The Free Quantum Electromagnetic Field in Smooth Space

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Abstract This article sketches a way of describing the quantum electromagnetic field in smooth D-dimensional space, without addressing the technical problems caused by trying to associate operators with individual points in continuous space. In the quantum model, the components of the electric and magnetic fields are operators on a Hilbert space, and they don't all commute with each other. This article uses the electric and magnetic field operators to construct operators that create photons. The relationship between photon polarizations and angular momenta in D-dimensional space is explored.

Contents

1 Introduction

This article describes a model of the free quantum electromagnetic field.[1] This is quantum electrodynamics without any matter, so the quantum electromagnetic field is the only player. To distinguish it from quantum electrodynamics with matter, the model described here will be called **quantum electromagnetism (QEM)**.

In quantum theory, observables are represented by operators on a Hilbert space. These operators typically don't commute with each other. Quantum field theory is a specialization of quantum theory in which observables are associated with regions of spacetime. We can try to associate observables with individual points in smooth spacetime, but then the commutator of two observables at the same point is not always defined. This problem can often be cured by treating space (or spacetime) as a discrete lattice instead of as a smooth manifold. Some models can even be constructed directly in continuous spacetime by smearing the observables slightly in space or time.[2] Both of those approaches work for QEM,[3,4] but they both take some extra effort.

This article does something easier: this article sketches a mathematically naïve formulation of the quantum electromagnetic field, using observables associated with points in smooth space. It's mathematically naïve because it ignores the problems caused by trying to associate non-commuting operators with individual points in smooth space. The rest of this article uses the words *define* and *construct* in this naïve sense. This easier approach still has merit, because many of its results remain essentially valid (with caveats) in mathematically legitimate formulations.

[1] Calling a field theory *free* means that the equations of motion are linear. For the electromagnetic field, *free* implies that electrically charged objects and currents are absent.

[2] Article 44563 describes one example.

[3] Article 51376 constructs QEM properly by treating space as a lattice.

[4] The lattice approach also works for quantum electrodynamics when interactions with quantum matter (like electrons) are included.

2 Outline

- Sections 3-4 will introduce some conventions.
- Sections 5-8 will review classical electrodynamics with a spatial metric that is not necessarily flat. This extra generality will help clarify the structure of the quantum model.
- Sections 9-13 will define the model (QEM). Its basic observables are the electric and magnetic fields, now represented as operators on a Hilbert space that don't all commute with each other.
- Sections 14-17 will show that the electric and magnetic field operators satisfy Maxwell's equations.
- Sections 18-23 will specialize to a flat but not-necessarily-diagonal spatial metric, construct the operators that generate translations in space, and relate the translation operators to the total momentum operators. This relationship involves the not-necessarily-diagonal spatial metric.
- Sections 24-26 will use the electric and magnetic field operators to introduce the concept of a *photon* and its polarization. Issues related to measurement will not be addressed here, because article 30983 already addresses that for particles associated with scalar fields, and similar principles apply to photons.
- Section 27 will mention how the model accounts for ordinary electromagnetic waves. Again, issues related to measurement will not be addressed here, because article 22792 already does that for particles associated with scalar fields, and similar principles apply to photons.
- After specializing the spatial metric to be proportional to the identity matrix so that rotation symmetry is manifest, sections 28-33 will explore a photon's angular momentum and its relationship to polarization.

3 Units conventions

This article uses a system of units in which the permittivity and permeability of free space (usually denoted ϵ_0 and μ_0) are both equal to 1, so the speed of light is also equal to 1.[5] Explicit factors of Planck's constant $\hbar$ will be retained.

This article doesn't include any electrically charged matter, but it is a step toward models that do. With that in mind, let q denote the magnitude of the smallest electric charge that we would eventually want to accommodate. This article uses a modified convention in which a factor of q is included in the definitions of the electric and magnetic fields. This convention is common in quantum field theory when we don't need to expand things in powers of q.[6] In this convention, the relationship between the total energy and the fields in flat spacetime with D spatial dimensions is[7]

$$\text{energy} = \int d^D x \, \frac{\mathbf{E}^2 + \mathbf{B}^2}{2q^2}. \tag{1}$$

Here's a summary of units in this convention, using $[X]$ to denote the units of the quantity X, and using M and L to denote units of mass length:[5,8]

$$[\mathbf{E}] = [\mathbf{B}] = M/L \qquad [q^2] = ML^{D-2} \qquad [\hbar] = ML$$
$$[\text{energy}] = [\text{momentum}] = M.$$

The case $D = 3$ is most important for physics, and it is also mathematically special: a dimensionless combination of q and $\hbar$ can be formed if and only if $D = 3$.

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[5]Article 00669

[6]With this convention, the coefficient of the kinetic term in the lagrangian is $1/4q^2$ (`https://physics.stackexchange.com/questions/244162/`). One example of a source that uses this convention is Tong (2018), specifically chapter 2 (Yang-Mills theory), chapter 7 (quantum field theory in two dimensions), and chapter 8 (quantum field theory in three dimensions).

[7]This equation is an abbreviation for equation (24). Article 78463 introduces this equation using a different system of units.

[8]In this convention, magnetic flux – the integral of the magnetic field over a two-dimensional surface – has the same units as $\hbar$. This will be convenient in article 51376.

4 Index and coordinate conventions

The number of dimensions of space will be denoted D. Index conventions:

- An index from the middle of the alphabet, namely i, j, k, ℓ, m, takes values in $\{1, 2, ..., D\}$.
- An index from the beginning of the alphabet, namely a, b, c, d, takes values in $\{0, 1, 2, ..., D\}$. The index 0 indicates the time coordinate.

The standard summation convention will be used: a sum over a spacetime or spatial index is implied whenever that index occurs both as a superscript and as a subscript in the same term. Examples:

- A sum over $a \in \{0, 1, 2, ..., D\}$ is implied in the expression $X^{ab}Y_{ac}$.
- A sum over $j \in \{1, 2, ..., D\}$ is implied in the expression $X^{jk}Y_{j\ell}$.

Each superscript in these expressions is an index, not an exponent. The coordinates of a point $\mathbf{x}$ in space will be denoted $(x^1, x^2, ..., x^D)$. The partial derivative with respect to x^k will be denoted ∂_k. The time coordinate will be denoted t, and the partial derivative with respect to t will be denoted either by ∂_0 or by an overhead dot ($\dot{\phi} \equiv \partial_0\phi \equiv d\phi/dt$).

The standard notation

$$\delta^j_k = \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

will also be used.

5 Generalized spacetime metric

Article 31738 introduces the classical electromagnetic field in flat spacetime, using a coordinate system in which the spacetime metric has the form:[9]

$$g_{00} = 1 \qquad g_{0k} = g_{k0} = 0 \qquad g_{jk} = \begin{cases} -1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The first part of this article will allow a more general metric. The conditions[10]

$$g_{00} = 1 \qquad g_{0k} = g_{k0} = 0 \qquad |\det g| = 1 \qquad \partial_0 g_{ab} = 0 \tag{4}$$

will be assumed, but no further conditions will be imposed. The spatial part is not necessarily diagonal, and its components g_{jk} are not necessarily independent of the spatial coordinates. This extra generality will help clarify the reasons for some details that involve the commutation relations.[11]

[9] This article and article 31738 both use the mostly-minus convention for the metric tensor.

[10] Recall (section 4) that $k \in \{1, 2, ..., D\}$ and $a, b \in \{0, 1, 2, ..., D\}$.

[11] Extra generality often helps clarify the reasons for some details that might otherwise seem haphazard. That's one of the reasons several articles in this series – including this one – allow an arbitrary number of dimensions.

6 Components of the electromagnetic field

In D-dimensional space, the electromagnetic field may be described using the **field strength tensor**[12] whose components F_{ab} are functions of space and time.[13] The components of F_{ab} satisfy the antisymmetry condition

$$F_{ab} = -F_{ba}. \tag{5}$$

The field strength tensor includes:[13]

- the **magnetic field**, whose components are $B_{jk} \equiv F_{jk}$,[14]
- the **electric field**, whose components are $E_k \equiv F_{k0}$.

With one exception (the quantity (2)), the metric is always used to raise or lower an index. Examples:

$$\begin{aligned} F^{ab} &\equiv g^{ac}g^{bd}F_{cd} && \text{(implied sums over } c,d \in \{0,1,2,...,D\}\text{)}, \\ F^{jk} &\equiv g^{j\ell}g^{km}F_{\ell m} && \text{(implied sums over } \ell,m \in \{1,2,...,D\}\text{)}, \\ F^{j0} &\equiv g^{jk}F_{k0} && \text{(implied sum over } k \in \{1,2,...,D\}\text{)}, \\ E^{j} &\equiv g^{jk}E_{k} && \text{(implied sum over } k \in \{1,2,...,D\}\text{)}. \end{aligned} \tag{6}$$

The definition $E_k \equiv F_{k0}$ implies $E^k = F^{k0}$. This article uses both E^k and E_k, whichever is more natural in each context.[15]

[12] Article 31738

[13] Remember the index conventions that were established in section 4.

[14] Two subscripts are used for the magnetic field to facilitate an arbitrary number D of spatial dimensions. Article 31738 explains this in more detail.

[15] When the metric is given by (3), the relationship (6) reduces to $E^k = -E_k$. (Article 31738 uses only E_k.) Using the mostly-plus convention for the metric would eliminate this inconvenient sign, but it would introduce inconvenient signs elsewhere. The best option is to be comfortable with both conventions.

7 Equations of motion with a general spatial metric

This section writes the equations of motion for the free electromagnetic field with a prescribed spacetime metric of the form (4), using the action principle from classical electrodynamics to motivate the form of the equations.

The action for the free classical electromagnetic field with a prescribed spacetime metric is

$$S = \frac{-1}{4q^2} \int d^{D+1}x \ \sqrt{|\det g\,|} \ g^{ab} g^{cd} F_{ac} F_{bd} \tag{7}$$

with

$$F_{ab} \equiv \partial_a A_b - \partial_b A_a. \tag{8}$$

When the metric has the form (4), the action (7) becomes[16]

$$S = \frac{-1}{q^2} \int d^{D+1}x \ \left(\frac{1}{2}\, g^{jk} F_{0j} F_{0k} + \frac{1}{4}\, g^{jk} g^{\ell m} F_{j\ell} F_{km} \right). \tag{9}$$

With this action, the classical equations of motion $\delta S/\delta A_0 = 0$ and $\delta S/\delta A_j = 0$ are

$$\partial_j E^j = 0 \tag{10}$$

$$\partial_0 E^j = \partial_k F^{kj}. \tag{11}$$

Equation (8) implies $\partial_{[a} F_{bc]} = 0$, where square brackets denote antisymmetrization of the indices. For later convenience, this can be separated into equations that involve a time-derivative and equations that don't:

$$\partial_0 B_{jk} + \partial_j E_k - \partial_k E_j = 0 \tag{12}$$

$$\partial_{[j} B_{k\ell]} = 0. \tag{13}$$

Equations (10)-(13) are the equations of motion for the free electromagnetic field with a prescribed spacetime metric of the form (4). Even with that metric, they are still called **Maxwell's equations**.

[16] Remember the index conventions that were established in section 4.

8 Energy density and momentum density

In classical electrodynamics, the (Hilbert) stress-energy tensor is[17]

$$T^{ab} = \frac{-2}{\sqrt{|\det g|}} \frac{\delta S}{\delta g_{ab}}$$

with S given by (7). Use the identities[18]

$$\frac{\delta}{\delta g_{ab}} |\det g| = |\det g|\, g^{ab} \qquad\qquad \frac{\delta}{\delta g_{ab}} g^{cd} = -g^{ac} g^{bd}$$

to get

$$T^{ab} = \frac{1}{q^2}\left(\frac{1}{4} g^{ab} F^{cd} F_{cd} - F^{ac} F^{bd} g_{cd}\right). \tag{14}$$

When the metric is given by (4), the 00 component of (14) reduces to

$$T^{00} = \frac{1}{2q^2}\left(\frac{1}{2} F^{jk} F_{jk} - E^j E^k g_{jk}\right), \tag{15}$$

and the $0k$ components reduce to

$$T^{0k} = \frac{1}{q^2} E^j F^{k\ell} g_{j\ell}. \tag{16}$$

The quantities (15) and (16) are the energy density and momentum density, respectively, in classical electrodynamics.

[17] Article 32191

[18] Article 11475

9 Observables and equal-time commutation relations

Sections 5-8 reviewed classical electrodynamics with a generic spatial metric, without any charged objects or currents. This section begins the task of constructing the quantum version of the model.

The quantum model's basic observables are the components $F_{ab}(\mathbf{x},t)$ of the electromagnetic field.[19] If R is any finite region of space, then all other observables in R at time t are expressed in terms of these, with $\mathbf{x}\in R$.

In classical electrodynamics, these observables all commute with each other, but in the quantum model, they don't. The equal-time commutation relations are[20,21]

$$\begin{aligned}
[E^i(\mathbf{y},t), B_{jk}(\mathbf{x},t)] &= i\hbar q^2(\delta^i_j\partial_k - \delta^i_k\partial_j)\delta(\mathbf{x}-\mathbf{y}) \\
[E^j(\mathbf{y},t), E^k(\mathbf{x},t)] &= 0 \\
[B_{j',k'}(\mathbf{y},t), B_{jk}(\mathbf{x},t)] &= 0.
\end{aligned} \tag{17}$$

The commutation relations (17) resemble the corresponding relations for a massless scalar field, which are[22,23]

$$\begin{aligned}
[\dot\phi(\mathbf{y},t), \partial_j\phi(\mathbf{x},t)] &= -i\partial_j\delta(\mathbf{x}-\mathbf{y}) \\
[\dot\phi(\mathbf{y},t), \dot\phi(\mathbf{x},t)] &= 0 \\
[\partial_j\phi(\mathbf{y},t), \partial_k\phi(\mathbf{x},t)] &= 0
\end{aligned} \tag{18}$$

with $\dot\phi \equiv \partial\phi/\partial t$.

[19] When space(time) has a nontrivial topology, the model also has other observables (called **line operators**) that cannot all be expressed in terms of these. This article assumes that the topology of spacetime is trivial.

[20] When used as a subscript or superscript, $i\in\{1,2,...,D\}$. Otherwise, i is a complex number satisfying $i^2=-1$.

[21] The notation $\partial_j\delta(\mathbf{x})$ means the derivative of $\delta(\mathbf{x})$ with respect to its argument x^j. The derivative of $\delta(\mathbf{x}-\mathbf{y})$ with respect to y^j is $-\partial_j\delta(\mathbf{x}-\mathbf{y})$.

[22] This comparison refers to what article 37301 calls the **trimmed variant** of the massless scalar quantum field, in which $\dot\phi$ and $\partial\phi$ are observables but ϕ itself is not.

[23] No explicit factors of $\hbar$ are shown here because they can be absorbed into the definition of ϕ, which doesn't correspond to any real-world field anyway.

10 Time dependence

In classical electrodynamics, the time dependence of the field's components $F_{ab}(\mathbf{x},t)$ is defined by Maxwell's equations. We could do that in the quantum model, too, but then we'd need to show that the commutation relations (17) are consistent with Maxwell's equations, which takes some work.[24] This section uses a different approach: the time dependence of the field operators $F_{ab}(\mathbf{x},t)$ will be defined in a way that is clearly consistent with the commutation relations (17). Showing that the field operators respect Maxwell's equations will take some work, but that will be deferred to in sections 14-17, after the model has been defined.

The time dependence of $F_{ab}(\mathbf{x},t)$ is defined by

$$F_{ab}(\mathbf{x},t) = U(-t)F_{ab}(\mathbf{x},0)U(t) \tag{19}$$

with

$$U(t) = e^{-iHt/\hbar}, \tag{20}$$

using the hamiltonian H that will be defined in section 11. Equation (20) implies

$$U(t)U(-t) = 1. \tag{21}$$

The hamiltonian H and the observables $F_{ab}(\mathbf{x},0)$ at $t=0$ are self-adjoint operators, so equations (19)-(20) imply that $F_{ab}(\mathbf{x},t)$ is self-adjoint for all t.

The commutation relations (17) are consistent with the time dependence defined by (19)-(20). This is clear because equations (19) and (21) imply

$$[X(t),Y(t)] = [U(-t)X(0)U(t),U(-t)Y(0)U(t)] = U(-t)[X(0),Y(0)]U(t),$$

so if the commutation relations (17) hold at $t=0$, then they hold for all t.

[24]This isn't an issue in classical electrodynamics, where the commutators are all zero.

11 The hamiltonian

Equations (19)-(20) express the time dependence of the field operators in terms of a hamiltonian H. The hamiltonian is

$$H = \int d^D x \; T^{00}(\mathbf{x}, 0) + \text{constant}, \tag{22}$$

where $T^{00}(\mathbf{x}, t)$ is given by equation (15) but with the classical version of the components F_{ab} replaced by the corresponding operators. When the metric has the form (4), the operator H is both the generator of translations in time (equations (19)-(20)) and the observable corresponding to the system's total energy. One of the general principles of quantum field theory is that the spectrum of energies among all states in the Hilbert space should have a lower bound. When the model is defined properly (say by treating space as a lattice), the constant term may be chosen so that the lower bound is zero. That value of the constant term diverges in the continuous-space limit, but the constant term doesn't affect anything in this article, so we can ignore that issue here.

Equation (20) implies

$$U(-t)HU(t) = H.$$

Use this together with (21) to infer that the hamiltonian (22) is unchanged when $T^{00}(\mathbf{x}, 0)$ is replaced with $T^{00}(\mathbf{x}, t)$, so we can also write

$$H = \int d^D x \; T^{00}(\mathbf{x}, t) + \text{constant}. \tag{23}$$

When g_{jk} is given by (3), this reduces to

$$H = \int d^D x \; \frac{\sum_j E_j^2(\mathbf{x}, t) + \sum_{j<k} B_{jk}^2(\mathbf{x}, t)}{2q^2} + \text{constant}, \tag{24}$$

as previewed in equation (1).

12 The field operators as differential operators

For a massless scalar field, the commutation relations (18) at time $t = 0$ may be enforced by using a Hilbert space in which a state-vector is a complex-valued function $\Psi[s]$ of a collection of real variables $s(\mathbf{x})$. The operators $\dot{\phi}$ and $\partial\phi$ in equations (18) are represented like this:

$$\dot{\phi}(\mathbf{x})\Psi[s] = -i\frac{\partial}{\partial s(\mathbf{x})}\Psi[s] \qquad\qquad \partial\phi(\mathbf{x})\Psi[s] = \partial s(\mathbf{x})\Psi[s].$$

Similarly, the commutation relations (17) for the electric and magnetic field operators may be implemented by using a Hilbert space in which a state-vector is a complex-valued function $\Psi[a]$ of a collection of variables $a_1(\mathbf{x})$, $a_2(\mathbf{x})$, ..., $a_D(\mathbf{x})$. At time $t = 0$, the operators E^k and B_{jk} are represented as[25]

$$\begin{aligned} E^k(\mathbf{x}, 0)\Psi[a] &= -i\hbar q^2\frac{\partial}{\partial a_k(\mathbf{x})}\Psi[a] \\ B_{jk}(\mathbf{x}, 0)\Psi[a] &= \big(\partial_j a_k(\mathbf{x}) - \partial_k a_j(\mathbf{x})\big)\Psi[a]. \end{aligned} \tag{25}$$

This implies the commutation relations (17) at $t = 0$.

By analogy with classical electrodynamics, the second equation in (25) says that the variables $a_k(\mathbf{x})$ correspond to a **gauge field** or **local potential**[26] – specifically in the **temporal gauge**, so that it has only spatial components.

[25]This representation of B_{jk} can be used everywhere in space because we're assuming that space is topologically trivial. This is an application of the **Poincaré lemma** (`https://ncatlab.org/nlab/show/Poincar%C3%A9+lemma`).

[26]Article 76708

13 Gauge invariance and the inner product

Instead of allowing arbitrary functions, the Hilbert space in QEM consists only of **gauge invariant** functions – functions that satisfy the condition

$$\Psi[a + \partial\varphi] = \Psi[a]. \tag{26}$$

The replacement

$$a_k(\mathbf{x}) \to a_k(\mathbf{x}) + \partial_k\varphi(\mathbf{x}) \tag{27}$$

is called a **gauge transformation**. Observables are represented by linear operators on the Hilbert space, so applying an observable to a gauge invariant function must always give another gauge invariant function. The operators E^k and B_{jk} defined in equations (25) satisfy this condition. Multiplication by $a_k(\mathbf{x})$ on its own doesn't satisfy this condition, so this does not represent an observable.

To finish defining the Hilbert space, we need to define the inner product. Schematically, the inner product has the form

$$\langle\Psi_1|\Psi_2\rangle \sim \int [da]\ \Psi_1^*[a]\Psi_2[a].$$

The integrand is gauge invariant, so the integral would be undefined if the as were unrestricted real variables, for the same reason that the integral $\int_{-\infty}^{\infty} dx\ f(x)$ is undefined when $f(x)$ is independent of x. In a standard formulation of QEM that treats space as a lattice, with distance ϵ between neighboring points, functions in the Hilbert space depend on the variables a only through $e^{ia\epsilon/\hbar}$, so we only need to integrate over $-\pi < a\epsilon/\hbar \le \pi$ for each of the integration variables.[27] The resulting inner product is finite, even though the integrand is gauge invariant.

This article won't use the inner product, except implicitly through the assertion that the operators $E_j(\mathbf{x}, 0)$ and $B_{jk}(\mathbf{x}, 0)$ defined in equations (25) are self-adjoint.

[27] Article 51376

14 Recovering Maxwell's equations: outline

In the quantum model that was defined in sections 9-13, the electric and magnetic field operators satisfy Maxwell's equations. This will be demonstrated in sections 15-17. Here's an outline:

- Two of Maxwell's equations don't involve time derivatives of F_{ab}, namely equations (10) and (13), These are called **constraint equations**. Section 15 will show that the electric and magnetic field operators satisfy the constraint equations.

- Section 16 will show that the electric and magnetic field operators satisfy equation (11).

- Section 17 will show that the electric and magnetic field operators satisfy equation (12).

The derivation of equation (12) in section 17 uses the hamiltonian H, but in classical electrodynamics, equation (12) is an identity that follows from equation (8) without ever involving the hamiltonian – or the action S, or the metric. That presents a paradox: why isn't the derivation of (12) that trivial in the quantum model? Actually, it is that trivial in the path integral formulation of the quantum model. One way to explore the paradox is to study how the Hilbert-space formulation is derived from the path integral formulation.[28] Making sense of the path integral formulation when gauge fields are involved involves its own challenges, though, so that won't be attempted here.

[28] The path integral formulation uses the fact that L_E is quadratic in $E \sim \partial_0 A$, which is why a generalization analogous to the one in section 16 – generalizing $L_E(t)$ to a not-necessarily-quadratic function of the electric field operators – will not be considered in section 17.

15 Recovering the constraint equations (10) and (13)

This section shows that the quantum model that was defined in sections 9-13 satisfies the constraint equations (10) and (13), first for $t = 0$ and then for all t.

When $t = 0$, the constraint equation (10) is enforced by the fact that the Hilbert space uses only gauge-invariant functions $\Psi[a]$. To see how this works, expand the left-hand side of equation (26) in powers of $\partial\varphi$ to get

$$\begin{aligned}\Psi[a + \partial\varphi] &= \Psi[a] + \int d^D x\ \big(\partial_k\, \varphi(\mathbf{x})\big)\frac{\partial}{\partial a_k(\mathbf{x})}\Psi[a] + O\big((\partial\varphi)^2\big) \\ &= \Psi[a] - \int d^D x\ \varphi(\mathbf{x})\partial_k\, \frac{\partial}{\partial a_k(\mathbf{x})}\Psi[a] + O\big((\partial\varphi)^2\big),\end{aligned}$$

with implied sums over k.[29] The function $\varphi(\mathbf{x})$ is arbitrary, so using this in (26) implies

$$\partial_k\, \frac{\partial}{\partial a_k(\mathbf{x})}\Psi[a] = 0. \tag{28}$$

Using the representation of $E^k(\mathbf{x}, 0)$ given in equations (25), this may be written

$$\partial_k E^k(\mathbf{x}, 0)\, \Psi = 0, \tag{29}$$

which is the quantum model's version of **Gauss's law** (equation (10)).

When $t = 0$, the other constraint equation – equation (13) – is also enforced by the representation (25), because the quantity $b_{jk} = \partial_j a_k - \partial_k a_j$ automatically satisfies $\partial_{[i} b_{jk]} = 0$.

To show that the constraint equations hold for all t, use the fact that they have the form $X(t) = 0$, where $X(t)$ is a linear combination of the components of F_{ab} at time t. For any such $X(t)$, if the equation $X(t) = 0$ holds at $t = 0$, then equation (19) implies that it holds for all t.

[29] This is a standard extension of the summation convention in section 6.

16 Recovering equation (11)

This section shows that equation (11) is automatically satisfied in the quantum model that was defined in sections 9-13. The derivation starts by taking the time derivative of equation (19), raising the indices, and setting $a, b = k, 0$ to get

$$i\hbar\partial_0 E^k(\mathbf{x}, t) = \big[E^k(\mathbf{x}, t),\, H\big]. \tag{30}$$

One way to proceed would be to use the equal-time commutation relations (17) and equation (23) for the hamiltonian to evaluate the right-hand side of (30). That works, but this section uses a different approach that provides more insight.

The insight is in the form of a generalization. Instead of focusing on the specific equation (11), which comes from the specific action (9) in classical electrodynamics, consider a more general action of the form

$$S = \int dt\ \big(L_E(t) + L_B(t)\big)$$

with

$$L_E(t) \equiv \frac{-1}{2q^2}\int d^D x\ g_{jk}E^j(\mathbf{x}, t)E^k(\mathbf{x}, t) \tag{31}$$

and where $L_B(t)$ depends only on A_k and its spatial derivatives at time t, with no time-derivatives. The original action (9) has this form with a specific L_B, but the derivation here will only assume that L_B is invariant under time-independent gauge transformations. With this generalization, the hamiltonian (23) becomes

$$H = L_E(t) - L_B(t),$$

and the equation of motion (11) becomes

$$\partial_0 E^j(\mathbf{x}, t) = Y(\mathbf{x}, t) \tag{32}$$

with

$$Y(\mathbf{x}, t) \equiv q^2 \frac{\partial}{\partial A_j(\mathbf{x}, t)} L_B(t). \tag{33}$$

Now the goal is to show that

$$\big[E^j(\mathbf{x},t), H\big] = i\hbar Y(\mathbf{x},t), \tag{34}$$

because using (34) in (30) gives (32).

Suppose that $L_B(t)$ is invariant under time-independent gauge transformations. Then $L_B(0)$ can be promoted to an operator on the Hilbert space by treating each $A_k(\mathbf{x},0)$ as multiplication by $a_k(\mathbf{x})$. Multiplication by $a_k(\mathbf{x})$ by itself doesn't define an operator on the Hilbert space, but multiplication by a gauge-invariant function of these variables does. When $L_B(0)$ is treated that way, the representation (25) combined with the definition (33) immediately shows that (34) holds at $t = 0$.

Now we just need to show that if (34) holds at $t = 0$, then it holds for all t. That follows from this more general result: if $X(t)$ and $Y(t)$ are any two quantities whose time dependence is defined as in equation (19), then equations (19) and (21) imply

$$\big[X(t), H\big] - i\hbar Y(t) = U(-t)\Big(\big[X(0), H\big] - i\hbar Y(0)\Big)U(t),$$

so if $\big[X(t), H\big] - i\hbar Y(t)$ is zero at $t = 0$, then it's zero for all t.

Altogether, this shows that the quantum model that was defined in sections 9-13 satisfies equation (11), even when equation (11) and the hamiltonian (23) are generalized as described above.

17 Recovering equation (12)

This section shows that equation (12) is automatically satisfied in the quantum model that was defined in sections 9-13.

The derivation starts by taking the time derivative of equation (19) and setting $a, b = j, k$ to get

$$i\hbar\partial_0 B_{jk}(\mathbf{x}, t) = \big[B_{jk}(\mathbf{x}, t),\ H\big]. \tag{35}$$

The commutation relation (17) says that the components of the magnetic field commute with each other, so equation (35) reduces to

$$i\hbar\partial_0 B_{jk}(\mathbf{x}, t) = \big[B_{jk}(\mathbf{x}, t),\ L_E(t)\big] \tag{36}$$

with $L_E(t)$ defined by (31). Now equation (12) can be reproduced by using the equal-time commutation relations (17) and equation (23) for the hamiltonian to evaluate the right-hand side of (36).

Instead of working that out directly, we can get the same result more easily by using the fact that any relationship of the form

$$X_k = Y_k \tag{37}$$

implies

$$\partial_j X_k - \partial_k X_j = \partial_j Y_k - \partial_k Y_j. \tag{38}$$

We can exploit this by writing

$$B_{jk}(\mathbf{x}, t) = \partial_j A_k(\mathbf{x}, t) - \partial_k A_j(\mathbf{x}, t) \tag{39}$$

and treating $A_\bullet(\mathbf{x}, t)$ formally as an operator, with the understanding that it's only meaningful in the combination (39).[30] Equation (36) is reproduced by applying the

[30]This is like the representation (25), but multiplication by $a_k(\mathbf{x})$ does not define an operator on the Hilbert space, because the Hilbert space defined in section 13 uses only gauge invariant functions $\Psi[a]$. We could try to define $A_k(\mathbf{x}, t)$ as an operator by extending the Hilbert space to a larger vector space that allows arbitrary functions $\Psi[a]$, but that's not worth the trouble, because we're only using the "operators" A_k here as formal devices to organize the calculation. Also, Gauss's law (29) would not hold in that larger vector space, because equation (29) says that $\Psi[a]$ is gauge invariant.

replacement (37)→(38) to the formal equation

$$i\hbar\partial_0 A_k(\mathbf{x},t) = \big[A_k(\mathbf{x},t),\ L_E(t)\big], \tag{40}$$

and the commutation relation (17) is reproduced by applying the replacement (37)→(38) to the formal equation[31]

$$[E^i(\mathbf{y},t), A_k(\mathbf{x},t)] = -i\hbar q^2 \delta^i_k \delta(\mathbf{x}-\mathbf{y}). \tag{41}$$

Now, instead of using the commutation relation (17) to evaluate the right-hand side of (36), we can use the simpler commutation relation (41) to evaluate the right-hand side of (40). This immediately gives

$$\partial_0 A_k = -E_k,$$

and applying the replacement (37)→(38) to this formal equation gives equation (12), as desired. The fact that we never really defined A_k as an operator isn't important here, because in hindsight, this is just convenient a way of organizing a calculation that really only involves the electric and magnetic field operators.

[31]The sign in (41) matches the sign in (18). Readers familiar with canonical quantization can check that this is as it should be, because in the expression $\Pi^k = \delta L/\delta(\partial_0 A_k) = \delta L_E/\delta F_{0k}$ for the canonical conjugate of A_k, the overall sign of L_E cancels the sign in $E^k \equiv F^{k0} = -F^{0k}$.

18 An abbreviation

The rest of this article is concerned only with observables that are all associated with the same time, say $t = 0$, so the abbreviations

$$E^j(\mathbf{x}) \equiv E^j(\mathbf{x}, 0) \qquad\qquad B_{jk}(\mathbf{x}) \equiv B_{jk}(\mathbf{x}, 0) \tag{42}$$

will be used from now on.

19 Translation symmetry in quantum field theory

The rest of this article assumes that the spatial metric is flat, so the system has translation symmetry. When the metric is flat, we can use a coordinate system in which its components are independent of the coordinates:

$$\partial_\ell \, g_{jk} = 0. \tag{43}$$

The rest of this article uses the condition (43). The specific case (3) will not be assumed until section 28.

The hamiltonian (22) is the operator that generates translations in time. It is also the observable corresponding to the system's total energy. Similarly, the operators that generate translations in space are closely related to the observables corresponding to the system's total momentum. If the spatial metric is proportional to the identity matrix, as in (3), then the translation operators and momentum operators are proportional to each other. More generally, if the spatial metric is constrained only by (43), then the relationship is[32]

$$P_k = g_{kj} P^j \tag{44}$$

where P_k is the operator that generates translations along the kth spatial coordinate, and P^j is the operator represents the jth component of the system's total momentum. If $\mathcal{O}(\mathbf{x})$ is an observable localized at $\mathbf{x}$, like $E^j(\mathbf{x})$ or $B_{jk}(\mathbf{x})$,[33] then the translation operators P_k and momentum operators P^j satisfy

$$[\mathcal{O}(\mathbf{x}), P_k] = i\hbar \, \partial_k \mathcal{O}(\mathbf{x}) \qquad\qquad [\mathcal{O}(\mathbf{x}), P^j] = i\hbar \, g^{jk} \, \partial_k \mathcal{O}(\mathbf{x}). \tag{45}$$

Even though they are related to each other by a linear transformation (equation (44)), this article will care to distinguish between them, just like this article takes care to distinguish between E^j and E_j.

[32] Reminder: a sum over the index j is implied.

[33] The assertion that the observables $E^j(\mathbf{x})$ and $B_{jk}(\mathbf{x})$ are localized at the point $\mathbf{x}$ is part of the model's definition.

20 The momentum and translation operators

The operators representing the system's total momentum are[34]

$$P^j \equiv \int d^D x \; T^{0j}(\mathbf{x}) + \text{constant}, \tag{46}$$

with T^{0j} given by (16) but with the classical version of the components F_{ab} replaced by the corresponding operators. The significance of the constant term here is like what section 11 explained: when the model is defined properly (which we haven't quite done in this article), the constant term can be chosen so that the state with the lowest possible energy has zero momentum. The value of the constant term doesn't affect anything in this article, though, so we'll leave it unspecified.

Combine (46) with (44) to get the general expression for the translation operators:

$$P_j \equiv \int d^D x \; T_{0j}(\mathbf{x}) + \text{constant}, \tag{47}$$

and use (16) to motivate the choice[35,36]

$$T_{0j} \equiv \frac{1}{q^2} B_{jk}(\mathbf{x}) E^k(\mathbf{x}). \tag{48}$$

I'm calling this a "choice" because the expression (16) in classical field theory doesn't tell us how the operators should be ordered in the quantum model. For the purposes of this article, any choice of ordering of the operators in (48) works equally well, because any change in the ordering of the operators in T^{0j} can be compensated by changing the value of the constant term, which commutes with everything.

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[34] Article 78463

[35] Compare this to the momentum in classical electromagnetism (article 78463). The extra factor of q^2 here comes from the convention described in section 3.

[36] The integral over unbounded space would be undefined, but when considering the commutator of P_j with a local observable, the integral only needs to include the part of space where the observable is localized.

21 Translation symmetry in QEM

To check that (47)-(48) is consistent with (45), use the abbreviation

$$\partial^\ell_{jk} \equiv \delta^\ell_j\,\partial_k - \delta^\ell_k\,\partial_j \tag{49}$$

and the commutation relations (17) to get

$$\begin{aligned}
[E^\ell(\mathbf{y}), T_{0j}(\mathbf{x})] &= i\hbar\, E^\bullet(\mathbf{x})\partial^\ell_{j\bullet}\delta(\mathbf{x}-\mathbf{y}) \\
[B_{\ell m}(\mathbf{y}), T_{0j}(\mathbf{x})] &= i\hbar\, B_{j\bullet}(\mathbf{x})\partial^\bullet_{\ell m}\delta(\mathbf{x}-\mathbf{y}),
\end{aligned} \tag{50}$$

using $\bullet$ as a spatial index to help make the index-heavy equations easier to parse. A sum over $\bullet \in \{1, 2, ..., D\}$ is implied. Use this in (47) to get

$$\begin{aligned}
\big[E^\ell(\mathbf{x}), P_j\big] &= -i\hbar\,\partial^\ell_{j\bullet}E^\bullet(\mathbf{x}) \\
\big[B_{\ell m}(\mathbf{x}), P_j\big] &= -i\hbar\,\partial^\bullet_{\ell m}B_{j\bullet}(\mathbf{x}).
\end{aligned}$$

Using the constraint equations (10) and (13) on the right-hand sides gives

$$\begin{aligned}
\big[E^\ell(\mathbf{x}), P_j\big] &= i\hbar\,\partial_j E^\ell(\mathbf{x}) \\
\big[B_{\ell m}(\mathbf{x}), P_j\big] &= i\hbar\,\partial_j B_{\ell m}(\mathbf{x}).
\end{aligned}$$

This confirms equations (45).

22 Translation symmetry and Fourier transforms

In the context of translation symmetry, the Fourier transform is often useful. It's useful because it expresses a function $f(\mathbf{x})$ of the spatial coordinates as a linear combination of functions of the form $e^{-ip_k x^k}$, which are eigenfunctions of the translation $\mathbf{x} \to \mathbf{x} + \mathbf{c}$ with eigenvalue $e^{-ip_k c^k}$.

To facilitate switching back and forth between the momentum operators and translation operators (equation (44)), this article will write the relationship between a function $f(\mathbf{x})$ and its Fourier transform $\tilde{f}(\mathbf{p})$ as[37]

$$f(\mathbf{x}) = \int \frac{d^D p}{(2\pi)^D}\; \tilde{f}(\mathbf{p}) e^{-i\langle \mathbf{p},\mathbf{x}\rangle/\hbar} \tag{51}$$

with

$$\mathbf{p} \equiv (p^1, ..., p^D)$$

and

$$\langle \mathbf{p},\mathbf{x}\rangle \equiv g_{jk} p^j x^k = p_k x^k, \tag{52}$$

using the standard relationship

$$p_k = g_{jk} p^j.$$

In section 23, the quantities p_k and p^j will occur as eigenvalues of the translation operator P_k and momentum operator P^j, respectively. When the metric is given by (3), the quantity (52) reduces to

$$\langle \mathbf{p},\mathbf{x}\rangle = -\mathbf{p}\cdot\mathbf{x} \equiv -\sum_k p^k x^k. \tag{53}$$

Equation (53) relates the notation $\langle \mathbf{p},\mathbf{x}\rangle$ that will be used here to the notation $\mathbf{p}\cdot\mathbf{x}$ that is used in articles 09193, 30983, and 58590.

[37] To make this completely unambiguous, we should specify whether the integration variables are p^k or p_k. The conditions (4) imply that these two versions of the integral are equal to each other except for an overall factor of $\det g = (-1)^D$, which doesn't affect any relationships or conclusions in this article.

23 Momentum-shifting operators

Suppose that $\mathcal{O}(\mathbf{x})$ is an observable localized at $\mathbf{x}$, and consider the **smeared operator**

$$\mathcal{O}(f) \equiv \int d^Dx\; f(\mathbf{x})\mathcal{O}(\mathbf{x})$$

where $f(\mathbf{x})$ is a complex-valued function that approaches zero rapidly enough as $|\mathbf{x}| \to \infty$ to make the integral converge. When $f(\mathbf{x})$ is written in terms of its Fourier transform, this becomes[38]

$$\mathcal{O}(f) = \int d^Dx \int \frac{d^Dp}{(2\pi)^D}\; \tilde{f}(\mathbf{p})e^{-i\langle\mathbf{p},\mathbf{x}\rangle/\hbar}\mathcal{O}(\mathbf{x}), \tag{54}$$

using the notation that was defined in section 22. We can write (54) formally as

$$\mathcal{O}(f) = \int \frac{d^Dp}{(2\pi)^D}\; \tilde{f}(\mathbf{p})\tilde{\mathcal{O}}(-\mathbf{p}) \qquad \text{with}\;\; \tilde{\mathcal{O}}(\mathbf{p}) \equiv \int d^Dx\; e^{i\langle\mathbf{p},\mathbf{x}\rangle/\hbar}\mathcal{O}(\mathbf{x}). \tag{55}$$

The integral that "defines" $\tilde{\mathcal{O}}(\mathbf{p})$ does not converge on its own, but we can still use it if we remember that it's really just part of the integrand in definition of $\mathcal{O}(f)$.

Use (45) and (55) to infer

$$[\tilde{\mathcal{O}}(\mathbf{p}), P_j] = p_j\tilde{\mathcal{O}}(\mathbf{p}) \qquad \Rightarrow \qquad P^j\tilde{\mathcal{O}}(-\mathbf{p}) = \tilde{\mathcal{O}}(-\mathbf{p})\big(P^j + p^j\big).$$

This shows that applying $\tilde{\mathcal{O}}(-\mathbf{p})$ to any state adds $\mathbf{p}$ to the state's momentum,[39] so $\tilde{\mathcal{O}}(-\mathbf{p})$ will be called a **momentum-shifting operator**.

[38] To be completely unambiguous, we should specify whether the integration variables are p^k or p_k. The conditions (4) imply that these two versions of the integral are equal to each other except for an overall factor of $\det g = (-1)^D$, which doesn't affect any relationships or conclusions in this article.

[39] The general rule is: if an operator ϕ satisfies $U(-t,-\mathbf{x})\phi U(t,\mathbf{x}) = e^{i(\omega t+\langle\mathbf{p},\mathbf{x}\rangle)/\hbar}\phi$ with $U(t,\mathbf{x}) \equiv e^{-i(Ht+\langle\mathbf{P},\mathbf{x}\rangle)/\hbar}$, then it adds energy ω and momentum $\mathbf{p}$ to any state on which it acts. This agrees with the example in article 00980.

24 Momentum-shifting operators and constraints

This section derives some consequences of the constraint equations (10) and (13) for the momentum-shifting operators[40]

$$\tilde{E}_j(\mathbf{p}) \equiv \int d^D x\ e^{i\langle \mathbf{p},\mathbf{x}\rangle/\hbar} E_j(\mathbf{x}) \qquad\qquad \tilde{B}_{jk}(\mathbf{p}) \equiv \int d^D x\ e^{i\langle \mathbf{p},\mathbf{x}\rangle/\hbar} B_{jk}(\mathbf{x}). \tag{56}$$

Section 32 will use the relationships derived here to study the angular momentum of a photon.

Equations (10), (13), and (56) imply

$$p_i \tilde{B}_{jk}(\mathbf{p}) + p_j \tilde{B}_{ki}(\mathbf{p}) + p_k \tilde{B}_{ij}(\mathbf{p}) = 0 \qquad\qquad \big\langle \mathbf{p}, \tilde{\mathbf{E}}(\mathbf{p})\big\rangle = 0. \tag{57}$$

If $\mathbf{p} = (p^1, 0, 0, 0, ..., 0)$, then these equations imply that $\tilde{B}_{jk}(\mathbf{p})$ is nonzero only if either j or k is equal to 1 and that $\tilde{E}_j(\mathbf{p})$ is nonzero only if $j \neq 1$.

Here's another consequence that isn't quite so obvious. Apply the partial derivative $\partial/\partial p^j$ to the second of equations (57) to get

$$\tilde{E}_j(\mathbf{p}) + \left\langle \mathbf{p},\ \frac{\partial}{\partial p^j}\tilde{\mathbf{E}}(\mathbf{p})\right\rangle = 0.$$

If $\mathbf{p} = (p^1, 0, 0, 0, ..., 0)$ and $j = 2$, then this reduces to

$$\tilde{E}_2(\mathbf{p}) + p^1 \frac{\partial}{\partial p^2}\tilde{E}_1(\mathbf{p})\bigg|_{\mathbf{p}=(p^1,0,0,0,...,0)} = 0.$$

In words: even though $\tilde{E}_1(\mathbf{p})$ is zero when $\mathbf{p} = (p^1, 0, 0, 0, ..., 0)$, its derivative with respect to the second component p^2 is not zero and in fact is proportional to $\tilde{E}_2(\mathbf{p})$. This will be used in section 32.

[40] Using the operators E_j instead of E^k will be convenient in section 25. They are related by $E_j = g_{jk}E^k$.

25 Energy-shifting operators and photons

This section constructs an operator $a_{jk}^\dagger(\mathbf{p})$ that adds an amount $|\mathbf{p}| \equiv \sqrt{-\langle \mathbf{p}, \mathbf{p}\rangle}$ to the energy of any state to which it is applied.

Sections 16-17 showed that the quantum model satisfies Maxwell's equations (11)-(12). Take the derivative of the generic time evolution equation (19) with respect to t and use Maxwell's equations (11)-(12) to get

$$\begin{aligned} \big[H, E_j(\mathbf{x})\big] &= -i\hbar g^{k\ell}\partial_k B_{\ell j}(\mathbf{x}) \\ \big[H, B_{jk}(\mathbf{x})\big] &= i\hbar\big(\partial_j E_k(\mathbf{x}) - \partial_k E_j(\mathbf{x})\big). \end{aligned} \tag{58}$$

Equations (56) and (58) imply

$$\begin{aligned} [H, \tilde{E}_j(\mathbf{p})] &= -p^k \tilde{B}_{kj}(\mathbf{p}) \\ [H, \tilde{B}_{jk}(\mathbf{p})] &= \big(p_j \tilde{E}_k(\mathbf{p}) - p_k \tilde{E}_j(\mathbf{p})\big). \end{aligned} \tag{59}$$

Now define[41]

$$a_{jk}^\dagger(\mathbf{p}) \equiv p_j \tilde{E}_k(-\mathbf{p}) - p_k \tilde{E}_j(-\mathbf{p}) - |\mathbf{p}|\tilde{B}_{jk}(-\mathbf{p}). \tag{60}$$

Use equations (59) and the first of equations (57) to get

$$\big[H, a_{jk}^\dagger(\mathbf{p})\big] = |\mathbf{p}|\, a_{jk}^\dagger(\mathbf{p}),$$

which may also be written

$$H a_{jk}^\dagger(\mathbf{p}) = a_{jk}^\dagger(\mathbf{p})\big(H + |\mathbf{p}|\big). \tag{61}$$

Section 23 already showed that applying the operator $a_{jk}^\dagger(\mathbf{p})$ to any state adds $\mathbf{p}$ to the state's momentum. Equation (61) shows that it also adds $|\mathbf{p}|$ to the state's energy. The excitation created by this operator is called a **photon**.

[41] The notation $a^\dagger$ is standard for a photon creation operator.

26 Photon polarizations

For a given momentum $\mathbf{p}$, the operators $a_{jk}^\dagger(\mathbf{p})$ are not all linearly independent. This section shows that in D-dimensional space, only $D-1$ of them are linearly independent. This is the number of linearly independent possible **polarizations** of a photon with momentum $\mathbf{p}$.

The energy of a photon can be arbitrarily small but not zero, because the operator (60) itself is zero when $\mathbf{p}=\mathbf{0}$. For $\mathbf{p}\neq\mathbf{0}$, we can write equation (60) like this:[42]

$$a_{jk}^\dagger(\mathbf{p}) = p_j a_k^\dagger(\mathbf{p}) - p_k a_j^\dagger(\mathbf{p})$$

with

$$a_k^\dagger(\mathbf{p}) = \tilde{E}_k(-\mathbf{p}) - \frac{p^\ell}{|\mathbf{p}|}\,\tilde{B}_{\ell k}(-\mathbf{p}). \tag{62}$$

The index k takes only D distinct values, so the number of linearly independent polarizations must be $\leq D$. The constraint equation (10) and the antisymmetry of $\tilde{B}_{jk}$ together imply the identity

$$p^k a_k^\dagger(\mathbf{p}) = 0,$$

which reduces the number of linearly independent polarizations to $D-1$, so all available polarizations may be expressed as linear combinations of a fixed set of $D-1$ polarizations. In the most important case $D=3$, two linearly independent polarizations are available for a photon with momentum $\mathbf{p}$.

[42] For the $\tilde{E}$ term, this is clear. For the $\tilde{B}$ term, use equation (57).

27 Essentially classical electromagnetic waves

Section 25 explained how to construct states with a single photon. Applying n of those photon creation operators to the vacuum state (the state of lowest energy) gives a state with n photons. States with a well-defined number of photons are exceptional, though. Most states don't have any definite number of photons, because most states are superpositions (linear combinations) of n-photon states with nonzero coefficients for infinitely many different values of n.

One special class of states that don't have any definite number of photons are the **coherent states**. These are eigenstates of the annihilation operators (the adjoints of the photon creation operators), typically with complex eigenvalues. Article 22792 studies coherent states in a model of a quantum scalar field. The analysis in QEM is essentially the same except for the existence of more than one polarization when $D \geq 3$, so it won't be repeated here. The most important conclusion is that this family of states includes states that behave like classical electromagnetic waves with respect to measurements of the electric and magnetic field observables that have sufficiently coarse spatial resolution. They behave like classical electromagnetic waves in the sense that the standard deviation in the measurement outcomes is much less than the expectation value. Classical electromagnetism is an approximation that works well for states like these.

28 Rotation symmetry in quantum field theory

Suppose that space is flat, as in section 19, but now choose a coordinate system in which the spatial metric is given by (3) so that rotation symmetry is manifest. Rotation symmetry about the origin in the j-k plane implies that the model includes an observable corresponding to the total angular momentum about the origin in the j-k plane. The operator representing that observable is[43,44]

$$J^{jk} = \int d^D x \; (T^{0j} x^k - T^{0k} x^j) + \text{constant}, \tag{63}$$

where T^{0j} are the operators representing momentum density.[45] With the metric (3), J^{jk} is equal to the generator J_{jk} of rotations about the origin in the j-k plane. The commutator of J_{jk} with an observable $\mathcal{O}(\mathbf{x})$ localized at $\mathbf{x}$ should be

$$\big[\mathcal{O}(\mathbf{x}), J_{jk}\big] = i\hbar\,(x^j\partial_k - x^k\partial_j)\mathcal{O}(\mathbf{x}) + \text{terms without derivatives}, \tag{64}$$

where the terms without derivatives depend on the tensor nature of $\mathcal{O}$. The sign convention is such that the effect of a rotation on the momentum generators is

$$[P^\ell, J_{jk}] = i\hbar(\delta^\ell_j P_k - \delta^\ell_k P_j). \tag{65}$$

When $\ell = j$, (65) is the derivative of this relationship with respect to θ at $\theta = 0$:

$$U(-\theta)P^j U(\theta) = P^j \cos\theta + P^k \sin\theta$$

with $U(\theta) \equiv e^{-iJ_{jk}\theta/\hbar}$. Using the momentum density given by equation (16), section 29 will confirm that equation (64) holds when $\mathcal{O}(\mathbf{x})$ is an electric or magnetic field operator, and the explicit expressions for the non-derivative terms will be used to confirm the sign convention (65).[46]

[43] The operator representing total angular momentum about some other point $\mathbf{c}$ is given by replacing the explicit factors of $x^\bullet$ in the integrand with $x^\bullet - c^\bullet$.

[44] Recall footnote 36 in section 20.

[45] Section 8

[46] In QEM, the derivative terms don't contribute to the commutator of J_{jk} with the momentum operators. (In models of scalar fields, they do.)

29 Rotation symmetry in QEM

With the metric (3), equation (63) and $J_{jk} = J^{jk}$ say that the generator of rotations about the origin in the j-k plane is

$$J_{jk} = -\int d^D x\ (T_{0j}x^k - T_{0k}x^j) + \text{constant}.$$

Use (50) to get

$$\begin{aligned}
[E^\ell(\mathbf{x}), J_{jk}] &= i\hbar\, \partial^\ell_{j\bullet}\big(x^k E^\bullet(\mathbf{x})\big) - (j \leftrightarrow k)\\
[B_{\ell m}(\mathbf{x}), J_{jk}] &= i\hbar\, \partial^\bullet_{\ell m}\big(x^k B_{j\bullet}(\mathbf{x})\big) - (j \leftrightarrow k).
\end{aligned} \tag{66}$$

Using the notation (49) and evaluating the derivatives of the x^k factors gives[47]

$$\begin{aligned}
[E^\ell(\mathbf{x}), J_{jk}] &= i\hbar\left(x^k \partial^\ell_{j\bullet} E^\bullet(\mathbf{x}) + \delta^\ell_j E^k(\mathbf{x})\right) - (j \leftrightarrow k)\\
[B_{\ell m}(\mathbf{x}), J_{jk}] &= i\hbar\left(x^k \partial^\bullet_{\ell m} B_{j\bullet}(\mathbf{x}) + \delta^k_m B_{j\ell}(\mathbf{x}) - \delta^k_\ell B_{jm}(\mathbf{x})\right) - (j \leftrightarrow k)
\end{aligned}$$

Using the constraint equations (10) and (13) to simplify the remaining derivatives gives

$$\begin{aligned}
[E^\ell(\mathbf{x}), J_{jk}] &= i\hbar\left(-x^k \partial_j E^\ell(\mathbf{x}) + \delta^\ell_j E^k(\mathbf{x})\right) - (j \leftrightarrow k)\\
&= i\hbar\left(\quad x^j \partial_k E^\ell(\mathbf{x}) + \delta^\ell_j E^k(\mathbf{x})\right) - (j \leftrightarrow k)\\
[B_{\ell m}(\mathbf{x}), J_{jk}] &= i\hbar\left(-x^k \partial_j B_{\ell m}(\mathbf{x}) + \delta^k_m B_{j\ell}(\mathbf{x}) - \delta^k_\ell B_{jm}(\mathbf{x})\right) - (j \leftrightarrow k)\\
&= i\hbar\left(\quad x^j \partial_k B_{\ell m}(\mathbf{x}) + \delta^k_m B_{j\ell}(\mathbf{x}) - \delta^k_\ell B_{jm}(\mathbf{x})\right) - (j \leftrightarrow k).
\end{aligned} \tag{67}$$

The derivative terms confirm equation (64), and the non-derivative terms may be used to confirm the sign convention (65).

[47] In the first equation, a term involving δ^k_j canceled because of the antisymmetry with respect to $j \leftrightarrow k$.

30 Rotation symmetry and photons

For any given value of $|\mathbf{p}|$, the photon creation operator $a_k^\dagger(\mathbf{p})$ that was defined in equation (62) is a linear combination of two terms: one proportional to $\tilde{E}_k(-\mathbf{p})$, and one proportional to $p^\ell \tilde{B}_{\ell k}(-\mathbf{p})$. Using $X_k(\mathbf{p})$ to denote either of these operators, section 31 will show that they both satisfy

$$\big[X_k(\mathbf{p}), J_{12}\big] = i\hbar\left(\left(p^1\tilde{\partial}_2 - p^2\tilde{\partial}_1\right)X_k(\mathbf{p}) + \delta_{k1}X_2(\mathbf{p}) - \delta_{k2}X_1(\mathbf{p})\right) \tag{68}$$

with

$$\tilde{\partial}_j \equiv \frac{\partial}{\partial p^j}.$$

This implies that the operators

$$X_\pm(\mathbf{p}) \equiv X_1(\mathbf{p}) \mp iX_2(\mathbf{p})$$

satisfy

$$[X_\pm(\mathbf{p}), J_{12}] = i\hbar\left(p^1\tilde{\partial}_2 - p^2\tilde{\partial}_1\right)X_\pm(\mathbf{p}) \pm \hbar X_\pm(\mathbf{p}). \tag{69}$$

In particular, the photon creation operators

$$a_\pm^\dagger(\mathbf{p}) \equiv a_1^\dagger(\mathbf{p}) \mp ia_2^\dagger(\mathbf{p}) \tag{70}$$

satisfy (69). Section 32 will explain the significance of this result.

31 Derivation of (68)

This section shows that the operators $\tilde{E}_k(-\mathbf{p})$, and $p^\ell \tilde{B}_{\ell k}(-\mathbf{p})$ both satisfy equation (68), where $X_k(\mathbf{p})$ denotes either of those operators.

The fact that $\tilde{E}_k(-\mathbf{p})$ satisfies (68) follows easily from equations (55) and (67) with the help of the identity

$$x^j \partial_k e^{i\langle \mathbf{p},\mathbf{x}\rangle/\hbar} = p^k \tilde{\partial}_j e^{i\langle \mathbf{p},\mathbf{x}\rangle/\hbar} \tag{71}$$

and the lemma[48]

$$\mathcal{O}'(\mathbf{p}) \equiv p^j \tilde{\partial}_k \mathcal{O}(\mathbf{p}) \qquad \text{implies} \qquad \mathcal{O}'(-\mathbf{p}) = p^j \tilde{\partial}_k \mathcal{O}(-\mathbf{p}). \tag{72}$$

Showing that $p^\ell \tilde{B}_{\ell k}(-\mathbf{p})$ satisfies (68) requires an extra step. Start by using (55), (67), and (71) to get

$$[p^\ell \tilde{B}_{\ell k}(\mathbf{p}), J_{12}] = i\hbar\, p^\ell \Big(p^1 \tilde{\partial}_2 \tilde{B}_{\ell k}(\mathbf{p}) + \delta_k^2 \tilde{B}_{1\ell}(\mathbf{p}) - \delta_\ell^2 \tilde{B}_{1k}(\mathbf{p}) \Big) - (1 \leftrightarrow 2).$$

Use the identity

$$p^\ell \left(p^1 \tilde{\partial}_2 - p^2 \tilde{\partial}_1 \right) f(\mathbf{p}) = \left(p^1 \tilde{\partial}_2 - p^2 \tilde{\partial}_1 \right) \left(p^\ell f(\mathbf{p}) \right) + (p^2 \delta_1^\ell - p^1 \delta_2^\ell) f(\mathbf{p})$$

to get

$$[p^\ell \tilde{B}_{\ell k}(\mathbf{p}), J_{12}] = i\hbar \left(p^1 \tilde{\partial}_2 \big(p^\ell \tilde{B}_{\ell k}(\mathbf{p}) \big) + \delta_k^1 p^\ell \tilde{B}_{\ell 2}(\mathbf{p}) \right) - (1 \leftrightarrow 2).$$

Now use (72) to deduce that $p^\ell \tilde{B}_{\ell k}(-\mathbf{p})$ satisfies (68).

[48] To check this, consider the case where $\mathcal{O}(\mathbf{p})$ is a product of n factors of the components of $\mathbf{p}$. Then $\mathcal{O}'(\mathbf{p})$ is also a product of n factors of the components of $\mathbf{p}$, so the sign ϵ in $\mathcal{O}'(-\mathbf{p}) = \epsilon \mathcal{O}'(\mathbf{p})$ is the same as the sign ϵ in $\mathcal{O}(-\mathbf{p}) = \epsilon \mathcal{O}(\mathbf{p})$.

32 The angular momentum of a photon

If a particle has nonzero mass, we can consider a single-particle state in which the particle's momentum is zero. The angular momentum of such a state defines the particle's **intrinsic angular momentum** or **spin**. We can't apply this definition to a photon, though, because a photon cannot have zero momentum: when $\mathbf{p} = 0$, the operators (60) are zero, and the operators (62) are undefined. This is related to the fact that the energy $|\mathbf{p}|$ of a photon approaches zero as its momentum $\mathbf{p}$ approaches zero.[49] In other words, photons are massless.

This section shows that we can still consider construct single-photon states with well-defined angular momentum, but intuition about *spin* that we developed based on experience with massive particles doesn't always apply to photons.[50] To help build some new intuition, this section considers two examples.

If the momentum is orthogonal to the 1-2 plane, so $p_1 = p_2 = 0$, then equation (69) says $J_{12}a_+^\dagger(\mathbf{p}) = a_+^\dagger(\mathbf{p})(J_{12} + \hbar)$, so the operator $a_+^\dagger(\mathbf{p})$ defined in (70) adds angular momentum $\hbar$ in the 1-2 plane to any state on which it acts. Similarly, the operator $a_-^\dagger(\mathbf{p})$ adds angular momentum $-\hbar$ in the 1-2 plane to any state on which it acts. This configuration is possible only if $D \geq 3$, because only then can a nonzero momentum be orthogonal to the 1-2 plane.

If the momentum is parallel to the 1-2 plane, say $\mathbf{p} = (p_1, 0, 0, ...)$, then section 24 showed that the right-hand side of (69) is zero when $X_\pm(\mathbf{p}) = a_\pm^\dagger(\mathbf{p})$, so in this case each of the operators $a_\pm^\dagger(\mathbf{p})$ creates a photon with zero angular momentum in the 1-2 plane. When $D = 2$, angular momentum doesn't have any other components, so this shows that the angular momentum of a photon is necessarily zero when $D = 2$.[51]

[49] Section 25

[50] Bekaert and Boulanger (2006) use group theory to study the angular momenta of both massive and massless particles in an arbitrary number of dimensions.

[51] Bekaert and Boulanger (2006), section B.3.1

33 Angular momentum and polarization

Section 26 showed that a photon with a given momentum has $D-1$ linearly independent polarizations. If the momentum is $\mathbf{p} = (0,0,0,...,0,1)$, so that the Dth component is the only nonzero component, then the operators $a_k^\dagger$ with $k \in \{1,2,3,...,D-1\}$ represent a set of $D-1$ linearly independent polarizations. When $D \geq 3$, a different set of $D-1$ linearly independent polarizations is represented by the operators

$$a_1^\dagger + ia_2^\dagger,\ a_2^\dagger + ia_3^\dagger,\ a_3^\dagger + ia_4^\dagger,\ ...,\ a_{D-2}^\dagger + ia_{D-1}^\dagger,\ a_{D-1}^\dagger \pm ia_1^\dagger.$$

These are linearly independent if the sign in the last operator is chosen appropriately.[52] This shows that when $D \geq 3$, we can choose a set of $D-1$ linearly independent polarizations that each have a definite nonzero angular momentum in one plane (but not all in the same plane if $D \geq 4$).

In the physically important case $D = 3$, this reduces to the statement that the two operators $a_1^\dagger \pm ia_2^\dagger$ create linearly independent polarizations that both have definite values of the angular momentum in the 1-2 plane, namely the values $\mp\hbar$. These are the single-photon versions of left-circular and right-circular polarization in classical electrodynamics.

[52] To deduce the appropriate sign, multiply the kth operator in the list by $(-i)^{k-1}$ and then add them all together. All of the $a_k^\dagger$ terms with $2 \leq k \leq D-1$ cancel, leaving the remainder $a_1^\dagger \pm i(-i)^{D-2}a_1^\dagger$. If we choose the sign so that this is nonzero, then the operators in the list are linearly independent.

34 References

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