

$$1,\ a,\ b,\ c,\ ab,\ ac,\ bc,\ abc$$

with coefficients in $\mathbb{Z}$ (the subscript on $\Lambda_{\mathbb{Z}}[\cdots]$). The grades of these elements are

$$0,\ 3,\ 3,\ 5,\ 6,\ 8,\ 8,\ 11,$$

respectively.[132] This tells us that every element of grade 6 is proportional to ab with a coefficient in $\mathbb{Z}$, so $H^6(X;\mathbb{Z}) \simeq \mathbb{Z}$. This also tells us that nonzero elements of grade 7 don't exist, so $H^7(X;\mathbb{Z}) = 0$. It also tells us that every element of grade 8 is a linear combination of ac and bc with coefficients in $\mathbb{Z}$, so $H^8(X;\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Equation (17) may be used to check that this is consistent with sections 25-26.

[130] Section 26

[131] Hatcher (2001), examples 3.13 and 3.16

[132] Example: the grade of bc is $|bc| = 8$ because $|b| + |c| = 3 + 5 = 8$.

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