Spinor Products and Lorentz Symmetry

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Abstract The Dirac equation is an equation of motion for a free spinor field. Article 21794 shows that the Dirac equation has symmetries corresponding to every spacetime isometry in the identity component of the Lorentz group. This article constructs two-spinor products that are invariant under those symmetry transformations, generalized to an arbitrary number of dimensions of spacetime and arbitrary signature. Some other properties of these two-spinor products are also explored.

Contents

1 Introduction

The Clifford algebra[1] Cliff(p, m) is an associative algebra generated by mutually anticommuting vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{p+m}$, each satisfying $\mathbf{e}_a^2 = \pm 1$, with p plus signs and m minus signs. The pair of integers (p, m) will be called the **signature**. A boldface letter like $\mathbf{v}$ will denote a linear combination of the $\mathbf{e}_a$s with real coefficients and will be called a **spacetime vector**, even though this article considers arbitrary signatures, not just lorentzian signatures.[2]

Let γ denote an irreducible representation of Cliff(p, m) on a complex vector space W. The matrix representing any $A \in \text{Cliff}(p, m)$ will be denoted $\gamma(A)$, and the abbreviation $\gamma_a \equiv \gamma(\mathbf{e}_a)$ will be used for the matrix representing a basis vector $\mathbf{e}_a$. Each γ_a is called a **Dirac matrix**, and an element $\psi \in W$ of the complex vector space on which these matrices act is called a **Dirac spinor**.[3]

The **Dirac equation** is an equation of motion for a free spinor field $\psi(x)$. Article 21794 showed that the Dirac equation in flat spacetime has symmetries corresponding to every Lorentz transformation that may be expressed as a composition of two reflections. Those symmetry transformations constitute the **spin group**.[4] Each of these transformations has the form

$$\psi(x) \to \gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)\psi(\bar{x}), \tag{1}$$

where the isometry $x \to \bar{x}$ is composed of reflections along the two spacetime directions $\mathbf{r}_1$ and $\mathbf{r}_2$. In quantum field theory (QFT), the equations of motion are only part of a given model's structure, so symmetries of the Dirac equation may or may not be symmetries of the full quantum model. This article focuses on the **identity component** of the spin group – the part of the spin group that is continuously

[1] Article 03910 introduces Clifford algebra.

[2] Article 21794 focused on **lorentzian** signatures (either $p = 1$ or $m = 1$).

[3] In quantum field theory, the components of a spinor field are operators on a Hilbert space, and they don't commute with each other. The manipulations in this article don't use the commutation relations, though, so here the components of a spinor may be viewed as ordinary complex numbers.

[4] Article 08264

connected to the identity transformation[5] – because the transformations in that subgroup are always symmetries of the full quantum model of a free Dirac spinor field in flat spacetime.

Instead of considering spinor *fields*, this article considers only spinors – individual elements of W, not parameterized by the spacetime coordinates x. Motivated by the context reviewed in the previous paragraph, this article constructs two-spinor products that are invariant under transformations of the form

$$\psi \to \gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)\psi \qquad\qquad \text{for all } \psi \in W, \tag{2}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are spacetime vectors with $\mathbf{r}_1^2 = \mathbf{r}_2^2 \in \{1, -1\}$ so that the transformation belongs to the identity component of the spin group.[6] This article writes $\langle\psi,\, \phi\rangle$ for a product of two spinors ψ and ϕ with particular properties.[7] The key property is[8]

$$\langle B\psi,\, B\phi\rangle = \langle\psi,\, \phi\rangle \qquad\qquad \text{for all } \psi, \phi \in W, \tag{3}$$

for all $B = \gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)$ with $\mathbf{r}_1^2 = \mathbf{r}_2^2 \in \{1, -1\}$. The two-spinor product is also required to be either **bilinear**, which means

$$\langle z\psi,\, \phi\rangle = z\langle\psi,\, \phi\rangle \quad \text{and} \quad \langle\psi,\, z\phi\rangle = z\langle\psi,\, \phi\rangle \qquad \text{for all complex numbers } z,$$

or **sesquilinear**, which (in this article) means

$$\langle z\psi,\, \phi\rangle = z^*\langle\psi,\, \phi\rangle \quad \text{and} \quad \langle\psi,\, z\phi\rangle = z\langle\psi,\, \phi\rangle \qquad \text{for all complex numbers } z,$$

where z^* is the complex conjugate of z.

[5]Sometimes the name *spin group* refers only to the identity component, as in Varadarajan (2004), section 5.4, page 193.

[6]The full spin group (as defined in article 08264) includes transformations for which $\mathbf{r}_1^2$ and $\mathbf{r}_2^2$ have opposite signs, but those transformations are not continuously connected to the identity transformation.

[7]The notation $\overline{\psi}\phi$ is more common in the physics literature, but the notation $\langle\psi,\, \phi\rangle$ has advantages when studying transformations. This use of angle-brackets should not be confused with the use of angle-brackets to denote the inner product of two state-vectors in quantum theory. Spinors are not state-vectors, at least not the way they're normally used in QFT. In QFT, each component of a spinor field is an operator that acts on state-vectors.

[8]This property is equivalent to $\langle B\psi,\, \phi\rangle = \langle\psi,\, B^{\text{rev}}\phi\rangle$, where $B^{\text{rev}} = \gamma(\mathbf{r}_2)\gamma(\mathbf{r}_1)$. This is proposition 4.1 in Deligne (1999) and lemma 6.5.1 in Varadarajan (2004), with the understanding that those authors define the *spin group* to be only the identity component of what article 08264 calls the *spin group*.

2 A sufficient condition

If the product $\langle\psi,\,\phi\rangle$ satisfies

$$\langle\gamma_a\psi,\,\phi\rangle = \sigma\langle\psi,\,\gamma_a\phi\rangle \qquad\qquad \text{for all } a \tag{4}$$

with $\sigma\in\{1,-1\}$, then it automatically has the property (3): it is automatically invariant under the identity component of the spin group. To prove this, use

$$\begin{aligned}\big\langle\gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)\psi,\,\phi\big\rangle &= \sigma\big\langle\gamma(\mathbf{r}_2)\psi,\,\gamma(\mathbf{r}_1)\phi\big\rangle\\ &= \sigma^2\big\langle\psi,\,\gamma(\mathbf{r}_2)\gamma(\mathbf{r}_1)\phi\big\rangle\\ &= \big\langle\psi,\,\gamma(\mathbf{r}_2)\gamma(\mathbf{r}_1)\phi\big\rangle.\end{aligned} \tag{5}$$

Now set

$$\phi = \gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)\phi'$$

on both sides of (5) and use $\mathbf{r}_1^2=\mathbf{r}_2^2\in\{1,-1\}$ to get

$$\big\langle\gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)\psi,\,\gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)\phi'\big\rangle = \big\langle\psi,\,\gamma(\mathbf{r}_2\mathbf{r}_1\mathbf{r}_1\mathbf{r}_2)\phi'\big\rangle = \langle\psi,\,\phi'\rangle. \tag{6}$$

This shows that (4) implies (3).

The requirement for $\langle\psi,\,\phi\rangle$ to be either bilinear or sesquilinear will be enforced by setting

$$\langle\psi,\,\phi\rangle = \begin{cases}\psi^T M\phi & \text{if bilinear,}\\ \psi^\dagger M\phi & \text{if sesquilinear.}\end{cases} \tag{7}$$

The right-hand sides are written using matrix notation: M is a square matrix, each spinor ψ,ϕ is a matrix with one column, ψ^T is the transpose of ψ, and $\psi^\dagger$ is the hermitian conjugate of ψ (the complex conjugate of ψ^T).

Given (7), the challenge is to find a matrix M that solves the constraint (4). The solution depends on the signature (p,m), on the sign σ, and on whether the bilinear or sesquilinear option is imposed. The first goal in this article is to determine when solutions exist. In cases where they do exist, solutions will be constructed, and their properties will be explored.

3 Non-existence of solutions in some cases

When $p+m$ is even, a matrix M for which (7) satisfies (4) always exists. That will be demonstrated later. This section shows that such a matrix M doesn't always exist when $p+m$ is odd.

Suppose that (4) holds with the same sign σ for all γ_a and all ψ, ϕ, but don't assume anything else about $\langle\psi,\,\phi\rangle$ yet. When $d \equiv p+m$ is odd, an irreducible matrix representation has the property that each Dirac matrix is proportional to the product of all of the others,[9] so we can write

$$\gamma_d = \epsilon\gamma_1\gamma_2\cdots\gamma_{d-1} \tag{8}$$

with[10]

$$\epsilon \equiv \begin{cases} \pm 1 & \text{if } d=4n+1 \text{ and } p \text{ is odd, or if } d=4n+3 \text{ and } m \text{ is odd,} \\ \pm i & \text{if } d=4n+1 \text{ and } m \text{ is odd, or if } d=4n+3 \text{ and } p \text{ is odd,} \end{cases} \tag{9}$$

where n is any nonnegative integer. Reversing the order of the factors in the product (8) may or may not change the overall sign, depending on the value of d:

$$\gamma_{d-1}\cdots\gamma_2\gamma_1 = \begin{cases} \gamma_1\gamma_2\cdots\gamma_{d-1} & \text{if } d=4n+1, \\ -\gamma_1\gamma_2\cdots\gamma_{d-1} & \text{if } d=4n+3. \end{cases} \tag{10}$$

Now calculate

$$\begin{aligned} \langle\gamma_d\psi,\,\phi\rangle &= \langle\epsilon\gamma_1\gamma_2\cdots\gamma_{d-1}\psi,\,\phi\rangle && \text{(equation (8))} \\ &= \sigma^{d-1}\langle\epsilon\psi,\,\gamma_{d-1}\cdots\gamma_2\gamma_1\phi\rangle && \text{(equation (4))} \\ &= \langle\epsilon\psi,\,\gamma_{d-1}\cdots\gamma_2\gamma_1\phi\rangle && (d \text{ is odd, so } \sigma^{d-1}=1) \\ &= \begin{cases} \langle\epsilon\psi,\,\epsilon^{-1}\gamma_d\phi\rangle & \text{if } d=4n+1 \\ -\langle\epsilon\psi,\,\epsilon^{-1}\gamma_d\phi\rangle & \text{if } d=4n+3. \end{cases} && \text{(equations (8) and (10))} \end{aligned} \tag{11}$$

[9] Article 86175
[10] Article 87696

To continue, consider the bilinear and sesquilinear options separately. If $\langle\psi,\,\phi\rangle$ is bilinear, then the factors ϵ and ϵ^{-1} on the last line of (11) cancel each other, leaving

$$\langle\gamma_d\psi,\,\phi\rangle = \begin{cases} \langle\psi,\,\gamma_d\phi\rangle & \text{if } d = 4n+1 \\ -\langle\psi,\,\gamma_d\phi\rangle & \text{if } d = 4n+3 \end{cases} \qquad \text{if bilinear.} \tag{12}$$

If $\langle\psi,\,\phi\rangle$ is sesquilinear, then

$$\langle\gamma_d\psi,\,\phi\rangle = \begin{cases} \epsilon^2\langle\psi,\,\gamma_d\phi\rangle & \text{if } d = 4n+1 \\ -\epsilon^2\langle\psi,\,\gamma_d\phi\rangle & \text{if } d = 4n+3, \end{cases} \qquad \text{if sesquilinear,}$$

and then using (9) to evaluate ϵ^2 gives

$$\langle\gamma_d\psi,\,\phi\rangle = \begin{cases} \langle\psi,\,\gamma_d\phi\rangle & \text{if } p \text{ is odd} \\ -\langle\psi,\,\gamma_d\phi\rangle & \text{if } m \text{ is odd} \end{cases} \qquad \text{if sesquilinear.} \tag{13}$$

We derived these results using the condition (4) for $a \neq d$, but the condition (4) is supposed to hold for all a, including $a = d$. Compare that requirement to the results (12) and (13) to deduce that a matrix M satisfying (4) and (7) does not exist for any of these four combinations:[11,12]

- bilinear, $d = 4n+1$, $\sigma = -1$,
- bilinear, $d = 4n+3$, $\sigma = 1$,
- sesquilinear, p odd and m even, $\sigma = -1$,
- sesquilinear, p even and m odd, $\sigma = 1$.

[11]This agrees with Hamilton (2017), tables 6.7 and 6.8.

[12]This implies that no solution (neither bilinear nor sesquilinear) exists for $\sigma = -1$ when $d = 4n+1$ and p is odd, and it also implies that no solution exists for $\sigma = 1$ when $d = 4n+3$ and p is even. At first, this might seem to contradict theorem 13.17 in Harvey (1990), which asserts the existence of products $\langle\psi,\,\phi\rangle$ for *both* signs σ for arbitrary (p, m). This apparent contradiction is resolved by the fact that in that theorem, ψ and ϕ are elements of what that author calls the *space of pinors*, which is not always the same as the space of Dirac spinors – even though much of the literature uses the name *pinor* for what I'm calling a *Dirac spinor*.

4 Summary of existence results

First consider the bilinear case. Then a two-spinor product of the form (7) will satisfy the condition (4) if the matrix M satisfies

$$\gamma_a^T M = \sigma M \gamma_a$$

with the same sign σ for all a. Article 87696 shows that in an irreducible representation of Cliff(p,m), the (non)existence of such a matrix M depends on the value of $d \equiv p + m$ as shown here:

$p+m$	σ	exists?
$4n$	$+1$	yes
$4n+1$	$+1$	yes
$4n+2$	$+1$	yes
$4n+3$	$+1$	**no**

$p+m$	σ	exists?
$4n$	-1	yes
$4n+1$	-1	**no**
$4n+2$	-1	yes
$4n+3$	-1	yes

Now consider the sesquilinear case. Then a two-spinor product of the form (7) will satisfy the condition (4) if the matrix M satisfies

$$\gamma_a^\dagger M = \sigma M \gamma_a$$

with the same sign σ for all a. Article 87696 shows that in an irreducible representation of Cliff(p,m), the (non)existence of such a matrix M depends on the values of p and m as shown here:

$p+m$	m	σ	exists?
even	even	$+1$	yes
even	odd	$+1$	yes
odd	even	$+1$	yes
odd	odd	$+1$	**no**

$p+m$	m	σ	exists?
even	even	-1	yes
even	odd	-1	yes
odd	even	-1	**no**
odd	odd	-1	yes

In these four tables, the cases marked "no" are precisely the cases that were ruled out in section 3. In all other cases, a matrix M with the required property does exist. Article 87696 uses a irreducible representation with standard properties (each Dirac matrix is either symmetric or antisymmetric and is also either hermitian or antihermitian) to construct them explicitly.

5 Building a vector from spinors

For spinors in an irreducible representation of Cliff(p, m), the analysis in sections 2-4 showed that bilinear and sesquilinear two-spinor products $\langle\psi,\,\phi\rangle$ both exist with the property[13]

$$\langle B\psi,\, B\phi\rangle = \langle\psi,\,\phi\rangle \qquad\qquad \text{for all } \psi,\phi \tag{14}$$

whenever

$$B = \gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2) \qquad\qquad \mathbf{r}_1^2 = \mathbf{r}_2^2 \in \{1, -1\}. \tag{15}$$

For any two-spinor product $\langle\cdot,\,\cdot\rangle$ with this property, the quantity

$$\langle\psi,\,\gamma(\mathbf{v})\phi\rangle$$

is affected by the transformation

$$\psi,\phi \to B\psi, B\phi \tag{16}$$

in a way that matches how the vector $\mathbf{v}$ would be affected by the same composition of reflections, namely $\mathbf{v} \to \mathbf{r}_2\mathbf{r}_1\mathbf{v}\mathbf{r}_1\mathbf{r}_2$. To demonstrate this, use the identities

$$B^{-1} = \gamma(\mathbf{r}_2)\gamma(\mathbf{r}_1) \tag{17}$$

and

$$B^{-1}\gamma(\mathbf{v})B = \gamma(\mathbf{r}_2\mathbf{r}_1\mathbf{v}\mathbf{r}_1\mathbf{r}_2) \tag{18}$$

to get

$$\begin{aligned}
\big\langle\psi,\,\gamma(\mathbf{v})\phi\big\rangle &\to \big\langle B\psi,\,\gamma(\mathbf{v})B\phi\big\rangle && \text{(using (16))}\\
&= \big\langle B\psi,\,BB^{-1}\gamma(\mathbf{v})B\phi\big\rangle && \text{(using (15) and (17))}\\
&= \big\langle\psi,\,B^{-1}\gamma(\mathbf{v})B\phi\big\rangle && \text{(using (14))}\\
&= \big\langle\psi,\,\gamma(\mathbf{r}_2\mathbf{r}_1\mathbf{v}\mathbf{r}_1\mathbf{r}_2)\phi\big\rangle. && \text{(using (18))}
\end{aligned}$$

[13]This works because the condition (4) is always satisfied for at least one sign σ and because the sign σ cancels when an even number of reflections is involved.

6 Reflections

Section 1 reviewed the motivation for considering transformations of the form (16) with B given by (15): those transformations correspond to symmetries of the Dirac equation. When $d \equiv p + m$ is even, the Dirac equation also has symmetries of the form[14] $\psi(x) \to \gamma(\mathbf{r})\omega\psi(\bar{x})$ corresponding to an individual reflection along the direction $\mathbf{r}$, where ω denotes the product of all Dirac matrices. With that motive, this section determines how a transformation of the form

$$\psi, \phi \to M\psi,\, M\phi$$

affects the quantity

$$\langle \psi,\, \gamma(\mathbf{v})\phi \rangle$$

when $M \propto \gamma(\mathbf{r})\omega$. The question is whether that effect matches the reflection[15]

$$\mathbf{v} \to \frac{-\mathbf{r}\mathbf{v}\mathbf{r}}{\mathbf{r}^2}. \tag{19}$$

The Dirac equation doesn't have any linear symmetries of the form $\psi(x) \to M\psi(\bar{x})$ corresponding to individual reflections when d is odd,[14] so this section only considers even values of d.

Define ω to be the product of all Dirac matrices in some given order, and let ω^{rev} denote the product of all Dirac matrices in the reverse order. Equation (4) implies

$$\langle \omega\psi,\, \phi \rangle = \langle \psi,\, \omega^{\text{rev}}\phi \rangle.$$

The signs σ cancel because d is even. Let

$$M = \epsilon'(\mathbf{r})\gamma(\mathbf{r})\omega$$

for some complex number $\epsilon'(\mathbf{r})$ that may depend on $\mathbf{r}$. (The matrix M also depends on $\mathbf{r}$, of course, but this dependence will be left implicit to streamline the equations.)

[14] Article 21794

[15] Article 08264

Then

$$\begin{aligned}\big\langle M\psi,\,\gamma(\mathbf{v})M\phi\big\rangle &= \big\langle \epsilon'(\mathbf{r})\psi,\,\omega^{\text{rev}}\gamma(\mathbf{r})\gamma(\mathbf{v})M\phi\big\rangle \\ &= \big\langle \epsilon'(\mathbf{r})\psi,\,\epsilon'(\mathbf{r})\omega^{\text{rev}}\gamma(\mathbf{rvr})\omega\phi\big\rangle \\ &= -\big\langle \epsilon'(\mathbf{r})\psi,\,\epsilon'(\mathbf{r})\gamma(\mathbf{rvr})\omega^{\text{rev}}\omega\phi\big\rangle \\ &= -(-1)^m\big\langle \epsilon'(\mathbf{r})\psi,\,\epsilon'(\mathbf{r})\gamma(\mathbf{rvr})\phi\big\rangle. \end{aligned} \tag{20}$$

The third step uses the fact that ω anticommutes with every Dirac matrix when d is even, and the m on the last line is the number of basis vectors whose square is negative (section 1).

When $\langle\cdot,\,\cdot\rangle$ is bilinear, the relationship (20) becomes

$$\big\langle M\psi,\,\gamma(\mathbf{v})M\phi\big\rangle = -(-1)^m\big(\epsilon'(\mathbf{r})\big)^2\big\langle\psi,\,\gamma(\mathbf{rvr})\phi\big\rangle.$$

To match (19), we must choose the complex coefficient $\epsilon'(\mathbf{r})$ so that

$$(-1)^m\big(\epsilon'(\mathbf{r})\big)^2 = \frac{1}{\mathbf{r}^2}.$$

This is clearly always possible.

When $\langle\cdot,\,\cdot\rangle$ is sesquilinear, the relationship (20) becomes

$$\big\langle M\psi,\,\gamma(\mathbf{v})M\phi\big\rangle = -(-1)^m\big|\epsilon'(\mathbf{r})\big|^2\big\langle\psi,\,\gamma(\mathbf{rvr})\phi\big\rangle.$$

To match (19), we would need

$$(-1)^m\big|\epsilon'(\mathbf{r})\big|^2 = \frac{1}{\mathbf{r}^2},$$

but now we don't have the ability to enforce this because the quantity $\big|\epsilon'(\mathbf{r})\big|^2$ is positive. The effect (19) will be matched only if $\mathbf{r}^2 = (-1)^m$. This is a condition on $\mathbf{r}$, so it works for some reflections, but not for all reflections.[16] When the signature is lorentzian, the premise that d is even implies that m is odd, so in that case this works if and only if $\mathbf{r}^2 = -1$.

[16] Here, we are only considering *linear* symmetries. In quantum field theory, we can also consider *antilinear* symmetries. Article 21794 shows that an antilinear reflection symmetry may exist even a linear reflection symmetry (with the reflection along the same direction) doesn't.

7 Chiral spinors

When $d \equiv p + m$ is even, each Dirac spinor ψ consists of two **chiral spinors** ψ_L and ψ_R that are not mixed with each other by transformations in the spin group. Some of the combinations

$$
\begin{array}{ll}
\langle \psi_L,\, \phi_L \rangle & \langle \psi_L,\, \gamma_a \phi_L \rangle \\
\langle \psi_L,\, \phi_R \rangle & \langle \psi_L,\, \gamma_a \phi_R \rangle \\
\langle \psi_R,\, \phi_L \rangle & \langle \psi_R,\, \gamma_a \phi_L \rangle \\
\langle \psi_R,\, \phi_R \rangle & \langle \psi_R,\, \gamma_a \phi_R \rangle
\end{array}
$$

are zero, depending on (p, m) and on whether $\langle \cdot, \cdot \rangle$ is bilinear or sesquilinear. This section determines which ones are zero.

Suppose that d is even. As in section 6, define ω to be the product of all Dirac matrices in some given order, and define[17]

$$
\Gamma \equiv \epsilon\omega \qquad\qquad \epsilon \equiv \begin{cases} 1 & \text{if } p - m = 4k \\ i & \text{if } p - m = 4k + 2 \end{cases}
$$

so that $\Gamma^2 = I$, where I is the identity matrix, which implies that the matrices

$$
P_\pm \equiv \frac{1 \pm \Gamma}{2}
$$

are mutually orthogonal projections.[18] Define the chiral spinors[19]

$$
\psi_L \equiv P_+\psi \qquad\qquad \psi_R \equiv P_-\psi.
$$

The identity $P_+ + P_- = I$ gives $\psi = \psi_L + \psi_R$. The fact that Γ anticommutes with every Dirac matrix (when d is even) implies that the chiral spinors $P_\pm\psi$ are not mixed with each other by transformations $\psi \to B\psi$ with B given by (15).

[17] The nonnegative integers p and m are defined as in section 1, and k is an arbitrary integer (positive or negative).

[18] This is also true when d is odd, but then one of them is I and the other is zero.

[19] Here, each chiral spinor is treated as a Dirac spinor (element of W) restricted to one of the two subspaces $P_\pm W$.

When $\langle\cdot,\,\cdot\rangle$ is bilinear,

$$\langle\Gamma\psi,\,\phi\rangle = \langle\psi,\,\Gamma^{\text{rev}}\phi\rangle = \begin{cases} \langle\psi,\,\Gamma\phi\rangle & \text{if } d = 4n, \\ -\langle\psi,\,\Gamma\phi\rangle & \text{if } d = 4n+2. \end{cases}$$

Combine this with $P_+P_- = P_-P_+ = 0$ to infer

$$\begin{aligned} &\langle\psi_L,\,\phi_R\rangle = \langle\psi_R,\,\phi_L\rangle = \langle\psi_L,\,\gamma_a\phi_L\rangle = \langle\psi_R,\,\gamma_a\phi_R\rangle = 0 \qquad \text{when } d = 4n \\ &\langle\psi_L,\,\phi_L\rangle = \langle\psi_R,\,\phi_R\rangle = \langle\psi_L,\,\gamma_a\phi_R\rangle = \langle\psi_R,\,\gamma_a\phi_L\rangle = 0 \qquad \text{when } d = 4n+2 \end{aligned}$$

when $\langle\cdot,\,\cdot\rangle$ is bilinear.

When $\langle\cdot,\,\cdot\rangle$ is sesquilinear,

$$\langle\Gamma\psi,\,\phi\rangle = \langle\psi,\,\epsilon^*\omega^{\text{rev}}\phi\rangle = (-1)^{(p-m)/2}(-1)^{(p+m)/2}\langle\psi,\,\Gamma\phi\rangle = (-1)^m\langle\psi,\,\Gamma\phi\rangle.$$

The signs in the third expression come from the relationships $\epsilon^* = (-1)^{(p-m)/2}\epsilon$ and $\omega^{\text{rev}} = (-1)^{(p+m)/2}\omega$. Use this to infer

$$\begin{aligned} &\langle\psi_L,\,\phi_R\rangle = \langle\psi_R,\,\phi_L\rangle = \langle\psi_L,\,\gamma_a\phi_L\rangle = \langle\psi_R,\,\gamma_a\phi_R\rangle = 0 \qquad \text{when } m \text{ is even} \\ &\langle\psi_L,\,\phi_L\rangle = \langle\psi_R,\,\phi_R\rangle = \langle\psi_L,\,\gamma_a\phi_R\rangle = \langle\psi_R,\,\gamma_a\phi_L\rangle = 0 \qquad \text{when } m \text{ is odd} \end{aligned}$$

when $\langle\cdot,\,\cdot\rangle$ is sesquilinear. When the signature is lorentzian, the premise that d is even implies that m is odd, so the last line is the relevant one for most applications in QFT.

8 References

Deligne, 1999. "Notes on Spinors." Pages 99-135 in *Quantum Fields and Strings: a Course for Mathematicians*, edited by Deligne *et al* (American Mathematical Society) https://publications.ias.edu/sites/default/files/79_NotesOnSpinors.pdf

Hamilton, 2017. *Mathematical Gauge Theory.* Springer

Harvey, 1990. *Spinors and Calibrations.* Academic Press

Varadarajan, 2004. *Supersymmetry for Mathematicians: An Introduction.* American Mathematical Society

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