Nonrelativistic Fermions and Bosons

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Abstract In quantum field theory, observables are associated with regions of spacetime, not with individual particles. This article introduces one of the simplest models in which that distinction really is important: a nonrelativistic model of multiple interacting particles with zero spin, all of the *same species*, sometimes called **identical** particles. A generalization to multiple species, with multiple particles of each species, is also introduced. Despite their shortcomings (they are nonrelativistic, they ignore the dynamics of the electromagnetic field, and their elementary particles all have zero spin), these models can still exhibit complex phenomena that resemble real molecules, fluids, and solids.

Article 15939 introduces a different way of constructing these models, using field operators.

Contents

1 An auxiliary Hilbert space

All infinite-dimensional separable Hilbert spaces over the complex numbers are isomorphic to each other (article 90771),[1] but different ways of constructing this Hilbert space are convenient for different purposes. Each of the models introduced in this article will be defined using a different subspace of an auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$, which in turn is given by the construction in article 41522 (reviewed below). These subspaces are isomorphic to each other (and to $\mathcal{H}_{\text{aux}}$), but these different ways of presenting them are convenient for constructing different models.

Let D denote the number of dimensions of space (normally $D = 3$). Boldface symbols will be used to denote quantities with D components: $\mathbf{x} = (x_1, ..., x_D)$. Subscripts on boldface symbols indicate different D-component quantities, so $\mathbf{x}_j$ and $\mathbf{x}_k$ each have D components.

A vector $|\psi\rangle$ in the auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ will be represented by a complex-valued function

$$\psi(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) \tag{1}$$

of $N \times D$ real variables, where N will correspond to the number of particles. A function (1) represents a vector $|\psi\rangle$ only if its norm is finite,[2] and two functions represent the same vector if the norm of their difference is zero. The inner product of two vectors $|\phi\rangle$ and $|\psi\rangle$ is defined by

$$\langle\phi|\psi\rangle \equiv \int \phi^*(\mathbf{x}_1, ..., \mathbf{x}_N)\psi(\mathbf{x}_1, ..., \mathbf{x}_N), \tag{2}$$

where $\int$ denotes the integral over all space for each argument $\mathbf{x}_n$.

[1] To prove this, let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two such Hilbert spaces. Choose any orthonormal basis for $\mathcal{H}_A$ and any orthonormal basis for $\mathcal{H}_B$. These two sets of elements are both countably infinite (because both spaces are infinite-dimensional and separable), and any one-to-one map between them defines a unitary equivalence (isomorphism) between the two Hilbert spaces.

[2] **Norm** is defined in article 41522.

2 Two special subspaces

Let $E_{j,k}$ be the linear operator that exchanges the arguments $\mathbf{x}_j$ and $\mathbf{x}_k$ with each other. Example:

$$E_{1,2}\,\psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_N) = \psi(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, ..., \mathbf{x}_N).$$

A function (1) representing a vector $|\psi\rangle$ will be called

- **symmetric** if $E_{j,k}|\psi\rangle = |\psi\rangle$ for all j, k
- **antisymmetric** if $E_{j,k}|\psi\rangle = -|\psi\rangle$ for all j, k.

The vectors represented by symmetric functions constitute a self-contained Hilbert space all by themselves, as do the vectors represented by antisymmetric functions. Let P_+ and P_- denote the projections onto these two subspaces, respectively. When $N \geq 2$, these projection operators are orthogonal to each other: $P_+P_- = 0$.[3]

For any N, the Hilbert spaces

$$\mathcal{H}_+ \equiv P_+\mathcal{H}_{\text{aux}} \qquad\qquad \mathcal{H}_- \equiv P_-\mathcal{H}_{\text{aux}}$$

are both isomorphic to the original Hilbert space $\mathcal{H}_{\text{aux}}$, but these different ways of describing them are convenient for constructing different models. The following sections use $\mathcal{H}_+$ to construct a model of bosons and use $\mathcal{H}_-$ to construct a model of fermions. Section 6 shows that these models are generally *not* equivalent to each other, even though the Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ are isomorphic to each other. A model is defined by the pattern of its observables, not just by the Hilbert space on which those observables are represented as linear operators.

[3] When $N = 1$, they are both equal to the identity operator: $P_+ = P_- = 1$.

3 Single species: hamiltonian

In a system of units in which $\hbar = 1$, let H be the operator on $\mathcal{H}_{\text{aux}}$ defined by

$$H = T + V \tag{3}$$

with T and V defined by[4]

$$T\psi(\mathbf{x}_1, ..., \mathbf{x}_N) \equiv \sum_n \frac{-(\nabla_n)^2}{2m}\psi(\mathbf{x}_1, ..., \mathbf{x}_N) \tag{4}$$

$$V\psi(\mathbf{x}_1, ..., \mathbf{x}_N) \equiv \sum_{j<k} V(\mathbf{x}_j - \mathbf{x}_k)\psi(\mathbf{x}_1, ..., \mathbf{x}_N), \tag{5}$$

where ∇_n is the gradient with respect to $\mathbf{x}_n$ and where $V(\mathbf{x})$ depends only on $|\mathbf{x}_j - \mathbf{x}_k|$. The operator H commutes with the exchange operators $E_{j,k}$. In other words, H is invariant under permutations of the arguments $\mathbf{x}_n$. This implies that H commutes with the projection operators $P_\pm$, so applying the unitary operator

$$U(t) \equiv e^{-iHt} \tag{6}$$

to any vector in $\mathcal{H}_+$ (respectively $\mathcal{H}_-$) gives another vector in $\mathcal{H}_+$ (respectively $\mathcal{H}_-$). This allows us to use H as the hamiltonian, the generator of time translations, for a model whose Hilbert space is only $\mathcal{H}_+$ or only $\mathcal{H}_-$.

[4] I'm using the notation that was introduced in article 20554: for any linear operator A on the Hilbert space, "$A\psi$" is the name of a function that represents the vector $A|\psi\rangle$.

4 Single species: local observables

For any time t and any region R of space, define a **counting operator** $C(R,t)$ by

$$C(R,t) \equiv U^{-1}(t)C(R,0)U(t) \tag{7}$$

with

$$C(R,0)\,\psi(\mathbf{x}_1,...,\mathbf{x}_N) = k\,\psi(\mathbf{x}_1,...,\mathbf{x}_N), \tag{8}$$

where $U(t)$ is the time translation operator defined in (6) and k is the number of $\mathbf{x}_n$s in R. Applying $C(R,t)$ to any vector in $\mathcal{H}_+$ (respectively $\mathcal{H}_-$) gives another vector in $\mathcal{H}_+$ (respectively $\mathcal{H}_-$), so we can use these operators to define two different single-species models, one based on $\mathcal{H}_+$ and one based on $\mathcal{H}_-$. In both models, the basic observables are these counting operators. They may be expressed in terms of mutually orthogonal projection operators like this:

$$C(R,t) = \sum_k Q^{(k)}(R,t)\,k,$$

where each projection operator $Q^{(k)}(R,t)$ is defined by

$$Q^{(k)}(R,t) \equiv U^{-1}(t)Q^{(k)}(R,0)U(t) \tag{9}$$

$$Q^{(k)}(R,0)\psi(\mathbf{x}_1,...,\mathbf{x}_N) \equiv \begin{cases} \psi(\mathbf{x}_1,...,\mathbf{x}_N) & \text{if exactly } k \text{ of the } \mathbf{x}_n\text{s are in } R, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

For each R and t, the possible outcomes when the observable $C(R,t)$ is measured are represented by the mutually orthogonal projection operators $Q^{(k)}(R,t)$, interpreted as "exactly k particles are in R at time t."

Any other observable A is interpreted as being localized in R at time t if and only if

$$A|\psi\rangle \propto |\psi\rangle \qquad \text{whenever } C(R,t)|\psi\rangle = 0,$$

for all $|\psi\rangle$ in the appropriate space ($\mathcal{H}_+$ or $\mathcal{H}_-$). The observables $C(R,t)$ and $Q^{(k)}(R,t)$ clearly satisfy this condition, as do many others.

With this association between regions of spacetime and observables, the model restricted to the subspace $\mathcal{H}_+$ is a model of N spinless **bosons**, and the model restricted to the subspace $\mathcal{H}_-$ is a model of N spinless **fermions**.[5] The observables defined above are associated with regions of spacetime, not with individual particles. That's what physicists mean when they say that particles of the same species are **identical** or **indistinguishable**.

[5]According to the spin-statistics theorem, "spinless fermions" would be an oxymoron in relativistic QFT, but the spin-statistics theorem does not apply in nonrelativistic QFT. A nonrelativistic model of spinless fermions may not be an approximation to any relativistic model, but it can still be a good warm-up for more realistic models.

5 The Pauli exclusion principle

The auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ includes states of the form

$$\psi(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) = g_1(\mathbf{x}_1) g_2(\mathbf{x}_2) \cdots g_N(\mathbf{x}_N). \tag{11}$$

Summing over all permutations of the arguments $\mathbf{x}_n$ (that is, applying the projection operator P_+) gives a state in the Hilbert space $\mathcal{H}_+$, which is used to construct the boson model. This state is nonzero even if two or more of the factors $g_n(\mathbf{x})$ are equal to each other, such as

$$\psi(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) = g(\mathbf{x}_1) g(\mathbf{x}_2) \cdots g(\mathbf{x}_N). \tag{12}$$

This is often expressed by saying that two or more particles of the same boson species can occupy the same "state" (using the word "state" for a single factor on the right-hand side, not to be confused with the full N-particle state).

In contrast, two or more particles of the same fermion species cannot occupy the same "state" in this sense, because applying P_- to (11) gives zero whenever two or more of the factors $g_n(\mathbf{x})$ are equal to each other. This is called the **Pauli exclusion principle**.

Electrons are fermions, so they respect the Pauli exclusion principle. The consequences of this in chemistry are widely appreciated. Section 10 highlights another important consequence that might not be so widely appreciated: it is essential for the stability of macroscopic matter.

6 Inequivalence of fermions and bosons

Let $\mathcal{L}(R,t)$ denote the set of local observables that a given model associates with region R at time t, and suppose that these observables are represented by operators on a Hilbert space $\mathcal{H}$. We can say that two models A and B are **equivalent** to each other if[6]

$$U^{-1}\mathcal{L}_A(R,t)U = \mathcal{L}_B(R,t) \tag{13}$$

for all R and all t, for some unitary transformation U between their Hilbert spaces:

$$U\mathcal{H}_B = \mathcal{H}_A \qquad\qquad U^{-1}\mathcal{H}_A = \mathcal{H}_B. \tag{14}$$

Now suppose that A and B are the fermion and boson models, respectively, so that $\mathcal{H}_{A/B} = \mathcal{H}_{\pm}$, and take them both to have the same values of D, N, m, and $V(\mathbf{x})$. Plenty of unitary transforms satisfying (14) exist in this case, so the *Hilbert spaces* are equivalent, but do any of those unitary transforms also satisfy (13)? If not, then the *models* are not equivalent, even though their Hilbert spaces are.

Here's an intuitive argument that the models are not equivalent when $N \geq 2$,[7] in the easiest case $H = T$ with T defined by (4), so the interaction term V is absent. Consider states of the form $Q^{(N)}(R,t)|\psi\rangle$, so that the probability $p(\overline{R},t)$ of detecting any of the particles outside of the region R is zero at time t. This probability will increase over time (article 20554). We can choose the vector $|\psi\rangle$ to minimize the rate at which it increases. For the boson model, we can do this by taking the function $\psi(\mathbf{x}_1,...,\mathbf{x}_N)$ that represents $|\psi\rangle$ to be the product of N copies of the function $\psi_1(\mathbf{x})$ that satisfies this rate-minimizing condition when $N = 1$, but the Pauli exclusion principle says that we can't do that in the fermion model. In the fermion model, the Pauli exclusion principle forces us to put some of the fermions in other states that disperse more quickly, so we expect that $p(\overline{R},t)$ must grow faster in the fermion model than its minimum rate of increase the boson model. This implies that the condition (13) does not hold, so the two models are not equivalent.

[6] $\mathcal{L}_A$ and $\mathcal{L}_B$ are sets of observables in models A and B, respectively.

[7] They are manifestly equivalent when $N = 1$.

To give that intuitive argument a little more substance, choose a region R and a time t, and let $i(T|R)$ denote the infimum of

$$\frac{\langle\psi|T|\psi\rangle}{\langle\psi|\psi\rangle} \tag{15}$$

among all vectors $|\psi\rangle$ in $\mathcal{H}_{\text{aux}}$ on which T is defined and that satisfy the constraint

$$Q^{(N)}(R,t)|\psi\rangle = |\psi\rangle. \tag{16}$$

In words: this is the lower bound of the energy's expectation value among states in which all of the particles are in R at time t. If we can show that $i(T|R)$ is greater in the fermion model than it is in the boson model, then we can infer that the probability $p(\overline{R},t)$ must grow more quickly in the fermion model than it does in the boson model (because greater $i(T|R)$ means higher momenta), which in turn shows that the two models are not equivalent.

In the boson model, we can deduce an *upper* bound on $i(T|R)$ by evaluating (15) for any state of the form (12), in which case (15) is N times its value for $N = 1$. By definition, the infimum $i(T|R)$ cannot exceed this, so

$$i(T|R) \leq \kappa_B N \qquad \text{(boson case)}, \tag{17}$$

where κ_B depends on R but not on N. In the fermion model, this *lower* bound on $i(T|R)$ can be derived:[8]

$$i(T|R) \geq \kappa_F N^{1+2/D} \qquad \text{(fermion case)}, \tag{18}$$

where κ_F depends on R but not on N.[9] Even without knowing the values of κ_B and κ_F, this shows that the infimum $i(T|R)$ in the fermion model is greater than

[8]The derivation for $D = 3$ is reviewed on pages 556-557 in Lieb (1976), and its generalization to arbitrary D is mentioned on pages 16-17 in Lieb (1990). In those theorems, the right-hand side of the inequality involves an integral $\int d^D x \ \left(\rho_\psi(\mathbf{x})\right)^{1+2/D}$, where $\rho_\psi(\mathbf{x})$ is the integral of $|\psi(\mathbf{x}_1, ..., \mathbf{x}_N)|^2$ over all but one of its N arguments. The constant κ_F in (18) is obtained by taking $\rho_\psi(\mathbf{x})$ to be uniform over the region R and zero outside of R, which minimizes this integral among states satisfying (16).

[9]Some constraints on the coefficients in inequalities like this are reviewed in Schimmer (2022).

it is in the boson model wnen N is sufficiently large. We could refine the argument to quantify "sufficiently large," but the point is already made: the fermion and boson models cannot always be equivalent to each other, even though their Hilbert spaces are.

7 Multiple species: Hilbert space

Section 9 describes a model with any number of distinct species, each of which may be either fermionic or bosonic. The number of species will be denoted N_{spec}. The special case $N_{\text{spec}} = 1$, in which all of the particles are the same species, was constructed in the preceding sections. The opposite extreme $N_{\text{spec}} = N$, which has just one particle of each species, was constructed in article 41522.

For the general case, start with the auxiliary Hilbert space constructed in section 1. Partition the index-set $\{1, ..., N\}$ into N_{spec} subsets, each corresponding to one of the N_{spec} species, and let let I_s be subset corresponding to species s. The number of elements in I_s, denoted $|I_s|$, is the number of particles of species s. These numbers satisfy

$$\sum_s |I_s| = N.$$

Define

$$\sigma_s \equiv \begin{cases} +1 & \text{if species } s \text{ is bosonic,} \\ -1 & \text{if species } s \text{ is fermionic.} \end{cases}$$

The model is based on a special subspace of the auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$, namely the subspace $\mathcal{H}$ represented by functions satisfying this condition:[10]

$$E_{jk}|\psi\rangle = \sigma_s|\psi\rangle \qquad \text{whenever } j, k \in I_s,$$

for every species s. In words: whenever the indices j and k are both associated with the same species, exchanging the arguments $\mathbf{x}_j$ and $\mathbf{x}_k$ either leaves $\psi(\mathbf{x}_1, ..., \mathbf{x}_N)$ unchanged or changes its sign, depending on whether the species is bosonic or fermionic.

[10] The exchange operators E_{jk} are defined in section 2.

8 Multiple species: hamiltonian

The hamiltonian is $H = T + V$ with[11]

$$T \equiv \sum_s \sum_{j \in I_s} \frac{-\nabla_j^2}{2m_s}$$
$$V \equiv \frac{1}{2} \sum_{s,s'} \sum_{j \in I_s} \sum_{k \in I_{s'}} V_{s,s'}(\mathbf{x}_j - \mathbf{x}_k),$$

where ∇_j is the gradient with respect to $\mathbf{x}_j$ and where each function $V_{s,s'}(\mathbf{x})$ depends only on $|\mathbf{x}|$. The sum over s, s' is over all ordered pairs of species, and the factor $1/2$ compensates for including both orderings (s, s') and $(s's)$ when $s \neq s'$, and for including both orderings of j, k when $s = s'$.

In words: all particles of species s have the same mass m_s, and all interactions between particles of the two given species s and s' have the same form $V_{s,s'}$. This ensures that applying the unitary operator $U(t) \equiv \exp(-iHt)$ to any vector in $\mathcal{H}$ gives another vector in $\mathcal{H}$.

[11] This generalizes equations (4)-(5) to multiple species. For brevity, the function $\psi(\mathbf{x}_1, ..., \mathbf{x}_N)$ is not written here.

9 Multiple species: local observables

The model has separate counting observables for each species, defined by

$$C_s(R,t) \equiv U^{-1}(t) C_s(R,0) U(t) \tag{19}$$

with

$$C_s(R,0)\,\psi(\mathbf{x}_1,...,\mathbf{x}_N) = k\,\psi(\mathbf{x}_1,...,\mathbf{x}_N),$$

where k is the number of $\mathbf{x}_n$s for which $n \in I_s$ and $\mathbf{x}_n \in R$. The operator $C_s(R,t)$ may be written in terms of mutually orthogonal projection operators like this:

$$C_s(R,t) = \sum_k Q_s^{(k)}(R,t)\,k$$

where $Q_s^{(k)}(R,t)$ is the projection operator defined by

$$Q_s^{(k)}(R,t) \equiv U^{-1}(t) Q_s^{(k)}(R,0) U(t)$$

and

$$Q_s^{(k)}(R,0)\psi(\mathbf{x}_1,...,\mathbf{x}_N) \equiv \begin{cases} \psi(\mathbf{x}_1,...,\mathbf{x}_N) & \text{if exactly } k \text{ of the } \mathbf{x}_n\text{s with } n \in I_s \text{ are in } R, \\ 0 & \text{otherwise.} \end{cases}$$

For each s, R, t, the observable (19) counts the number of particles of species s in the region R at time t. An observable A is localized in R at time t if and only if it acts like something proportional to the identity operator on all states in which the particles are all localized elsewhere, outside of R. This condition be expressed more precisely as

$$A\,|\psi\rangle \propto |\psi\rangle \qquad \text{whenever } C_s(R,t)|\psi\rangle = 0 \text{ for all } s.$$

This defines a model of N_{spec} different species, each either bosonic or fermionic. The number of particles of species s is $|I_s|$. Distinct species are detected by different observables, but those observables are associated with regions of spacetime, not with individual particles.

10 The stability of macroscopic matter

The family of models defined in sections 7-9 includes an important special case, defined by these conditions:

- Space is three-dimensional: $D = 3$.
- The interactions are Coulomb interactions: $V_{s,s'}(\mathbf{x}) = e_s e_{s'} / \big(4\pi\epsilon_0 |\mathbf{x}|\big)$.
- Two species, both called *electrons*,[12] are fermions with equal values of e_s and the same single-particle mass m.
- The rest of the species, called *nuclei*, have values of e_s with the opposite sign and masses that are $\gg m$ but otherwise arbitrary. They may be fermions or bosons.

This special case is important because it captures some of the qualities of electrons and nuclei that are most important for matter under ordinary conditions. It doesn't include electromagnetic interactions (except for the static Coulomb interactions), but it can still exhibit complex phenomena that resemble real molecules, fluids, and solids.[13]

This simplified model of matter predicts that matter is stable, in this sense: the lowest possible energy E_0 of a system with N_- electrons of charge e_- and N_+ nuclei is bounded by[14] $E_0 \geq -\kappa(N_- + N_+)$ with $\kappa \sim m e_-^4 / (\epsilon_0 \hbar)^2$, so bringing two chunks of matter together can't release tremendous amounts of energy if they were already in their respective lowest-energy states. The fact that electrons are fermions is essential: if they were bosons, then matter would not be stable in this sense.[15]

[12] Real electrons have nonzero spin. In the family of models defined in sections 7-9, the (elementary) particles all have zero spin, but since the model excludes non-Coulomb electromagnetic interactions, we can account for the electron's spin by using two species of spinless electrons.

[13] This type of model is commonly used in quantum chemistry and in solid-state physics, often using a further approximation in which each nucleus has infinite mass (doesn't move) and is localized at a point.

[14] The lower bound is negative because the model ignores the particles' rest-energies (article 15939).

[15] Both of these results are reviewed in Lieb (1976): the stability of ordinary matter, and the fact that it would be unstable if electrons were bosons. Both results are derived using models of the type described here in sections 7-9. Lieb and Seiringer (2010) gives a comprehensive review of related results.

11 The behavior of well-localized particles

Article 41522 showed that in a model with only one particle of each species, the particles' behavior can be described to a good approximation by a classical model, as long as they're not significantly entangled with each other and as long as they each remain well-localized. In that context, using the Schrödinger picture, the quantity

$$\overline{\mathbf{x}}_n \equiv \frac{\int \left|\psi(\mathbf{x}_1, ..., \mathbf{x}_N, t)\right|^2 \mathbf{x}}{\int \left|\psi(\mathbf{x}_1, ..., \mathbf{x}_N, t)\right|^2} \tag{20}$$

provides a convenient way to keep track of the nth particle's location – the time-dependent location of a small region R in which the probability of the outcome "the particle is in R at time t" would be practically equal to 1 if that observable were measured. The quantity (20) is meaningful in a context where each particle – the only one of its species – has its own associated detection observables, as in article 41522.[16]

A similar result also applies when multiple particles of each fermion/boson species are involved, with one caveat: using the quantities (20) to keep track of the particles' locations no longer makes sense. It no longer makes sense because of the (anti)symmetry conditions introduced in section 2. Each species has its own associated set of detection observables, but the individual particles do not. Observables are associated with regions of spacetime, not with individual particles. A similar result still holds, though, because as long as each particle remains well-localized, we can still describe its motion in terms of a small region R that moves with time so that a probability of detecting one particle in that region (if that observable were measured) remains close to 1.[17]

[16] That article emphasized the idea of associating observables with regions of spacetime instead of associating them with particles, but since the models constructed there had only one particle of each species, the two perspectives were interchangeable. In a model with two or more particles of the same species, the two concepts are no longer interchangeable.

[17] We can solve the Schrödinger equation in $\mathcal{H}_{\text{aux}}$ first and then (anti)symmetrize the solution, because the projection onto the properly (anti)symmetrized subspace of $\mathcal{H}_{\text{aux}}$ commutes with the time-evolution operators $U(t) = \exp(-iHt)$.

12 References

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13 References in this series

Article **15939** (https://cphysics.org/article/15939):
"Field Operators for Nonrelativistic Fermions and Bosons" (version 2023-11-12)

Article **20554** (https://cphysics.org/article/20554):
"The Free Nonrelativistic Quantum Particle" (version 2023-11-12)

Article **41522** (https://cphysics.org/article/41522):
"Nonrelativistic Interacting Particles, All of Different Species" (version 2023-11-12)

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