The Free Scalar Quantum Field: Waves

Randy S

Abstract Article 30983 showed that the model of a free scalar quantum field includes states that behave like localized particles. This article shows that the same model includes states that behave more like classical waves. This article also derives two examples of how the field is affected by an **external source**, an entity that influences the field but is not influenced by the field. (This is used as an easy approximation to a more complete – but also more difficult – model in which the influences go both ways.) One example shows that a time-independent source is surrounded by a "force field," analogous to the static electric field that surrounds a point charge in electrodynamics. Another example shows that a time-dependent source generates a state of the quantum field that behaves like a classical wave.

Contents

1 Introduction

In quantum theory, observables are represented by operators on a Hilbert space.[1] In quantum field theory (QFT), observables are usually expressed in terms of auxiliary operators called field operators. The field operators themselves may or may not be observables, depending on the model. In the free scalar model, the field operators $\phi(t, \mathbf{x})$ can be included among the model's observables.[2,3] A measurement of the observable $\phi(t, \mathbf{x})$ can be described as a measurement of the amplitude of the field at the location $\mathbf{x}$ at time t.

In classical physics, *particles* and *fields* are distinct concepts. A given classical model may include both, but they are separate entities. In QFT, the distinction is blurred: a model may have some states that behave approximately like classical fields and some states that behave approximately like localized particles.[4] This happens even in the free scalar model, which has only a single quantum field. Article 30983 explained how the field operators may be used to construct observables that act like particle detectors. This article describes states that behave more like classical fields, to a good approximation.

Any given observable can be measured in any given state, at least in what article 03431 calls the *artificial approach* to measurement. In that trivial sense, we can always apply both field-like concepts or particle-like concepts to the same state, just by deciding what to measure. However, in most states, the outcomes of particle-counting measurements will have a large variance, as will the outcomes of field-amplitude measurements. States that give consistent outcomes with respect to particle-counting observables were constructed for the free scalar model in article 30983. This article constructs states that give consistent outcomes (low variance) with respect to field-amplitude observables instead.

[1] Article 03431

[2] The technical caveat described in article 37301 will be ignored here. It only matters in the infinite-space limit.

[3] Some aspects of the free scalar model are studied in articles 00980 and 30983. As in those articles, $\mathbf{x} = (x_1, x_2, ..., x_D)$ denotes a point in D-dimensional space, and t is the time coordinate.

[4] Most states don't behave like either one, although dynamic phenomena like **decoherence** (article 03431) may tend to select states that do behave approximately like one or the other, depending on the circumstances.

2 The model

A model of a single free scalar quantum field can be constructed in a relatively striaightforward way by treating space as a lattice.[5] This article writes and manipulates integrals and derivatives of the field operators as though the operators were defined at individual points in space. This can be justified by using the lattice-based definition and then agreeing that only low-resolution observables should be taken seriously.

Let $\mathbf{x} = (x_1, ..., x_D)$ denote a point in D-dimensional space, and let t denote the time coordinate. The equation of motion for the free scalar field $\phi(t, \mathbf{x})$ has the form

$$\ddot{\phi} - \nabla^2\phi + m^2\phi = 0. \tag{1}$$

where each overhead dot denotes a derivative with respect to t, ∇ denotes the gradient with respect to $\mathbf{x}$, and m is the mass of one particle. For any m, the field operators satisfy the equal-time commutation relations[6]

$$\begin{aligned} \big[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})\big] &= 0 \\ \big[\dot{\phi}(t,\mathbf{x}), \dot{\phi}(t,\mathbf{y})\big] &= 0 \\ \big[\phi(t,\mathbf{x}), \dot{\phi}(t,\mathbf{y})\big] &= i\delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{2}$$

Observables localized in a spacetime region R are constructed from the field operators $\phi(t, \mathbf{x})$ with $(t, \mathbf{x}) \in R$. The hamiltonian, the operator that generates translations in time, is

$$H = \int d^D x \, \left(\frac{\dot{\phi}^2(t,\mathbf{x}) + \big(\nabla\phi(t,\mathbf{x})\big)^2}{2} + m^2\phi(t,\mathbf{x}) \right) + \text{constant}. \tag{3}$$

The hamiltonian is independent of time,[5] even though the integrand is not. The

[5] Article 52890

[6] $[A, B] \equiv AB - BA$

equation of motion and commutation relations are both satisfied by[7]

$$\phi(t,\mathbf{x}) = \int \frac{d^Dp}{(2\pi)^D}\,\frac{a(\mathbf{p})e^{-i\omega t+i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega}} + \text{adjoint} \tag{4}$$

with $\omega(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2+m^2}$, if the operators $a(\mathbf{p})$ satisfy

$$[a(\mathbf{p}), a(\mathbf{p}')] = 0 \tag{5}$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^D\,\delta(\mathbf{p}'-\mathbf{p}). \tag{6}$$

[7] Article 37301 addresses technicalities in the massless case ($m=0$). Those technicalities won't be important here.

3 Field-amplitude measurements

The field operator $\phi(t,\mathbf{x})$ is the observable that would correspond to a measurement of the field at the spacetime point $(t,\mathbf{x})$, but a real measurement can't actually resolve anything at a *point*. Mathematically, measurements of this observable have huge variance, becoming infinite in the continuum limit (section 5). This is true in any state, including even the vacuum state. That's a feature, not a flaw: it's a sign that the theory "knows" that a real measurement can't actually resolve anything at a *point*. If we ask the theory to predict the outcome of an unrealistic measurement, we shouldn't be surprised when it gives us an unrealistic answer.

Real measurements have finite resolution in space. To construct an observable corresponding to a field-amplitude measurement with finite spatial resolution, let $g(\mathbf{x})$ is a real-valued function with a finite width, and define

$$\phi(t,g) \equiv \int d\mathbf{x}\; g(\mathbf{x})\phi(t,\mathbf{x}). \tag{7}$$

This is called a **smeared** field operator. The width of the function $g(\mathbf{x})$ corresponds to the spatial resolution of the measurement. A convenient example of such a function is

$$g(\mathbf{x}) \propto e^{-\mathbf{x}^2/\sigma^2} \qquad\qquad \int d^Dx\; |g(\mathbf{x})|^2 = 1. \tag{8}$$

Sections 5-6 construct a family of states in which measurements of such observables have low variance. They are states in which the field has a relatively well-defined amplitude. The fact that such states exist helps explain why classical field theory is often a good-enough approximation.[8] Section 7 explains how these states are related to states that behave like particles.

[8]Footnote 4 mentioned another part of the explanation.

4 States that behave like waves: criterion

Let $\rho(\cdots)$ denote any state (article 03431), so that $\rho\big(\phi(t,\mathbf{x})\big)$ is the expectation value of the field operator $\phi(t,\mathbf{x})$. The fact that the equation of motion (1) is linear in the field implies

$$\left(\left(\frac{d}{dt}\right)^2 - \nabla^2 + m^2\right)\rho\big(\phi(t,\mathbf{x})\big) = 0. \tag{9}$$

This is true for any state, but it doesn't necessarily mean that the state behaves like a classical wave, because it doesn't imply that any collection of field-amplitude measurements would be consistent with each other. A necessary condition for a state $\rho(\cdots)$ to have an approximately well-defined value of an observable A is that the variance

$$\rho(A^2) - \rho^2(A)$$

should be small.[9] The variance of the raw field operator $\phi(t,\mathbf{x})$ is not small in any state,[10] but the variance of the smeared field operator (7) can be small in some states if the smearing function is wide enough. Section (5) describes a family of states in which smeared field operators have relatively small variance for all times t. For such states, the concept of a classical field[11] is a good approximation.

[9]In a more realistic model, *small* could be quantified by comparison to something from familiar experience. In this toy model, it can be compared to $\rho^2(A) \equiv \big(\rho(A)\big)^2$.

[10]It scales like $1/\epsilon^{D-1}$ (or like $\log\epsilon$ if $D = 1$), where ϵ is the distance between neighboring points in space. This is finite when space is treated as a lattice, but it diverges in the continuum limit. To derive how it scales with ϵ, start with a result derived in article 00980, which says that the variance of $\phi(t,\mathbf{x})$ is $\int \frac{d^Dp}{(2\pi)^D}(\mathbf{p}^2+m^2)^{-1/2}$ in the infinite-volume limit, where the integral is over a Brillouin zone, whose linear size scales like $1/\epsilon$ (article 71852). For large $\mathbf{p}$, this scales like the integral of p^{D-2} from zero to $1/\epsilon$, which gives the result quoted at the beginning of this footnote. This is consistent with dimensional analysis of the canonical commutation relations, which says that the product $\phi\dot{\phi}$ has the same units as $1/\epsilon^D$.

[11]Article 49705

5 States that behave like waves: construction

For the model defined in section 2, a state $|0\rangle$ satisfying

$$a(\mathbf{p})|0\rangle = 0 \tag{10}$$

has the lowest possible energy, so it represents completely empty space (vacuum). This section constructs a family of states with classical-wave-like properties by applying unitary operators to the vacuum state.

Given any complex-valued function $\beta(\mathbf{p})$, Section 6 constructs a unitary operator U that satisfies

$$U^\dagger a(\mathbf{p}) U = a(\mathbf{p}) + \beta(\mathbf{p}) \tag{11}$$

for all $\mathbf{p}$. If $|0\rangle$ is the vacuum state, then equation (11) implies that the state

$$|\beta\rangle \equiv U|0\rangle \tag{12}$$

satisfies[12]

$$a(\mathbf{p})|\beta\rangle = \beta(\mathbf{p})|\beta\rangle. \tag{13}$$

This is called a **coherent state**.[13] This section shows that $|\beta\rangle$ behaves like a classical wave, according to the criterion in section 4, if $\big|\beta(\mathbf{p})\big|$ is large enough.

Equations (4) and (11) imply

$$U^\dagger \phi(t,\mathbf{x}) U = \phi(t,\mathbf{x}) + \phi_\beta(t,\mathbf{x}) \tag{14}$$

with

$$\phi_\beta(t,\mathbf{x}) \equiv \int \frac{d^D p}{(2\pi)^D}\, \frac{\beta(\mathbf{p}) e^{-i\omega t - i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega}} + \text{complex conjugate}. \tag{15}$$

Using the abbreviations

$$\rho_\beta(\cdots) \equiv \frac{\langle\beta|\cdots|\beta\rangle}{\langle\beta|\beta\rangle} \qquad\qquad \rho_0(\cdots) \equiv \frac{\langle 0|\cdots|0\rangle}{\langle 0|0\rangle},$$

[12] More detail: $a(\mathbf{p})|\beta\rangle = a(\mathbf{p})U|0\rangle = U\big(a(\mathbf{p}) + \beta(\mathbf{p})\big)|0\rangle = \beta(\mathbf{p})|\beta\rangle$.

[13] This name is standard, even though the word *coherent* is heavily overloaded. It's also called a **Glauber state**.

Equation (10) implies

$$\rho_0\big(\phi(t,\mathbf{x})\big) = 0, \tag{16}$$

and combining this with (14) gives

$$\rho_\beta\big(\phi(t,\mathbf{x})\big) = \rho_0\big(\phi(t,\mathbf{x}) + \phi_\beta(t,\mathbf{x})\big) = \phi_\beta(t,\mathbf{x}).$$

This is consistent with equation (9), which holds for all states.

The next goal is to calculate the variance of the smeared field operator (7) in this state. First consider the correlation function

$$f(\mathbf{x}-\mathbf{y}) \equiv \rho_\beta\big(\phi(t,\mathbf{x})\phi(t,\mathbf{y})\big) - \rho_\beta\big(\phi(t,\mathbf{x})\big)\rho_\beta\big(\phi(t,\mathbf{y})\big). \tag{17}$$

Use (14) and (16) to get

$$f(\mathbf{x}-\mathbf{y}) = \rho_0\big(\phi(t,\mathbf{x})\phi(t,\mathbf{y})\big). \tag{18}$$

All dependence on the function β has cancelled. Use equations (6), (4), and (10) in (18) to get

$$f(\mathbf{x}-\mathbf{y}) = \int \frac{d^Dp}{(2\pi)^D}\; \frac{e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega}. \tag{19}$$

This diverges as $\mathbf{y}\to\mathbf{x}$, so the variance of the raw field operator is infinite.[14] On the other hand, the variance of the smeared field operator is

$$\begin{aligned} v &\equiv \rho_\beta\big(\phi^2(t,g)\big) - \rho_\beta^2\big(\phi(t,g)\big) \\ &= \rho_0\big(\phi^2(t,g)\big) \\ &= \int d^Dx\, d^Dy\; g(\mathbf{x})f(\mathbf{x}-\mathbf{y})g(\mathbf{y}). \end{aligned}$$

For the smearing function (8), evaluating the integrals over $\mathbf{x}$ and $\mathbf{y}$ gives

$$v \propto \int \frac{d^Dp}{(2\pi)^D}\; \frac{e^{-\sigma^2\mathbf{p}^2/2}}{2\omega(\mathbf{p})}. \tag{20}$$

[14] Footnote 10 in section 4

To evaluate the integral (20), consider these two cases:

- $m > 0$ and $\sigma \gg 1/m$.
- $m = 0$.

In the first case, the factor $e^{-\sigma^2\mathbf{p}^2/2}$ in the integrand enforces the condition $\mathbf{p}^2 \ll m^2$, so (20) becomes

$$v \approx \frac{1}{2m}\int \frac{d^Dp}{(2\pi)^D}\; e^{-\sigma^2\mathbf{p}^2/2} \propto \frac{1}{m\sigma^D}.$$

The last expression follows just from dimensional analysis. In the second case ($m = 0$), dimensional analysis implies

$$v \propto \frac{1}{\sigma^{D-1}}.$$

In both cases, the variance is small if σ is sufficiently large.

Altogether, this shows that a coherent state of sufficiently large amplitude behaves like a classical wave with respect to sufficiently coarse field-amplitude measurements: the standard deviation in the measurement outcomes is much less than the expectation value.

6 Coherent states

This section explains how to construct a unitary operator that satisfies equation (11).

According to the commutation relations (6), the operator

$$B \equiv \int \frac{d^D p}{(2\pi)^D}\; \beta(\mathbf{p}) a^\dagger(\mathbf{p}) \tag{21}$$

satisfies

$$[a(\mathbf{p}),\, B] = \beta(\mathbf{p}). \tag{22}$$

The operator $B - B^\dagger$ is self-adjoint, so the operator defined by

$$U(\theta) \equiv e^{(B-B^\dagger)\theta} \tag{23}$$

is unitary:

$$U^\dagger(\theta) = U^{-1}(\theta) = U(-\theta).$$

Equation (22) implies[15]

$$\frac{d}{d\theta} U(-\theta) a(\mathbf{p}) U(\theta) = U(-\theta)[a(\mathbf{p}),\, B]U(\theta) = \beta(\mathbf{p}).$$

This is a first-order differential equation for $U(-\theta)a(\mathbf{p})U(\theta)$, with initial condition

$$U(-\theta)a(\mathbf{p})U(\theta)\big|_{\theta=0} = a(\mathbf{p}).$$

The unique solution of this differential equation with this initial condition is

$$U(-\theta)a(\mathbf{p})U(\theta) = a(\mathbf{p}) + \theta\beta(\mathbf{p}). \tag{24}$$

Set $\theta = 1$ to get equation (11).

[15]The derivative was calculated without changing the order of any operators that don't commute with each other.

7 Coherent states and particles

This section explains how to express a coherent state as a superposition of n-particle states. The superposition involves all values of $n \in \{0, 1, 2, ...\}$. More carefully: a coherent state may be written as a superposition of state-vectors that *would, by themselves*, represent different numbers of particles. Conversely, an n-particle state like $B^n|0\rangle$ could be written as a superposition of state-vectors that would, by themselves, represent classical-ish waves with different amplitudes. Remember, though, that thinking of a coherent state as being "made of" particles – or conversely – is no more (or less) correct than thinking of a diagonal vector as being "made of" vertical and horizontal vectors.

Consider the operators

$$V(\theta) \equiv \exp\left(\frac{\theta^2}{2}[B, B^\dagger]\right) e^{\theta B} e^{-\theta B^\dagger}. \tag{25}$$

Clearly, $V(0) = 1$. The commutator $[B, B^\dagger]$ is proportional to the identity operator (it commutes with everything), so $V(\theta)$ satisfies[16]

$$\frac{d}{d\theta} V(\theta) = \big(\theta[B, B^\dagger] + B\big) V(\theta) - V(\theta) B^\dagger.$$

A derivation similar to the one that led from (23) to (24) gives

$$e^{\theta B} B^\dagger e^{-\theta B} = B^\dagger + \theta[B, B^\dagger],$$

and using this on the right-hand side of the previous equation gives

$$\frac{d}{d\theta} V(\theta) = (B - B^\dagger) V(\theta).$$

The operators $U(\theta)$ defined by equation (23) clearly also satisfy

$$\frac{d}{d\theta} U(\theta) = (B - B^\dagger) U(\theta) \qquad\qquad U(0) = 1.$$

[16]The derivative was calculated without changing the order of any operators that don't commute with each other.

These conditions can only have one solution, so

$$U(\theta) = V(\theta).$$

Together with equations (12), (25), and $a(\mathbf{p})|0\rangle = 0$, this implies

$$|\beta\rangle \propto e^B|0\rangle \equiv \sum_n \frac{B^n}{n!}|0\rangle.$$

Article 30983 shows that $B^n|0\rangle$, by itself, would be a state with n particles, but remember: thinking of a coherent state as being "made of" particles – or conversely – is no more (or less) correct than thinking of a diagonal vector as being "made of" vertical and horizontal vectors.

8 Classical versus quantum superposition

Let $U(\beta)$ be the unitary operator defined as in sections 5-6, with $\theta = 1$, for a given amplitude-function $\beta(\mathbf{p})$.[17]

A *classical* superposition of two waves $\beta(\mathbf{p})$ and $\beta'(\mathbf{p})$ is given by

$$U(\beta)U(\beta')|0\rangle = U(\beta + \beta')|0\rangle. \tag{26}$$

This is another coherent state, now with amplitude-function $\beta+\beta'$. That's different from the *quantum* superposition

$$U(\beta)|0\rangle + U(\beta')|0\rangle. \tag{27}$$

This is *not* a coherent state (unless $\beta' = \beta$).

Even when $\beta' = \beta$, the two states (26) and (27) are different from each other. The quantum superposition (27) is $2U(\beta)|0\rangle$, which is proportional to $U(\beta)|0\rangle$. Mutually proportional state-vectors are physically equivalent: they both represent precisely the same physical scenario.[18] In contrast, the classical superposition (26) is $U(2\beta)|0\rangle$, which is physically distinct from $U(\beta)|0\rangle$, because the amplitude of the wave's oscillations – which is observable – is twice as great as what it would be in the state $U(\beta)|0\rangle$.

[17] Here, the argument of $U(\beta)$ specifies the function $\beta(\mathbf{p})$. This is different than the notation $U(\theta)$ that was used in section 6.

[18] Article 03431

9 The concept of an external source

In electrodynamics, influences go both ways: charges and currents influence the behavior of the electromagnetic field, and the electromagnetic field influences the behavior of charges and current. When learning classical electrodynamics, we often use a simplified model in which the influence goes only one way. In particular, we often use Maxwell's equations with charges and currents whose behavior is prescribed instead of being influenced by the field.

We can use a similar simplification in quantum physics: instead of using a model in which two quantum entities influence each other, we can use a model in which an external entity influences a quantum entity but is not influenced by the quantum entity. We can take the external entity to be a *classical* entity, meaning that all of its associated observables commute with each other.[19]

Sections 10-13 study the effect of a classical **external source** – a source that influences the scalar field but but is not influenced by the scalar field. The external source is analogous to a prescribed distribution of charge or current in classical electrodynamics. We will see that the effect of the external source is similar to its effect in the classical model: a time-independent source is surrounded by a static field (like the Coulomb field), and an oscillating source generates a propagating wave in the field.

[19]We cannot do the opposite: mathematically, we cannot define a model in which a quantum entity (one whose associated observables don't all commute with each other) influences a classical entity (one whose associated observables do all commute with each other), because the influence would necessarily push noncommutativity into the "classical" entity's future observables, contradicting the assertion that it is a "classical" entity. Some of the literature about measurement in quantum theory refers to the idea of a quantum system influencing a classical one. That makes sense if we read "classical" to mean "large and complex, but still quantum." It doesn't make sense if we read "classical" to mean "involving only observables that strictly commute with each other."

10 The model with an external source

The external source is a prescribed function $J(t,\mathbf{x})$. In the model with this external source, the commutation relations are the same as before (equations (2)), but the field's time-dependence is governed by the modified equation of motion[20]

$$\ddot{\phi}(t,\mathbf{x})-\nabla^2\phi(t,\mathbf{x})+m^2\phi(t,\mathbf{x})=J(t,\mathbf{x}). \tag{28}$$

The observable representing the system's total energy is[21]

$$H(t)=\int d^Dx\;\left(\frac{\dot{\phi}^2(t,\mathbf{x})+\big(\nabla\phi(t,\mathbf{x})\big)^2+m^2\phi^2(t,\mathbf{x})}{2}\right.\\ \left.-\,\phi(t,\mathbf{x})J(t,\mathbf{x})\right)+\text{constant}. \tag{29}$$

The equation of motion (28) can be used to show that if $J(t,\mathbf{x})$ is independent of time, then the hamiltonian (29) is also independent of time. If the external source $J(t,\mathbf{x})$ is not independent of time, then neither is the hamiltonian $H(t)$, so in this case the total energy of the quantum system is not conserved. Intuitively, this is because the external source is prescribed: it can influence, but cannot be influenced by, the quantum system. One-way influences break symmetries that would otherwise lead to conservation laws, like they do in classical physics.[22]

As usual, each local algebra $\Omega(R)$ is defined to be the algebra generated by the field operators $\phi(t,\mathbf{x})$ for all points $(t,\mathbf{x})$ inside R.

[20]Compare this to (1).

[21]Compare this to (3). Article 85870 briefly explains the justification for this model.

[22]Article 33629

11 How the source affects the field operator

The commutation relations (2) and the new equation of motion (28) are both satisfied by

$$\phi(t,\mathbf{x}) = \phi_0(t,\mathbf{x}) + \phi_J(t,\mathbf{x}), \tag{30}$$

where $\phi_0(t,\mathbf{x})$ is a new notation for the original operator (4) and $\phi_J(t,\mathbf{x})$ is an ordinary function (not operator)[23] that satisfies

$$\ddot{\phi}_J(t,\mathbf{x}) - \nabla^2\phi_J(t,\mathbf{x}) + m^2\phi_J(t,\mathbf{x}) = J(t,\mathbf{x}). \tag{31}$$

Equation (31) has infinitely many different solutions, but the effect of choosing a different solution can be compensated by choosing a different state (equation (14)) if the difference between the two solutions has the form (15).[24]

[23] In equation (30), the term denoted ϕ_J is understood to mean the function ϕ_J times the identity operator.

[24] This doesn't work if the difference between the two solutions has the time-independent form $\phi_J \propto \exp(m\mathbf{u}\cdot\mathbf{x})$ with $\mathbf{u}^2 = 1$ (or the corresponding form $\phi_J \propto \mathbf{u}\cdot\mathbf{x}$ in the case $m = 0$), which satisfies (31) with $J = 0$, but such a configuration has infinite energy, which is why it was excluded from (4) in the first place.

12 The effect of a time-independent source

Consider the case where the external source is independent of time. In this case, we might as well choose the classical solution ϕ_J to be time-independent. This section shows that the minimum-energy state in this case includes a non-zero field-amplitude and that the variance is small according to measurements with coarse enough resolution.

By definition, ϕ_J solves (31). Use the time-independence of ϕ_J to see that the field operator (30) may be written

$$\phi(t,\mathbf{x}) = \int \frac{d^Dp}{(2\pi)^D}\,\frac{\big(a(\mathbf{p})e^{-i\omega t}+\beta(\mathbf{p})\big)e^{-i\mathbf{p}\cdot\mathbf{x}} + \text{adjoint}}{\sqrt{2\omega}}, \tag{32}$$

where the complex-valued function $\beta(\mathbf{p})$ is defined by

$$\phi_J(\mathbf{x}) = \int \frac{d^Dp}{(2\pi)^D}\,\frac{\beta(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega}} + \text{complex conjugate} \tag{33}$$

and the operators $a(\mathbf{p})$ still satisfy the same commutation relations (6). Use (32) to see that the hamiltonian may be written

$$H = \int \frac{d^Dp}{(2\pi)^D}\,\omega\, a^\dagger(\mathbf{p})a(\mathbf{p}) + \text{constant}. \tag{34}$$

This can also be derived by substituting the modified field operator (32) into the modified hamiltonian (29), which shows that the constant term is (the identity operator times) an integral involving J and ϕ_J.[25] Equation (34) shows that the spectrum condition is still satisfied and that a non-zero state-vector $|0\rangle$ satisfying

$$a(\mathbf{p})|0\rangle = 0 \tag{35}$$

still gives a state of the lowest possible energy (zero).

[25] Article 85870 derives an explicit expression for this integral.

Now consider the mean and variance of the field operator (30) with respect to this state. Use the abbreviation

$$\rho(\cdots) \equiv \frac{\langle 0|\cdots|0\rangle}{\langle 0|0\rangle}, \tag{36}$$

and use (35) to get

$$\rho\big(\phi_0(t,\mathbf{x})\big) = 0.$$

This shows that the expectation value of the new field operator is

$$\rho\big(\phi(t,\mathbf{x})\big) = \phi_J(\mathbf{x}), \tag{37}$$

and it can also be used to show that the variance

$$\rho\big(\phi^2(t,g)\big) - \rho^2\big(\phi(t,g)\big)$$

is not affected by the external source, where $\phi(t,g)$ is a smeared version of (32).

Altogether, in the presence of the external source, the minimum-energy state includes a non-zero field-amplitude and that the variance is small according to measurements with sufficiently coarse resolution. In other words, an isolated charge is surrounded by a "force field," a special configuration of the quantum field that remains after all temporary disturbances have radiated away. This is analogous to the static electric field that surrounds a point charge in electrodynamics. To make the analogy more explicit, consider a point-like source

$$J(\mathbf{x}) \propto \delta(\mathbf{x}),$$

and suppose that the field is massless ($m = 0$) and that space is three-dimensional ($D = 3$). Then equation (31) reduces to $-\nabla^2\phi_J(\mathbf{x}) = J(\mathbf{x})$, which is satisfied by

$$\phi_J(\mathbf{x}) \propto \frac{1}{|\mathbf{x}|}. \tag{38}$$

This is form of the "force field" (37) surrounding a pointlike charge when $m = 0$ and $D = 3$.

13 Quantum radiation from a classical source

Now suppose that the external source and the classical solution ϕ_J are independent of time for $t < 0$, but become time-dependent for $t > 0$. In this case, we can interpret the state $|0\rangle$ satisfying (10) as representing empty space for $t < 0$. How does this state behave for $t > 0$?

Using the abbreviation (36) again, the expectation value of ϕ_0 in this state is

$$\rho\big(\phi_0(t, \mathbf{x})\big) = 0$$

for all t. As in sections 5 and (12), we can use this to confirm that the expectation value of the new field operator is

$$\rho\big(\phi(t, \mathbf{x})\big) = \phi_J(t, \mathbf{x}) \tag{39}$$

and that the variance is not affected by the external source. This shows that if the classical model predicts radiation, then so does the quantum model.

Here's another way to reach the same conclusion.[26] When the external source depends on time, the time-evolution of the field operator $\phi(t, \mathbf{x})$ is no longer generated by any time-independent hamiltonian,[27] but we can still define unitary operators $U(t)$ parameterized by t so that

$$\phi(t, \mathbf{x}) = U^\dagger(t)\phi(0, \mathbf{x})U(t). \tag{40}$$

Then $U(t)|0\rangle$ is the time-dependent state in the Schrödinger picture. To construct $U(t)$, define $\beta(t, \mathbf{p})$ so that

$$\phi_J(\mathbf{x}) = \int \frac{d^Dp}{(2\pi)^D}\, \frac{\beta(t, \mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2\omega}} + \text{complex conjugate.} \tag{41}$$

[26] A similar analysis with more narration is posted here: https://physics.stackexchange.com/q/443760

[27] In the present example, the hamiltonian (34) works for $t < 0$, but no time-independent hamiltonian works for $t > 0$.

For each time t, we can use same approach as in section 6 – after replacing $\beta(\mathbf{p})$ with $\beta(t, \mathbf{p})$ in equation (21) – to construct unitary operators $U_J(t)$ satisfying

$$U_J^\dagger(t) a(\mathbf{p}) U_J(t) = a(\mathbf{p}) + \beta(t, \mathbf{p}).$$

Then the operators[28]

$$U(t) \equiv U_J^\dagger(0) U_J(t) e^{-iHt}$$

satisfy (40), with H given by (34). The alternate expression for $U_J(t)$ derived in section 7 then gives this expression for the Schrödinger-picture state:[29]

$$U(t)|0\rangle = U_J^\dagger(0) U_J(t)|0\rangle \propto e^{B(t)-B(0)}|0\rangle,$$

where $B(t)$ is defined by equation (21) with $\beta(\mathbf{p})$ replaced by $\beta(t, \mathbf{p})$. This shows that in the Schrödinger picture, starting with the vacuum state $|0\rangle$ and then turning on an external source at time $t > 0$ produces a coherent state (Glauber state).

[28]The factor $U_J^\dagger(0)$ is not needed if $J(t, \mathbf{x}) = 0$ for $t \leq 0$, but it is included here so that it works for arbitrary $J(t, \mathbf{x})$.

[29]This result uses $H|0\rangle = 0$.

14 References in this series

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"The Free Scalar Quantum Field: Vacuum State" (version 2023-11-12)

Article **03431** (https://cphysics.org/article/03431):
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