

# Normal Ordering and Composite Operators for Free Scalar Quantum Fields

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**Abstract** In quantum field theory, many models can be defined by treating spacetime as a very fine lattice, but operators that are well-defined on the lattice don't always remain well-defined in the continuous-spacetime limit. We can fix this by smearing the operator over a time interval that remains finite in the continuum limit, but without smearing the best we can do is construct operators whose  $n$ -point correlation functions remain well-defined as long as the points remain separated from each other in spacetime, even though the operators themselves do not remain well-defined as ordinary operators on the Hilbert space. In a model with no interactions, the construction uses a prescription called **normal ordering**, which is a way of modifying the original operator to make its correlation functions well-defined in the continuum limit. This article introduces the concept and shows how to efficiently work out explicit expressions for the normal-ordered versions of arbitrary powers of a free scalar quantum field.

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# 1 Introduction

Consider the free scalar quantum field in  $D$ -dimensional space. Treat space as a lattice so that the field operator  $\phi(x)$  at an individual point  $x$  in space is well-defined as an operator on the Hilbert space.<sup>1,2</sup> If  $|0\rangle$  is the vacuum state, then  $\phi(x)|0\rangle$  is another state-vector in the Hilbert space. Its norm

$$r \equiv \frac{\langle 0|\phi^2(x)|0\rangle}{\langle 0|0\rangle} \quad (1)$$

is a finite real number whose value will be estimated in section 7 as a function of the lattice spacing  $\epsilon$ . In the continuous-space limit ( $\epsilon \rightarrow 0$ ), the norm  $r$  becomes undefined ( $r \rightarrow \infty$ ), which implies that  $\phi(x)$  is not well-defined as an operator on the Hilbert space when space is continuous. In contrast, the **smear**ed field operator<sup>3</sup>

$$\phi(f) \equiv \epsilon^D \sum_x f(x, \epsilon) \phi(x)$$

does remain well-defined<sup>4</sup> if the function  $f(x, \epsilon)$  approaches a smooth function of nonzero width as  $\epsilon \rightarrow 0$ .

Squaring does not commute with smearing. In particular, even though  $(\phi(f))^2$  remains well-defined as an operator on the Hilbert space in the continuous-space limit, the combination

$$\int d^D x f(x) \phi^2(x) \quad (2)$$

(the result of squaring-and-then-smearing  $\phi(x)$ ) does not. We can improve the behavior by subtracting its vacuum expectation value, because the correlation func-

<sup>1</sup>Article [52890](#)

<sup>2</sup>In this article, all points are at the same time.

<sup>3</sup>The sum is over all points in space at a given time.

<sup>4</sup>In the infinite-volume limit, if the single-particle mass  $m$  is zero and space is one-dimensional ( $D = 1$ ), then the smeared field operator becomes ill-defined (article [37301](#)), but it remains well-defined if either  $m > 0$  or  $D \geq 2$  (or both).

tion

$$\langle 0 | (\phi^2(x) - r)(\phi^2(y) - r) | 0 \rangle$$

has a well-defined continuum limit as long as  $|x - y| > 0$ . The combination  $\phi^2(x) - r$  still doesn't have a well-defined continuum limit, not even if it's smeared over a finite region of space,<sup>5</sup> but it is still useful in many contexts where an ultraviolet cutoff (like the lattice spacing  $\epsilon$ ) is understood to be present. This is an example of a **composite operator**. More generally, (renormalized) composite operators are things whose  $n$ -point correlation functions in the vacuum state remain well-defined in the continuum limit, as long as the points remain separated from each other.<sup>6</sup> The behavior of such correlation functions as any two of the points approach each other is described by the **operator product expansion**<sup>7</sup>

$$\mathcal{O}_A(x)\mathcal{O}_B(y) \sim \sum_C w_{AB}^C(x - y)\mathcal{O}_C(x),$$

where each  $\mathcal{O}_\bullet$  is a (composite) operator and the  $w$ s are complex-valued functions.

In general, just subtracting the vacuum expectation value is not sufficient: it is sufficient for the special case  $\phi^2(x)$ , but not for  $\phi^n(x)$  when  $n \geq 3$ . To ensure that  $n$ -point correlation functions have well-defined continuum limits when the points are separated, we can use a prescription called **normal ordering**. One definition of *normal ordering* refers to the operator product expansion, specifically to the first nonsingular term in an expansion in powers of  $|x - y|$ . That definition is described in section 6.5 of Di Francesco *et al* (1997). This article uses another standard definition, one that refers to creation/annihilation operators instead of to the operator product expansion. This definition will be introduced in section 2.

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<sup>5</sup>If it's smeared in time, then it remains well-defined as an operator on the Hilbert space in the continuum limit (article 10690).

<sup>6</sup>A composite operator is not necessarily well-defined as an operator on the Hilbert space in the continuum limit. The name *operator* is used more generally both in physics (where people often don't precisely define the symbols that they're manipulating) and in mathematics (because the elements of a  $*$ -algebra – defined in article 74088 – are often called *operators* even if they're not represented as linear operators on a Hilbert space).

<sup>7</sup>The convergence of this expansion in a particular model in four-dimensional spacetime is analyzed in Hollands *et al* (2014). Pappadopulo *et al* (2012) explores its convergence in CFTs (quantum field theories with conformal symmetry).

## 2 Normal ordering

This section gives the general definition of normal ordering that will be used in the rest of this article.

First, here's a special case: the normal ordered version of  $\phi^2(x)$  is<sup>8</sup>

$$\Phi_2(x) \equiv \phi^2(x) - r \quad (3)$$

with  $r$  defined by (1).

For the general definition, choose a point  $x$  in spacetime and use the abbreviation

$$\phi \equiv \phi(x).$$

Write

$$\phi = C + A$$

where  $C$  is the adjoint of  $A$ , and  $A|0\rangle = 0$ . Mnemonic:  $C$  is the creation-operator part of  $\phi$ , and  $A$  is the annihilation-operator part of  $\phi$ . The commutator

$$r \equiv [A, C] \equiv AC - CA = \frac{\langle 0|\phi^2|0\rangle}{\langle 0|0\rangle}$$

is a positive real number.<sup>9,10</sup> Define  $\Phi_n$  to be the operator obtained from the formal expression  $(C + A)^n$  by writing all factors of  $C$  to the left of all factors of  $A$  in each term.<sup>11</sup> Example:

$$\Phi_2 = C^2 + 2CA + A^2.$$

The goal is to derive an expression for  $\Phi_n$  as a polynomial in  $\phi$ . Section 4 will show a few examples, and section 6 will show another way of writing them.

<sup>8</sup>The standard notation for the normal-ordered version of  $\mathcal{O}(x)$  is  $:\mathcal{O}(x):$ . The non-standard notation used in this article is meant to make the equations easier to read.

<sup>9</sup>This is only true in a model without any interactions. This article is limited to *free* scalar quantum fields.

<sup>10</sup>Section 7 estimates the value of this number when  $D$ -dimensional space is treated as an infinite lattice.

<sup>11</sup>The motive for this definition is the fact that  $\Phi_n(x)$  (the normal ordered version of  $\phi^n(x)$ ) has well-defined vacuum correlation functions in the continuum limit, as long as the points are all distinct from each other.

### 3 The key identity

The key identity is<sup>12</sup>

$$\Phi_n = \phi\Phi_{n-1} - (n-1)r\Phi_{n-2}. \quad (4)$$

To derive this, start with the obvious identity

$$\phi\Phi_{n-1} = C\Phi_{n-1} + \Phi_{n-1}A + [A, \Phi_{n-1}],$$

which may also be written

$$\phi\Phi_{n-1} = \Phi_n + [A, \Phi_{n-1}]. \quad (5)$$

Now use the identity

$$[A, C^k] = krC^{k-1}$$

to see that  $[A, \dots]$  acts formally like  $r$  times the derivative with respect to  $C$ :

$$[A, \dots] \sim r \frac{\partial}{\partial C} \dots$$

Use this in (5) to get (4).

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<sup>12</sup>Brunetti *et al* (1996) uses a more general version of this as the *definition* of normal ordering (definition 5.1 in that paper), but they express the definition in terms of  $\phi(x_1) \cdots \phi(x_n)$  instead of  $\phi^n(x)$  because they work directly in continuous spacetime. Then they define a normal-ordered version of  $\phi^n(f)$  (they call it a **Wick monomial**), where  $\phi(f)$  is the smeared field operator  $\int f(x)\phi(x)$ . This is expressed more concisely but less explicitly by equation (7) in the preprint version of Pinamonti (2008). Their approach is explored further in Hollands and Ruan (2002).

## 4 Results

For  $n = 2$ , the key identity (4) reduces to

$$\Phi_2 = \phi^2 - r. \quad (6)$$

For  $n = 3$ , the key identity (4) reduces to

$$\Phi_3 = \phi\Phi_2 - 2r\phi.$$

Use (6) in the right-hand side to get

$$\Phi_3 = \phi^3 - 3r\phi. \quad (7)$$

For  $n = 4$ , the key identity (4) reduces to

$$\Phi_4 = \phi\Phi_3 - 3r\Phi_2.$$

Use (6) and (7) in the right-hand side to get

$$\Phi_4 = \phi^4 - 6r\phi^2 + 3r^2 = (\phi^2 - 3r)^2 - 6r^2. \quad (8)$$

Similarly,

$$\begin{aligned} \Phi_5 &= \phi^5 - 10r\phi^3 + 15r^2\phi \\ \Phi_6 &= \phi^6 - 15r\phi^4 + 45r^2\phi^2 - 15r^3 \\ \Phi_7 &= \phi^7 - 21r\phi^5 + 105r^2\phi^3 - 105r^3\phi \\ \Phi_8 &= \phi^8 - 28r\phi^6 + 210r^2\phi^4 - 420r^3\phi^2 + 105r^4. \end{aligned}$$

## 5 Relationship to Bell polynomials

The **Bell polynomial**  $B_n(a_1, \dots, a_n)$  is defined by the condition<sup>13</sup>

$$\sum_{n=0}^{\infty} \frac{1}{n!} B_n(a_1, \dots, a_n) z^n = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} a_n z^n \right). \quad (9)$$

This section shows that  $\Phi_n$  can be written<sup>14</sup>

$$\Phi_n = B_n(\phi, -r, 0, \dots, 0). \quad (10)$$

Use (10) in (9) to get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Phi_n z^n = \exp \left( \phi z - \frac{r}{2} z^2 \right), \quad (11)$$

which may also be written

$$\Phi_n = \left( \frac{d}{dz} \right)^n \exp \left( \phi z - \frac{r}{2} z^2 \right) \Big|_{z=0}. \quad (12)$$

Use the identity

$$\left( \frac{d}{dz} \right)^n \exp \left( \phi z - \frac{r}{2} z^2 \right) = \left( \frac{d}{dz} \right)^{n-1} \left( (\phi - rz) \exp \left( \phi z - \frac{r}{2} z^2 \right) \right)$$

to deduce that (12) implies (4).

<sup>13</sup>The study of relationships like this is called **umbral calculus** (Roman and Rota (1978)). Some references are linked in <https://ncatlab.org/nlab/show/umbral+calculus>.

<sup>14</sup>This is stated without proof in Ellis *et al* (2016).

## 6 Another way to write the results

The generating function (11) can also be written

$$\sum_{n=0}^{\infty} \frac{1}{n!} \Phi_n z^n = \exp(\phi z) \exp\left(-\frac{r}{2} z^2\right). \quad (13)$$

Expand the right-hand side in powers of  $z$  to get

$$\begin{aligned} \frac{1}{4!} \Phi_4 &= \frac{\phi^4}{4!} - \left(\frac{r}{2}\right) \frac{\phi^2}{2!} + \text{const} \\ \frac{1}{6!} \Phi_6 &= \frac{\phi^6}{6!} - \left(\frac{r}{2}\right) \frac{\phi^4}{4!} + \frac{1}{2} \left(\frac{r}{2}\right)^2 \frac{\phi^2}{2!} + \text{const} \\ \frac{1}{8!} \Phi_8 &= \frac{\phi^8}{8!} - \left(\frac{r}{2}\right) \frac{\phi^6}{6!} + \frac{1}{2} \left(\frac{r}{2}\right)^2 \frac{\phi^4}{4!} - \frac{1}{3!} \left(\frac{r}{2}\right)^3 \frac{\phi^2}{2!} + \text{const} \end{aligned}$$

and so on. These should be compared to the expressions derived in article [79649](#) for the eigenfunctions of the linearized renormalization group equations.

## 7 Evaluating $r$

Use the same formulation and notation as article [00980](#), treating  $D$ -dimensional space as a lattice of infinite size. In the massless version of the model with  $D \geq 2$ ,<sup>15</sup> the constant  $r$  defined in sections 1 and 2 is

$$r = \int \frac{d^D p}{(2\pi)^D} \frac{1}{2\omega(\mathbf{p})} \quad (14)$$

with

$$\omega(\mathbf{p}) \equiv \sqrt{\sum_k \left( \frac{2 \sin(\mathbf{e}_k \cdot \mathbf{p}/2)}{\epsilon} \right)^2}.$$

The domain of integration is the Brillouin zone (article [71852](#))

$$|p_k| \leq \frac{\pi}{\epsilon}.$$

Instead of trying to evaluate the integral (14) exactly, this section derives easy upper and lower bounds. The inequalities

$$\frac{2\theta}{\pi} \leq \sin \theta \leq \theta$$

hold for all  $0 \leq \theta \leq \pi/2$ , which implies

$$\rho \leq r \leq \frac{\pi}{2}\rho \quad (15)$$

with<sup>16</sup>

$$\rho \equiv \int \frac{d^D p}{(2\pi)^D} \frac{1}{2|\mathbf{p}|} \quad |\mathbf{p}| \equiv \sqrt{\sum_k p_k^2}. \quad (16)$$

<sup>15</sup>For  $D = 1$ , the integral (14) is undefined when the lattice has infinite size. Article [37301](#) explains how to modify the model so that this doesn't cause any problems.

<sup>16</sup>Equation (2.125) in Repko (2016) shows the exact value of the integral (16) when  $D = 3$ , but without the factor of  $2(2\pi)^D$  in the denominator. After adjusting for that factor, the result in Repko (2016) gives  $\rho \approx 0.189$ .

The domain of integration is a  $D$ -dimensional cube with edge-length  $2\pi/\epsilon$ . Define  $\rho_0$  to be the integral with the same integrand but whose domain is the largest sphere contained within that cube. Define  $\rho_1$  to be the integral with the original domain of integration but with the integrand replaced by  $\epsilon/(2\pi)$  wherever  $|\mathbf{p}| > \pi/\epsilon$ . Then

$$\rho_0 \leq \rho \leq \rho_1. \quad (17)$$

The integrals  $\rho_0$  and  $\rho_1$  are both easy to evaluate. Define  $\Omega_D$  by the relationship

$$\rho_0 = \frac{\Omega_D}{(2\pi)^D} \int_0^{\pi/\epsilon} \frac{p^{D-1} dp}{2p}.$$

In words,  $\Omega_D$  is the  $D$ -dimensional version of the “surface area” of the unit sphere. (Examples:  $\Omega_2 = 2\pi$  and  $\Omega_3 = 4\pi$ .) Then

$$\rho_0 = \frac{\Omega_D}{(2\pi)^D} \frac{(\pi/\epsilon)^{D-1}}{2(D-1)}$$

and<sup>17</sup>

$$\rho_1 = \rho_0 + \frac{\epsilon/(2\pi)}{(2\pi)^D} \left( (2\pi/\epsilon)^D - \Omega_D \frac{(\pi/\epsilon)^D}{D} \right).$$

Combine (15) and (17) to get the final result

$$\rho_0 \leq r \leq \frac{\pi}{2} \rho_1.$$

When  $D = 3$ , this becomes

$$\frac{1}{8\epsilon^2} \leq r \leq \frac{1}{4\epsilon^2} \left( \frac{\pi}{12} + 1 \right).$$

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<sup>17</sup>The first term in large parentheses is the volume of the cube with edge-length  $2\pi/\epsilon$ , and the second term is the volume of the sphere with radius  $\pi/\epsilon$ .

## 8 References

(Open-access items include links.)

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## 9 References in this series

Article **00980** (<https://cphysics.org/article/00980>):

“The Free Scalar Quantum Field: Vacuum State”

Article **10690** (<https://cphysics.org/article/10690>):

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