The Wilsonian Effective Action with Scalar Quantum Fields

Randy S

Abstract In quantum field theory, we don't yet know how to define most models of interest directly in continuous spacetime. As a workaround, we can define many models of interest by treating spacetime as a lattice, with the understanding that the model is only meant to be used at resolutions much coarser than the lattice scale. This works well because of **universality**: many of a model's details don't have any significant effect on the model's predictions at such low resolutions. Article 10142 introduced some of the general concepts used in the study of universality. This article introduces a specific method for studying universality in greater quantitative detail.

Contents

1 Introduction

Thanks to **universality**, different models may be practically indistinguishable from each other at sufficiently low energy.[1] For models that differ only slightly from a given scale-invariant model, we can use the concept of **scaling dimension** to anticipate which kinds of changes are likely to be important at low energies and which ones are not. In d-dimensional spacetime, if the model's action is modified by adding an operator $\mathcal{O}$ with scaling dimension Δ, then this change tends to be **relevant** (increasingly important at lower energies) if $\Delta < d$, and it tends to be **irrelevant** (decreasingly important at lower energies) if $\Delta > d$. This is explained in article 10142.

When $\Delta = d$, the argument based on scaling dimensions does not discern whether the change's importance increases or decreases importance as the energy is decreased. This article uses models of a single scalar field to explain how the **(wilsonian) effective action**[2] can be used to quantify the rate at which the effect of a given change increases or decreases as the energy is decreased,[3] at least in the vicinity of a trivial[4] scale-invariant model, where approximations are weak enough to justify expanding in powers of the interaction strength(s). Results from this method corroborate the rule that perturbations with $\Delta < d$ and $\Delta > d$ are *relevant* and *irrelevant*, respectively, and they reveal which way the balance tips when $\Delta = d$.

[1]In this article, *low energy*, *low momentum*, and *low resolution* are all synonymous.

[2]The adjective *wilsonian* (or *Wilson*) is meant to distinguish this effective action from the generator of *one-particle irreducible (1PI) functions*, which is also often called an *effective action*. Both kinds of effective action are useful in the study of renormalization and universality, but they are distinct concepts.

[3]An effective action can also be used as an ansatz for the action of an approximate low-energy model whose coefficients are to be determined either by fitting the model directly to experimental data or by comparing selected low-energy predictions to those of a more fundamental model (Lepage (1989)). An example is non-relativistic quantum electrodynamics (NRQED). NRQED uses an ansatz with a state-dependent low-energy cutoff, the cutoff depending on the number of each species of particle in the state. Choosing the energy cutoff to be just above the combined rest-energy of the assumed set of particles is equivalent to restricting each particle's momentum to be much less than its mass – the condition that defines the nonrelativistic approximation. NRQED uses an ansatz that exploits this state-dependent low-energy condition.

[4]Here, *trivial* means that interactions are absent.

2 The models used in this article

Consider a model involving only a single scalar field[5] in d-dimensional spacetime, which will be treated as a lattice, and let I be some time-ordered product of field operators and their expectation values.[6] This article focuses on models in which the vacuum expectation value of the operator I can be reconstructed from the **euclidean path integral**[7]

$$\langle I \rangle \equiv \frac{\int [d\phi]\; e^{-S[\phi]} I[\phi]}{\int [d\phi]\; e^{-S[\phi]}} \tag{1}$$

using Wick rotation. This includes models that are effectively Lorentz symmetric at resolutions much coarser than the lattice scale. The insertion $I[\phi]$ (which represents the operator I) and the euclidean action $S[\phi]$ are both expressed in terms of the scalar field variables $\phi(x)$ and their discretized derivatives with respect to x. In this article, the euclidean action $S[\phi]$ will just be called the **action**.

[5]The generalization to multiple scalar fields is straightforward.

[6]Mnemonic: I stands for *insertion* or *integrand*, because of the way this operator is represented in the path integral (1).

[7]Article 63548

3 Momentum-domain field variables

Let ϵ denote the lattice spacing, the distance between nearest-neighbor points in the lattice. The original field variables $\phi(x)$, one for each point in the lattice, can be written in terms of a different set of variables $\tilde{\phi}(p)$ like this:[8]

$$\phi(x) = \frac{1}{L^d}\sum_p \; e^{ip\cdot x}\tilde{\phi}(p), \tag{2}$$

where $L \equiv K\epsilon$ is the linear size of the lattice and K is the number of lattice sites along any of the canonical axes. The components of the **momentum**[9] p are integer multiples of $2\pi/L$.

The path integral (1) may also be written as an integral over the momentum-domain field variables $\tilde{\phi}(p)$, because the integration measures $[d\tilde{\phi}]$ and $[d\phi]$ are proportional to each other.[10,11] This gives the equivalent expression

$$\langle I\rangle = \frac{\int [d\tilde{\phi}]\; e^{-S[\phi]}I[\phi]}{\int [d\tilde{\phi}]\; e^{-S[\phi]}}. \tag{3}$$

[8]The coefficient is chosen so that this relationship becomes $\phi(x) = \int \frac{d^dp}{(2\pi)^d}\; e^{ip\cdot x}\tilde{\phi}(p)$ in the infinite-volume limit.

[9]Here, the word *momentum* is being used as in article 71852. This is related to, but only rarely equivalent to, the *momentum* defined by the operators that generate translations in space (article 30983).

[10]The Jacobian of the transformation (2) is a constant (independent of the field variables).

[11]The variables $\tilde{\phi}(p)$ are complex-valued, with $\tilde{\phi}^*(p) = \tilde{\phi}(-p)$. The measure $[d\tilde{\phi}]$ is defined to be the product of the measures of the real and imaginary parts for each $\tilde{\phi}(p)$.

4 Restricting resolution by restricting momentum

For predictions involving only resolutions much coarser than the lattice scale, the insertions $I[\phi]$ used in (3) only need to involve momenta whose components all have magnitude much less than $1/\epsilon$. This condition can also be written

$$|p| \ll \frac{1}{\epsilon} \qquad \text{with} \quad |p| \equiv \left(\sum_k |p_k|^2\right)^{1/2}, \tag{4}$$

where the sum is over the d components of p. This way of expressing the low-resolution condition works in the context of the lorentzian version of the path integral, too, even though the quantity $|p|$ is not Lorentz invariant.[12] When the euclidean version of the path integral is used, as in this article, the original Lorentz symmetry of the model in d-dimensional spacetime becomes rotational symmetry in d-dimensional euclidean space, and then expressing the low-resolution condition in terms of $|p|$ is natural because $|p|$ is invariant under those rotations – at least in the infinite-volume limit, where the components of p become continuous real variables.

[12] *Low resolution* is not a Lorentz symmetric concept anyway.

5 A generalized UV cutoff

The lattice acts as a **UV cutoff**, because it limits the set of momenta to a bounded domain. In equation (2), the sum is over all momenta p in the **Brillouin zone**: each component of p is an integer multiple of $2\pi/L$ in the range between $-\pi/\epsilon$ and π/ϵ. Geometrically, the **Brillouin zone** has the shape of a hypercube (the d-dimensional analog of a cube).

More generally, we can define modified field variables

$$\phi_\Gamma(x) \equiv \frac{1}{L^d} \sum_{p \in \Gamma} e^{ip\cdot x} \tilde{\phi}(p) \tag{5}$$

for any given subset Γ of the Brillouin zone. If Γ is not the full Brillouin zone, then the variables $\phi_\Gamma(x)$ are not all linearly independent of each other, and they are not strictly local: for each spacetime point x, the quantity $\phi_\Gamma(x)$ may be written in terms of the original field variables $\phi(x')$ (by writing $\tilde{\phi}(p)$ in terms of $\phi(x')$), and that expression may involve points x' from all of spacetime. However, if Γ only excludes momenta with large magnitudes $|p| \sim 1/\epsilon$, then $\phi_\Gamma(x)$ is still effectively local as far as the model's low-resolution predictions are concerned.

When the variables $\phi_\Gamma(x)$ are used, the lattice (the set of allowed values of x) is still the same as it was before, but the reduced domain Γ acts as a more restrictive UV cutoff. This allows the UV cutoff and the lattice spacing ϵ to be varied independently of each other,[13] subject only to the constraint that Γ is a subset of the Brillouin zone (not a superset). This liberty will be used to implement a generalized version of the *momentum-shell renormalization group*.

Conceptually, the purpose of treating spacetime as a lattice is to ensure that everything is mathematically well-defined. The UV cutoff defined by Γ is an additional concept used to implement the renormalization group. Section 6 will highlight one benefit of distinguishing between these two concepts.

[13] Footnote 54 in section 22

6 The goal

With ϕ_Γ defined as in equation (5), suppose that the original action has the form

$$S[\phi_\Gamma] = \epsilon^d \sum_x \left(\frac{\big(\partial\phi_\Gamma(x)\big)^2}{2} + \sum_n c_n \mathcal{O}_n[\phi_\Gamma(x)] \right) \tag{6}$$

where each $\mathcal{O}_n[\phi_\Gamma(x)]$ is a product of the variables $\phi_\Gamma(x)$ and its derivatives of arbitrarily high order: $\partial_a\phi_\Gamma(x)$, and $\partial_a\partial_b\phi_\Gamma(x)$, and so on. In this case, equation (3) is generalized to[14]

$$\langle I\rangle \equiv \frac{\int [d\tilde{\phi}]_\Gamma \; e^{-S[\phi_\Gamma]} I[\phi_\Gamma]}{\int [d\tilde{\phi}]_\Gamma \; e^{-S[\phi_\Gamma]}}. \tag{7}$$

Now the path integral is only over the quantities $\tilde{\phi}(p)$ with $p \in \Gamma$. If the coefficients c_n were all zero,[15] then taking the straightforward continuum limit would give a scale-invariant model.

The goal is to quantify the degree to which the various perturbations $\mathcal{O}_n$ tend to affect the model's predictions at low resolution, compared to the predictions of that scale-invariant model. Expressing the goal this way, using a domain Γ that is not necessarily the full Brillouin zone, allows using a domain with a more convenient shape – namely a sphere[16] – while still treating spacetime as a lattice. The quantitative details of the renormalization group analysis will depend on the shape of Γ,[17] because the perturbations $\mathcal{O}_n$ in equation (7) do, but the implications for the model's low-resolution "physical" predictions do not.[18]

[14] The integration measure is written $[d\phi]_\Gamma$ instead of $[d\phi_\Gamma]$ because the latter notation would be technically incorrect – because the variables $\phi_\Gamma(x)$ are not all independent of each other (section 5).

[15] In this case, the zero-momentum part must be excluded from $[d\tilde{\phi}]_\Gamma$.

[16] In the infinite-volume limit, the components of p become continuous variables.

[17] Even if Γ is the largest sphere that can be inscribed in the Brillouin zone, the difference between Γ and the full Brillouin zone is still significant. Define $\rho(d)$ be the d-dimensional version of the ratio (volume of inscribed sphere)/(volume of cube). Then $\rho(d) = \frac{\pi}{2n}\rho(d-2)$ with $\rho(1) = \rho(0) = 1$, so $\rho(d)$ is a monotonically decreasing function of d for all integers $d \geq 1$, and it approaches zero as $d \to \infty$. Examples: $\rho(2) = \frac{\pi}{4} \approx 0.79$, $\rho(3) = \frac{\pi}{6} \approx 0.52$, $\rho(4) = \frac{\pi^2}{32} \approx 0.31$, and $\rho(5) = \frac{\pi^2}{60} \approx 0.16$.

[18] Models involving only scalar fields are just toy models, not meant to have realistic applications, but the concepts

7 The momentum-shell renormalization group

Article 10142 introduced the idea of the **renormalization group**, expressed in terms of observables. The renormalization group formalizes the question "how do a model's predictions vary with resolution?" This sections outlines a particular way of implementing the renormalization group, and the rest of this article will describe it in more detail.

Let Γ_0 be a subset of the Brillouin zone, and let $\Gamma_1 \subset \Gamma_0$ be a slightly smaller subset. Use the abbreviations

$$\phi_0 \equiv \phi_{\Gamma_0} \qquad\qquad \phi_1 \equiv \phi_{\Gamma_1} \tag{8}$$

with $\phi_\Gamma(x)$ defined by equation (5). Two examples are worth keeping in mind:[19]

- Γ_0 could be the spherical region with radius $|p| = \pi/\epsilon$ (the largest spherical region that fits inside the Brillouin zone), and Γ_1 could be a spherical region with a slightly smaller radius.

- Γ_0 could be the whole Brillouin zone, and Γ_1 could be the smaller domain defined by restricting each component of p to be something slightly less than π/ϵ. In this case, Γ_0 and Γ_1 each have the shape of a d-dimensional cube.

In either case, let λ denote the ratio $\Lambda_0/\Lambda_1 > 1$, where Λ_k is the linear size of the domain Γ_k. The ratio λ will be called the **scale factor**.

Start with the action

$$S[\phi_0] = \epsilon^d \sum_x \left(\frac{\big(\partial\phi_0(x)\big)^2}{2} + \sum_n c_n \mathcal{O}_n[\phi_0(x)], \right) \tag{9}$$

which is equation (6) with $\Gamma = \Gamma_0$. The goal is to quantify the degree to which the various perturbations $\mathcal{O}_n$ tend to affect the model's predictions at low resolution, compared to what those predictions would be if the perturbations $\mathcal{O}_n$ were absent.

introduced in this article also apply to models that do have realistic applications.

[19] Special attention will be given to spherical domains, but most of the analysis in this article is valid for either shape.

The first step is to construct an action with the slightly lower UV cutoff but that still gives the same low-energy predictions as the original model. This will be done by (approximately) evaluating the path integral over the field variables $\tilde{\phi}(p)$ with p in the **shell**[20] – in Γ_0 but not in Γ_1. Suppose that the collection of $\mathcal{O}_n$s is complete, in the sense that the result of integrating out the those high-energy modes is equivalent to replacing the original action (9) with an *effective action*[21]

$$S_{1,\epsilon}[\phi_1] = \epsilon^d \sum_x \left(Z(\lambda) \frac{\big(\partial\phi_1(x)\big)^2}{2} + \sum_n \tilde{c}_n(\lambda)\mathcal{O}_n[\phi_1(x)] \right) \tag{10}$$

with λ-dependent coefficients. For later convenience, the lattice spacing ϵ – which has not changed (yet) – is indicated by a subscript in the notation for the effective action. By construction, for any $\lambda > 1$, the actions (9) and (10) both produce the same predictions for low-energy observables, which can be represented by insertions of the form $I[\phi_1]$. Using (10) in place of (9) is equivalent to discarding observables that involve higher energies, so this implements the first step in what article 10142 calls the *irreversible* renormalization group.[22]

The next step is to replace ϵ with $\lambda\epsilon$ so that the new UV cutoff in units of $1/(\lambda\epsilon)$ is numerically equal to the original UV cutoff in units of $1/\epsilon$. This replacement is made in two places: in the factor ϵ^d that multiples the overall sum over points in spacetime, and in the denominator of every (discretized) derivative with respect to a spacetime coordinate. After this replacement, the effective action is

$$S_{1,\lambda\epsilon}[\phi_1] = (\lambda\epsilon)^d \sum_x \left(\frac{Z(\lambda)}{\lambda^2} \frac{\big(\partial\phi_1(x)\big)^2}{2} + \sum_n \frac{\tilde{c}_n(\lambda)}{\lambda^{N_\partial}} \mathcal{O}_n[\phi_1(x)] \right) \tag{11}$$

[20] In other contexts, the name *momentum shell* is also used for something different, namely for momenta satisfying $p^2 = M^2$ for some specified mass scale M. In contrast, the momentum shell defined in this article has nonzero thickness (a range of values of p^2).

[21] Section 8 will describe this step in more detail.

[22] This implementation is called the **momentum shell renormalization group** (chapter 15 in Fradkin (2021), and the online version Fradkin (2022)), because it involves integrating over the field variables in a thin shell in the momentum (energy) domain.

where N_∂ is the number of derivatives in the term $\mathcal{O}_n$. This implements the second step in what article 10142 calls the irreversible renormalization group.

The goal is to quantify the rate at which the effect of a given term $\mathcal{O}_n$ increases or decreases as the UV cutoff is decreased. The "effect" is relative to the scale-invariant model that we would have if all of the coefficients c_n were zero. If the coefficients of the $(\partial\phi)^2$ terms in equations (9) and (11) were equal to each other, then the λ-dependence of the other terms would quantify the tendency of their effects to increase or decrease as the UV cutoff is decreased. We can equalize the coefficients of the $(\partial\phi)^2$ terms by rescaling the field variables in the effective action. This doesn't affect the model's predictions because the field variables are just integration variables in the path integral. In terms of the rescaled field variables

$$\phi_1'(x) \equiv \phi_1(x) \times \big(Z(\lambda)\,\lambda^{d-2}\big)^{1/2}, \tag{12}$$

the effective action (11) is

$$S_{1,\lambda\epsilon}[\phi_1] = \epsilon^d \sum_x \left(\frac{\big(\partial\phi_1'(x)\big)^2}{2} + \sum_n c_n(\lambda)\mathcal{O}_n[\phi_1'(x)] \right) \tag{13}$$

with

$$c_n(\lambda) \equiv \frac{\lambda^d}{\big(Z(\lambda)\,\lambda^{d-2}\big)^{N_\phi/2}\,\lambda^{N_\partial}}\,\tilde{c}_n(\lambda), \tag{14}$$

where N_ϕ is the number of factors of ϕ in the term $\mathcal{O}_n$. Now the coefficient of the $(\partial\phi)^2$ term is the same as in the original action (9), so we can quantify the rate at which the effect of another term $\mathcal{O}_n$ increases or decreases as the cutoff is decreased by comparing the λ-dependent coefficient $c_n(\lambda)$ to the corresponding coefficient $c_n = c_n(\lambda)\big|_{\lambda=1}$ in the original action (9).

Since the goal is to understand the degree to which the various perturbations $\mathcal{O}_n$ affect the model's low-energy predictions, a more direct approach would be to derive explicit expressions for all of the model's low-energy predictions as functions of the coefficients c_n in the original action (9). That would be more direct, but the calculations would be more difficult. The approach outlined here is easier.

8 The effective action

Write

$$\phi_0(x) = \phi_1(x) + \chi(x) \tag{15}$$

with

$$\chi(x) \equiv \frac{1}{L^d} \sum_{p \in \text{shell}} e^{ip\cdot x} \tilde{\phi}(p) \tag{16}$$

In words, $\chi(x)$ involves only momenta that are in Γ_0 but not in Γ_1. Then the path integral (7) may also be written

$$\langle I \rangle = \frac{\int [d\tilde{\phi}]_0 \ \exp\big(-S[\phi_1 + \chi]\big) I[\phi]}{\int [d\tilde{\phi}]_0 \ \exp\big(-S[\phi_1 + \chi]\big)} \qquad\qquad [d\tilde{\phi}]_0 \equiv [d\tilde{\phi}]_{\Gamma_0} \tag{17}$$

This way of writing the path integral is useful when the insertion $I[\phi]$ involves only low momenta – that is, when

$$I[\phi] = I[\phi_1]. \tag{18}$$

In this case, the action S is only thing in the integrand (17) that depends on the variables $\tilde{\phi}(p)$ with $p \notin \Gamma_1$, so (17) may be written

$$\langle I \rangle = \frac{\int [d\tilde{\phi}]_1 \ \exp\big(-S_{1,\epsilon}[\phi_1]\big) I[\phi_1]}{\int [d\tilde{\phi}]_1 \ \exp\big(-S_{1,\epsilon}[\phi_1]\big)} \tag{19}$$

where the **effective action** $S_{1,\epsilon}[\phi_1]$ is defined by the result of doing the integrals over the high-momentum parts:

$$\boxed{\exp\big(-S_{1,\epsilon}[\phi_1]\big) \equiv \int [d\tilde{\phi}]_{\text{shell}} \ \exp\big(-S[\phi_1 + \chi]\big).} \tag{20}$$

The path integral in (19) is only over the variables $\tilde{\phi}(p)$ with $p \in \Gamma_1$, and the path integral in (20) is only over the variables $\tilde{\phi}(p)$ with p in the momentum shell (in Γ_0 but not in Γ_1).

9 Separating the high- and low-momentum parts

As an example, the analysis from here through section 26[23] starts with the action

$$S[\phi_0] = S^{\text{free}}[\phi_0] + V[\phi_0] \tag{21}$$

with[24]

$$\begin{aligned} S^{\text{free}}[\phi] &\equiv \epsilon^d \sum_x \left(\frac{\big(\partial\phi(x)\big)^2}{2} + c_2 \frac{\phi^2(x)}{2} \right) \\ V[\phi] &\equiv \epsilon^d \sum_x c_4 \frac{\phi^4(x)}{4!}. \end{aligned} \tag{22}$$

To begin evaluating the right-hand side of (20), the write $\phi_0 = \phi_1 + \chi$ as before, and expand the action in powers of χ. Writing S^{free} in terms of the Fourier-transformed variables $\tilde\phi$ gives[25]

$$S^{\text{free}}[\phi_0] = \frac{1}{L^d} \sum_{p\,\in\,\Gamma_0} \frac{p^2 + c_2}{2} \big|\tilde\phi(p)\big|^2. \tag{23}$$

Each term in the sum involves the variables $\tilde\phi$ for only one value of $|p|$, so

$$S^{\text{free}}[\phi_1 + \chi] = S^{\text{free}}[\phi_1] + S^{\text{free}}[\chi].$$

Use this to get

$$S[\phi_1 + \chi] = S[\phi_1] + S^{\text{free}}[\chi] + \delta V[\phi_1, \chi]$$

with

$$\delta V = \epsilon^d \sum_x c_4 \times \left(\frac{\phi_1^3(x)\,\chi(x)}{3!} + \frac{\phi_1^2(x)\,\chi^2(x)}{2} + \frac{\phi_1(x)\,\chi^3(x)}{3!} + \frac{\chi^4(x)}{4!} \right). \tag{24}$$

[23] Chapter 15 in Fradkin (2021) (or the online version Fradkin (2022)) shows some additional details.

[24] Section 28 will start with a more general action.

[25] In the euclidean action, $p^2 \equiv |p|^2$, with $|p|$ defined as in (4). After Wick-rotating back to lorentzian spacetime, the quantity p^2 becomes the Lorentz-invariant product of p with itself, namely $p^2 = \eta^{ab} p_a p_b$ where η^{ab} are the components of the Minkowski metric. In both cases, the low-resolution condition may still be written as (4).

To continue, use the abbreviation[26]

$$\langle\cdots\rangle \equiv \frac{\int [d\tilde{\phi}]_{\text{shell}}\ \exp\big(-S^{\text{free}}[\chi]\big)\cdots}{\int [d\tilde{\phi}]_{\text{shell}}\ \exp\big(-S^{\text{free}}[\chi]\big)} \tag{25}$$

and define $R[\phi_1]$ by

$$\exp\big(-R[\phi_1]\big) \propto \big\langle \exp\big(-\delta V[\phi_1,\chi]\big)\big\rangle \tag{26}$$

with proportionality factor chosen so that equation (20) gives

$$\exp\big(-S_{1,\epsilon}[\phi_1]\big) = \exp\big(-S[\phi_1]\big)\exp\big(-R[\phi_1]\big). \tag{27}$$

This relationship can also be written

$$S_{1,\epsilon}[\phi_1] = S[\phi_1] + R[\phi_1]. \tag{28}$$

If c_4 is not zero, then c_2 must be negative to approach a limit in which spacetime is continuous or at least practically continuous.[27] If $|c_2|$ is less than the minimum value of p^2 in the shell, then $S^{\text{free}}[\chi]$ is still positive for all $\chi \neq 0$ because $p^2 + c_2 > 0$ (equation (23)). For the rest of this article, suppose that $|c_2|$ satisfies this condition so that the integrals in (25) are well-defined.

[26] In previous sections, the notation $\langle\cdots\rangle$ was used instead for the path integral over the field variables $\tilde{\phi}(p)$ with p a simply-connected domain Γ, but here it is used only for the integral over the field variables $\tilde{\phi}(p)$ with p in the shell. This new definition of $\langle\cdots\rangle$ is in effect for the rest of this article.

[27] Article 10142

10 The tree-level approximation

If $c_4 = 0$, then R would be independent of ϕ_1, so

$$S_{1,\epsilon}[\phi_1] = S[\phi_1] + \text{constant} \qquad\qquad (\text{if } c_4 = 0). \tag{29}$$

If $c_4 > 0$, then equation (29) is no longer true, but we can contemplate ignoring R as an approximation. This is called the **tree-level approximation**.[28] In that approximation, the coefficients $\tilde{c}_n(\lambda)$ in the effective action (10) would be equal to the coefficients c_n in the original action (9), and $Z(\lambda)$ would likewise be equal to 1, so equation (14) would imply

$$c_n(\lambda) = \frac{\lambda^d}{\lambda^{(d-2)N_\phi/2}\lambda^{N_\partial}}\, c_n \tag{30}$$

in the generic case (13). Specialized to the case (21)-(22), this gives

$$c_2(\lambda) = \lambda^2 c_2 \qquad\qquad c_4(\lambda) = \lambda^{4-d} c_4.$$

This approximation suggests that the ϕ^2 term is a relevant perturbation (its importance grows with increasing λ) and that the relevance of the ϕ^4 term depends on the number d of spacetime dimensions: relevant if $d \leq 3$, irrelevant if $d \geq 5$, and the case $d = 4$ is too close to call.

The tree-level approximation might seem crude, but these conclusions turn out to be accurate, with one caveat: they become accurate after we replace ϕ^4 with $\phi^4 + \gamma\phi^2$ with a specially-tuned negative value of γ. Without this subtraction, the ϕ^4 term by itself is actually relevant for every d, effectively morphing into a ϕ^2 term at low energies when $d \geq 4$. This is emphasized in article 10142, and it will be derived in section 24. The required value of γ depends on the shape of the domain Γ_0 in the original action 9.

[28] The approximation is called *tree-level* because it doesn't include contributions from *loops* (section 20).

11 The small-coupling approximation

Expand the right-hand side of (26) in powers of δV to get

$$\exp\big(-R[\phi_1]\big) \propto \left\langle 1 - \delta V + \frac{1}{2}(\delta V)^2 + O\big((\delta V)^3\big)\right\rangle \tag{31}$$

Take the log of both sides of (26) and use $\log(1+x) = x - x^2/2 + O(x^3)$ to get

$$R[\phi_1] = \text{constant} - \langle \delta V\rangle + \frac{\big\langle(\delta V)^2\big\rangle - \langle \delta V\rangle^2}{2} + O\big((\delta V)^3\big). \tag{32}$$

Let δV_n denote the part of δV with n factors of χ. The integral (25) is zero if "$\cdots$" is the product of an odd number of χs, so

$$\begin{aligned}
\langle \delta V\rangle &= \big\langle \delta V_2 + \delta V_4\big\rangle \\
\big\langle(\delta V)^2\big\rangle &= \Big\langle \big(\delta V_1\big)^2 + \big(\delta V_2\big)^2 + \big(\delta V_3\big)^2 + \big(\delta V_4\big)^2 + 2(\delta V_1)\,(\delta V_3) + 2(\delta V_2)\,(\delta V_4)\Big\rangle.
\end{aligned}$$

Section 12 will show more explicit expressions for each of these terms.

12 Terms in the small-coupling approximation

Use the abbreviation

$$v_n(x) \equiv \frac{1}{n!}\left(\frac{\partial}{\partial\phi_1(x)}\right)^n V[\phi_1] \tag{33}$$

so that

$$\delta V_n = \epsilon^d \sum_x \chi^n(x) v_n(x). \tag{34}$$

Then

$$\langle \delta V_2\rangle = \epsilon^d \sum_x \big\langle \chi^2(x)\big\rangle\, v_2(x) \tag{35}$$

$$\big\langle (\delta V_1)^2\big\rangle = \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi(x)\chi(y)\big\rangle\, v_1(x)v_1(y) \tag{36}$$

$$\big\langle (\delta V_2)^2\big\rangle = \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^2(x)\chi^2(y)\big\rangle\, v_2(x)v_2(y) \tag{37}$$

$$\big\langle (\delta V_3)^2\big\rangle = \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^3(x)\chi^3(y)\big\rangle\, v_3(x)v_3(y) \tag{38}$$

$$\big\langle (\delta V_1)(\delta V_3)\big\rangle = \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi(x)\chi^3(y)\big\rangle\, v_1(x)v_3(y) \tag{39}$$

$$\big\langle (\delta V_2)(\delta V_4)\big\rangle = \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^2(x)\chi^4(y)\big\rangle\, v_2(x)v_4. \tag{40}$$

Terms involving only δV_4 can be ignored, because they are independent of ϕ_1 and so only contribute to the constant term in (32).

13 The momentum shell correlation functions

To evaluate $\langle\chi^j(x)\chi^k(y)\rangle$, use equations (23) and (25) to get

$$\langle\chi^j(x)\chi^k(y)\rangle = \left(\epsilon^{-d}\frac{\partial}{\partial J(x)}\right)^j \left(\epsilon^{-d}\frac{\partial}{\partial J(y)}\right)^k Z[J]\bigg|_{J=0} \tag{41}$$

with

$$Z[J] \equiv \frac{\int[d\tilde{\phi}]_{\text{shell}}\; \exp\big(-S^{\text{free}}[\chi]+\epsilon^d\sum_x \chi(x)J(x)\big)}{\int[d\tilde{\phi}]_{\text{shell}}\; \exp\big(-S^{\text{free}}[\chi]\big)}, \tag{42}$$

where $J(x)$ is a collection of independent variables, one for each lattice site x. This works even though the quantities $\chi(x)$ are not all independent. To evaluate the generating functional (42), use equation (16) to write $\chi(x)$ in terms of the independent variables $\tilde{\phi}(p)$ and rewrite the exponent of (42) as

$$\begin{aligned}-S^{\text{free}}[\chi]+\epsilon^d\sum_x\chi(x)J(x) = &-\frac{1}{L^d}\sum_{p\,\in\,\text{shell}}\tilde{\phi}'(-p)\frac{p^2+c_2}{2}\tilde{\phi}'(p)\\ &+\frac{1}{2}(\epsilon^d)^2\sum_{x,y}J(x)G(x-y)J(y)\end{aligned}$$

with

$$\tilde{\phi}'(p)\equiv\tilde{\phi}(p)-\frac{\tilde{J}(p)}{p^2+c_2} \qquad\qquad G(x-y)\equiv\frac{1}{L^d}\sum_{p\,\in\,\text{shell}}\frac{e^{ip\cdot(x-y)}}{p^2+c_2}. \tag{43}$$

where $\tilde{J}(p)$ is the Fourier transform of $J(x)$. The shift $\tilde{\phi}(p)\to\tilde{\phi}'(p)$ doesn't affect the integrals over $\tilde{\phi}(p)$, so the result is simply

$$Z[J]=\exp\left(\frac{1}{2}(\epsilon^d)^2\sum_{x,y}J(x)G(x-y)J(y)\right). \tag{44}$$

Thanks to equation (44), the correlation function (41) reduces to a sum of products of the two-point functions $G(x-y)=\langle\chi(x)\chi(y)\rangle$ and $G(0)$.

14 Connected terms

Section 11 showed that the quantity $R[\phi_1]$ in equation (32) is a sum of terms with the form

$$\Big\langle(\delta V_{n_1})(\delta V_{n_2})\cdots(\delta V_{n_N})\Big\rangle \tag{45}$$

with δV_n given by equation (34). Use definition of δV_n to write (45) as[29]

$$(\epsilon^d)^N \sum_{x_1,x_2,...,x_N} \Big\langle\chi^{n_1}(x_1)\chi^{n_2}(x_2)\cdots\chi^{n_N}(x_N)\Big\rangle v_{n_1}(x_1)v_{n_2}(x_2)\cdots v_{n_N}(x_N). \tag{46}$$

Section 13 showed that the factor $\langle\cdots\rangle$ as a sum of products of two-point functions. The structure of any one term in that sum can be represented as a **graph** (points connected to each other by lines) in which each of the spacetime points x_j is drawn as a dot (also called a **vertex** in graph theory) and each two-point function $\langle\chi(x_j)\chi(x_k)\rangle$ is drawn as a line (also called an **edge** in graph theory) connecting the points x_j and x_k. In the context of the quantity (46), each point x in the graph is associated with a factor $v_n(x)$, and a sum over each argument x is implied.

A graph is called **disconnected** if its vertices can be separated into two nonempty subsets such that no edge connects any vertex in one subset to any vertex in the other. Otherwise, the graph is called **connected**. A term represented by a (dis)connected graph is called a **(dis)connected term**. The quantity $e^{-R[\phi_1]}$ (equation (31)) involves both connected and disconnected terms, but all of the disconnected terms cancel in the quantity $R[\phi_1]$ (equation (32)). This implies that the effective action $S_{1,\epsilon}[\phi_1]$ defined in equation (20) can be calculated by evaluating only the connected terms.

[29] Here, each x_j is a point in spacetime, a collection of d coordinates: $x_j = \big((x_j)_1, ..., (x_j)_d\big)$.

15 The momentum-shell two-point function

The infinite-volume limit of the two-point function (43) is

$$G(x-y)=\int_{p\,\in\,\text{shell}}\frac{d^dp}{(2\pi)^d}\;\frac{e^{ip\cdot(x-y)}}{p^2+c_2}. \tag{47}$$

This is clearly finite when $x=y$, because the domain of integration is finite. This section evaluates this integral in a special case,[30] namely the case of spherical momentum domains Γ_0 and Γ_1 (section 7) with $d=3$. The key message is that $G(x-y)$ is a decreasing function of $|x-y|$, at least when $d\geq 3$. This will be demonstrated here only in a special case, but the key message holds more generally.

Let Λ_n be the radius of the spherical domain Γ_n. Then the integral is only over momenta with magnitudes in the range $\Lambda_1<|p|<\Lambda_0$. The **thin-shell approximation** will be used, so that Λ_1 is only slightly different than the original scale Λ_0:

$$\delta\Lambda\equiv\Lambda_0-\Lambda_1\ll\Lambda_0. \tag{48}$$

To evaluate (47), write $p\cdot(x-y)=|p|\,|x-y|\,\cos\theta$ so that

$$G(x-y)=\frac{\Omega_{d-1}}{(2\pi)^d}\int_{\Lambda_1}^{\Lambda_0}|p|^{d-1}\,d|p|\int_0^\pi(\sin\theta)^{d-2}\,d\theta\;\frac{e^{i|p|\,|x-y|\,\cos\theta}}{|p|^2+c_2}$$

where Ω_d is the "surface area" of the unit sphere in d-dimensional space.[31] To continue, specialize to $d=3$ to make the θ-integral easy. Then

$$\begin{aligned}G(x-y)&=\frac{\Omega_2}{(2\pi)^3}\int_{\Lambda_1}^{\Lambda_0}|p|^2\,d|p|\int_{-1}^{1}d\cos\theta\;\frac{e^{i|p|\,|x-y|\,\cos\theta}}{|p|^2+c_2}\\&=\frac{1}{(2\pi)^2}\int_{\Lambda_1}^{\Lambda_0}|p|^2\,d|p|\;\frac{e^{i|p|\,|x-y|}-e^{-i|p|\,|x-y|}}{\big(|p|^2+c_2\big)\big(i|p|\,|x-y|\big)}.\end{aligned}$$

[30] If the domain of integration were infinite, then dimensional analysis could be used to deduce $G(x-y)\propto|x-y|^{2-d}$. The fact that the domain of integration is finite makes the analysis more challenging.

[31] Examples: $\Omega_1=2$, $\Omega_2=2\pi$, $\Omega_3=4\pi$, and $\Omega_4=2\pi^2$.

If $|c_2| \ll \Lambda_1$, then

$$\begin{aligned}
G(x-y) &= \frac{1}{2\pi^2}\int_{\Lambda_1}^{\Lambda_0} \frac{d|p|}{|p|}\,\frac{\sin\big(|p|\,|x-y|\big)}{|x-y|} \\
&\approx \frac{1}{\pi^2(\Lambda_1+\Lambda_0)}\int_{\Lambda_1}^{\Lambda_0} d|p|\,\frac{\sin\big(|p|\,|x-y|\big)}{|x-y|} \\
&= \frac{\cos\big(\Lambda_1\,|x-y|\big)-\cos\big(\Lambda_0\,|x-y|\big)}{\pi^2(\Lambda_1+\Lambda_0)\,|x-y|^2}.
\end{aligned}$$

The numerator is an oscillating function of $|x-y|$ with magnitude ≤ 2, so the denominator makes $G(x-y)$ fall of as $|x-y|$ increases. At $x=y$, its value is $G(0)\approx(\Lambda_0-\Lambda_1)/2\pi^2$, so

$$\frac{G(x-y)}{G(0)} \sim \frac{\cos\big(\Lambda_1\,|x-y|\big)-\cos\big(\Lambda_0\,|x-y|\big)}{(\Lambda_0^2-\Lambda_1^2)|x-y|^2}$$

This becomes negligible compared to its maximum value when

$$(\Lambda_0^2-\Lambda_1^2)|x-y|^2 \gg 1.$$

This shows that $G(x-y)$ is a decreasing function of $|x-y|$, as claimed.

16 $G(0)$ and $\sum_y G^n(y)$

For use in section 22, this section evaluates the quantities $G(0)$ and[32]

$$\epsilon^d \sum_y G^n(x-y) \tag{49}$$

for $n=1$ and $n=2$, using a spherical momentum shell as in section 15.

Use the abbreviation[33,34]

$$\omega_d \equiv \frac{\Omega_d}{2(2\pi)^d}. \tag{50}$$

Using the thin-shell approximation (48), the quantity $G(0)$ is

$$G(0) = \int_{\text{shell}} \frac{d^d p}{(2\pi)^d}\, \frac{1}{p^2+c_2} \approx \frac{\delta\Lambda}{\Lambda_0} \frac{\Lambda_0^d}{\Lambda_0^2+c_2}\, 2\omega_d \tag{51}$$

if the domain of integration is a spherical shell of radius Λ_0 and thickness $\delta\Lambda$.

To evaluate (49), start with equation (47) and use

$$\epsilon^d \sum_y e^{iy\cdot\sum p} \sim \delta\left(\sum p\right)$$

where $\sum p$ is the sum of the n momenta occurring in the Fourier transforms of the n factors G. The quantity (49) is zero when $n=1$, because $G(x-y)$ is an integral over only nonzero momenta, where $\delta(p)$ is zero. When $n=2$,

$$\epsilon^d \sum_y G^2(x-y) = \int_{\text{shell}} \frac{d^d p}{(2\pi)^d}\, \frac{1}{(p^2+c_2)^2} \approx \frac{\delta\Lambda}{\Lambda_0} \frac{\Lambda_0^d}{(\Lambda_0^2+c_2)^2}\, 2\omega_d \tag{52}$$

for a spherical shell.

[32] The quantities (49) will appear in section 17 (equation (54)).

[33] The extra factor of 2 is included in the denominator for later convenience. It is compensated by the factors of 2 in equations (51) and (52).

[34] Ω_n is defined as in section 15.

17 The no-derivatives approximation

Sections 11-14 showed that the effective action involves terms of the form

$$\left(\epsilon^d\right)^2 \sum_{x,y} \phi^j(x) G^n(x-y) \phi^k(y)$$

with $G(x-y) = \big\langle \chi(x)\chi(y) \big\rangle$. The function $G(x-y)$ is nonzero even when $x \neq y$, so $R[\phi_1]$ may involve products of ϕ_1s at widely separated points in spacetime.

The fact that ϕ_1 involves only momenta in a finite domain (equations (5) and (8)) implies that we could write

$$\phi_1(y) = \left[\phi_1(x) + a\cdot\partial\phi_1(x) + \frac{1}{2}(a\cdot\partial)^2\phi_1(x) + \cdots\right]_{a=y-x} \tag{53}$$

if the arguments x, y were continuous. This still works when spacetime is treated as a lattice so that the derivatives ∂ are finite differences.[35] We can use this to write $R[\phi_1]$ as a sum over x of products of $\phi_1(x)$ and its derivatives, as in (9).

The rest of this article uses the **no-derivatives approximation**[36]

$$\left(\epsilon^d\right)^2 \sum_{x,y} \phi^j(x) G^n(x-y) \phi^k(y) \approx \left(\epsilon^d \sum_{x} \phi^{j+k}(x)\right)\left(\epsilon^d \sum_{y} G^n(x-y)\right), \tag{54}$$

which discards all terms involving derivatives on the right-hand side of (53). This has a chance of being a reasonable approximation when $d \geq 3$, because then terms with $N_\partial \geq 2$ have scaling dimension $\Delta > d$ (except when $N_\phi = 2$), and terms with $\Delta > d$ are expected to be irrelevant at sufficiently low resolution.[37,38,39] The real reason for using this approximation here, though, is that it simplifies the analysis.

[35] Rota *et al* (1973), theorem 2

[36] This is also called the **local potential approximation (LPA)** (section 2.4.4 in Pelissetto and Vicari (2002), section I.A in Hellwig *et al* (2015), section 3 in Codello *et al* (2018), and section 3 in Bagnuls and Bervillier (2001), but the text below equation (2) in https://arxiv.org/abs/1307.3679 challenges this language).

[37] Section 1

[38] Heuristically, this is consistent with – but not necessarily implied by – the fact that $G(x-y)$ approaches zero for large $|x-y|$ (section 15), at least when $d \geq 3$.

[39] Derivative terms can contribute to $(\partial\phi)^2$, but not in the one-loop approximation (section 20).

18 The no-derivatives approximation: a consequence

In the no-derivatives approximation, all terms involving δV_1 can be discarded.[40] In other words, the part of δV that is linear in χ can be discarded. To see why, start with the derivative expansion

$$\begin{aligned}\big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^k(x)\chi(y)\big\rangle\, v_k(x)v_1(y) \\ = \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^k(x)\chi(y)\big\rangle\, v_k(x)\big(v_1(x) + (y-x)\cdot\partial v_1(x) + \cdots\big),\end{aligned} \tag{55}$$

and then use the no-derivatives approximation to get

$$\big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^k(x)\chi(y)\big\rangle\, v_k(x)v_1(y) \approx \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^k(x)\chi(y)\big\rangle\, v_k(x)v_1(x).$$

The right-hand side is zero because $\epsilon^d \sum_y \chi(y)$ would be the zero-momentum part of $\chi(y)$, but $\chi(y)$ doesn't have a zero-momentum part because its momenta are restricted to the shell. The rest of this article uses the no-derivatives approximation, so terms involving factors of δV_1 will be discarded. In particular, the quantities (36) and (39) will be discarded.

We can reach the same conclusion by applying the no-derivatives approximation to the v_k factor instead, which would give

$$\big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^k(x)\chi(y)\big\rangle\, v_k(x)v_1(y) \approx \big(\epsilon^d\big)^2 \sum_{x,y} \big\langle \chi^k(x)\chi(y)\big\rangle\, v_k(y)v_1(y).$$

This is zero because of equation (23). That equation implies that when "$\cdots$" is a product of $\tilde\phi$s, the quantity $\langle\cdots\rangle$ is zero unless the momenta sum to zero pairwise (meaning that each factor of $\tilde\phi(p)$ occurs together with a factor of $\tilde\phi(-p)$). The quantity $\epsilon^d \sum_x \langle \chi^k(x)\chi(y)\rangle$ doesn't have any terms that satisfy this condition, because the individual momenta are all restricted to the shell, and the sum $\epsilon^d \sum_x \chi^k(x)$ involves only products of k $\tilde\phi$s whose momenta already sum to zero.

[40] The quantity δV_1 is defined in section 11.

19 A warning

Section 18 showed that terms involving δV_1 cannot contribute to the effective action when the no-derivatives approximation is used. This section shows that terms involving δV_1 can contribute when the no-derivatives approximation is not used. The word *warning* is in the title of this section because some authors discard terms involving δV_1 without explaining why – or when – this is allowed.

Factors of δV_1 enter through quantities of the form

$$\left(\epsilon^d\right)^2 \sum_{x,y} \left\langle \chi^k(x)\chi(y)\right\rangle v_k(x)v_1(y).$$

The correlation function factorizes into a product of the two-point factor $\langle\chi(x)\chi(y)\rangle$ with $\langle\chi^{k-1}(x)\rangle$, so the sum over y involves only the combination

$$\epsilon^d \sum_y G(x-y)v_1(y).$$

Let $\tilde{v}_1(p)$ denote the Fourier transform of $v_1(y)$. Then

$$\begin{aligned}\epsilon^d \sum_y G(x-y)v_1(y) &= \epsilon^d \sum_y \left(\frac{1}{L^d}\sum_{p\,\in\,\text{shell}} \frac{e^{ip\cdot(x-y)}}{p^2+c_2}\right)\left(\frac{1}{L^d}\sum_q f e^{iq\cdot y}\tilde{v}_1(q)\right)\\ &= \frac{1}{L^d}\sum_{p\,\in\,\text{shell}} \frac{e^{ip\cdot x}\,\tilde{v}_1(p)}{p^2+c_2}\end{aligned}$$

This is zero if $v_1(y) \propto \phi_1(y)$, because $\phi_1(y)$ involves only momenta in Γ_1 (not in the shell),[41] but it can be nonzero if $v_1(y)$ involves higher powers of $\phi_1(y)$, because the sum of three or more momenta in Γ_1 can equal a momentum in the shell.

[41] This is the situation illustrated by figure 15.10 in Fradkin (2021).

20 The loop expansion

Consider the quantities $\langle(\delta V_2)^2\rangle$ and $\langle(\delta V_3)^2\rangle$ shown in equations (37) and (38), respectively, which involve the correlation functions $\langle\chi^k(x)\chi^k(y)\rangle$ with $k = 2, 3$. According to the result derived in section 13, the quantity $\langle\chi^2(x)\chi^2(y)\rangle$ is a sum of terms of these forms:

$$\langle\chi^2(x)\rangle\langle\chi^2(y)\rangle \qquad\qquad \langle\chi(x)\chi(y)\rangle^2.$$

In the graph representation described in section 14, the form shown on the left is disconnected, so it does not contribute to the effective action. The form shown on the right has two lines connecting the points x and y to each other, so it has one **loop**. Similarly, the quantity $\langle\chi^3(x)\chi^3(y)\rangle$ is a sum of terms of these forms:[42]

$$\langle\chi^2(x)\rangle\langle\chi^2(y)\rangle\langle\chi(x)\chi(y)\rangle \qquad\qquad \langle\chi(x)\chi(y)\rangle^3.$$

Graphically, the form shown on the left has a line connecting x to itself, a line connecting y to itself, and a line connecting x to y, so the graph is connected with two loops. The form shown on the right has three lines connecting x to y, so it is again connected with two independent loops.[43]

The rest of this article uses the one-loop approximation, so terms with two or more loops will be ignored. The expansion in the number of loops is related to an expansion in powers of the coefficient(s) c_n, so the one-loop approximation can be viewed as a kind of small-coupling approximation.[44] Section 21 will show that when the no-derivatives approximation is used, the number of loops is equal to the number of independent integrals over the momentum shell, so the loop expansion can also be viewed as an expansion in powers of the ratio $\delta\Lambda/\Lambda_0 \ll 1$.[45]

[42]Other cases are zero because $\langle\chi^k\rangle = 0$ when k is odd.

[43]The third loop is a composition of the first two loops, so it doesn't count as an independent loop.

[44]The relationship is simplest when V has only a single coefficient, as in equation (22).

[45]Equation (48)

21 The loop expansion with no derivatives

In the no-derivatives approximation,[46] the quantities $\langle(\delta V_2)^2\rangle$ and $\langle(\delta V_3)^2\rangle$ shown in equations (37) and (38) each become

$$\left\langle(\delta V_k)^2\right\rangle \approx \left(\epsilon^d\right)^2 \sum_{x,y} \left\langle\chi^k(x)\chi^k(y)\right\rangle v_k(x)v_k(x) \tag{56}$$

After χ is written in terms of its momentum components $\tilde{\phi}(p)$ as in equation (16), the quantity $\chi^k(y)$ is proportional to[47]

$$\sum_{p_1,p_2,\ldots,p_k} \exp\left(iy\cdot\sum_k p_k\right)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\cdots\tilde{\phi}(p_k),$$

where the "integral" (sum) over each momentum p_j is restricted to the shell. The sum over y in equation (56) enforces the condition $p_1+p_2+\cdots+p_k=0$, which reduces the number of momentum integrals to $k-1$. The momentum integrals coming from the $\chi^k(x)$ factor are all eliminated by the fact that $\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle$ is zero unless $p_1+p_2=0$. Altogether, even though the number of factors of χ in (56) is $2k$, the number of independent momentum integrals is only $k-1$, which is the same as the number of loops.

This example illustrates the general fact that the no-derivatives approximation can be arranged to make the number of independent momentum integrals equal to the number of loops.[48] Each momentum integral is over a shell of thickness $\delta\Lambda \ll \Lambda_0$ (equation (48)), so the loop expansion can also be viewed as an expansion in powers of the small ratio $\delta\Lambda/\Lambda_0$.[49]

---

[46]The lowest-order terms with derivatives are treated in Fradkin (2021), chapter 15 (and the online version Fradkin (2022)).

[47]Here, each p_j represents a point in the momentum domain, so it is a collection of d components: $p_j = ((p_j)_1, ..., (p_j)_d)$.

[48]They would automatically be equal if the quantities $v_k(x)$ were independent of x.

[49]The analysis still relies on a small-coupling approximation. Even if the initial action only has a single interaction

22 The flow equations

In the no-derivatives and one-loop approximations, the quantities (36), (38), and (39) may all be discarded. The quantity (32) then reduces to[50,51]

$$
\begin{aligned}
R[\phi_1] &\approx \text{constant} - \langle \delta V_2 \rangle + \frac{\big\langle (\delta V_2)^2 \big\rangle - \langle \delta V_2 \rangle^2}{2} \\
&\approx \text{constant} - \epsilon^d \sum_x \big\langle \chi^2(x) \big\rangle v_2(x) \\
&\qquad + \big(\epsilon^d\big)^2 \sum_{x,y} \frac{\big\langle \chi^2(x)\chi^2(y) \big\rangle - \big\langle \chi^2(x) \big\rangle \big\langle \chi^2(y) \big\rangle}{2} \big(v_2(x)\big)^2 \\
&= \text{constant} - \epsilon^d \sum_x G(0) v_2(x) + \big(\epsilon^d\big)^2 \sum_{x,y} G^2(x-y) \big(v_2(x)\big)^2.
\end{aligned}
$$

The quantities $G(0)$ and $\epsilon^d \sum_y G^2(y)$ were evaluated in section 16 for the case of a spherical momentum shell. Use those results to get

$$
R[\phi_1] \approx \text{constant} + \epsilon^d \sum_x \left(-\frac{v_2(x)}{\Lambda_0^2 + c_2} + \left(\frac{v_2(x)}{\Lambda_0^2 + c_2} \right)^2 \right) \frac{\delta\Lambda}{\Lambda_0} \Lambda_0^d \omega_d.
$$

Use the definition of $v_2(x)$ in equations (33) and (22) to get[52]

$$
v_2(x) = c_4 \frac{\phi_1^2(x)}{4},
$$

term (the ϕ^4 term in equation (22)), integrating over a momentum shell generates higher powers of ϕ even in the one-loop approximation. Also, the no-derivatives approximation relies on scaling dimensions to argue that most terms with derivatives are *a priori* irrelevant (section 17), and the accuracy of that correspondence already depends on a small-coupling approximation (section 15.6 in Fradkin (2021), mentioned in article 10142).

[50] The quantity (40) does not contribute because δV_4 is a constant when V has the form shown in 9, so (40) does not have a connected part (section 14).

[51] The last step accounts for a factor of 2 from the combinatorics of equations (41) and (44).

[52] One factor of 1/2 comes from the factor of $1/n!$ in equation (33), and the other comes from the factor of $1/4!$ in equation (22).

and write the coefficients c_n as

$$c_2 = g_2\Lambda_0^2 \qquad\qquad c_4 = g_4\Lambda_0^{4-d} \tag{57}$$

to get[53]

$$R[\phi_1] \approx \text{constant} + \epsilon^d \sum_x \left(-\frac{g_4}{1+g_2}\,\Lambda_0^2\,\frac{\phi_1^2(x)}{2} + \left(\frac{g_4}{1+g_2}\right)^2 \Lambda_0^{4-d}\,\frac{\phi_1^4(x)}{8}\right) \frac{\delta\Lambda}{\Lambda_0}\,\omega_d.$$

Altogether, the effective action (28) is

$$S_{1,\epsilon}[\phi_1] = S[\phi_1] + R[\phi_1]$$

with

$$\begin{aligned} S[\phi_1] &= \epsilon^d \sum_x \left(\frac{\big(\partial\phi_1(x)\big)^2}{2} + c_2\frac{\phi_1^2(x)}{2} + c_4\frac{\phi_1^4(x)}{4!}\right) \\ &= \epsilon^d \sum_x \left(\frac{\big(\partial\phi_1(x)\big)^2}{2} + g_2\Lambda_0^2\,\frac{\phi_1^2(x)}{2} + g_4\Lambda_0^{4-d}\,\frac{\phi_1^4(x)}{4!}\right). \end{aligned}$$

As in section 7, define $\lambda \equiv \Lambda_0/\Lambda_1$. As explained in that section, the last step in this implementation of the renormalization group is to replace $\epsilon \to \lambda\epsilon$ and express the resulting effective action $S_{1,\lambda\epsilon}[\phi_1]$ in terms of the rescaled field variables $\phi_1'(x)$ defined in equation (12). This gives[54]

$$\begin{aligned} S_{1,\lambda\epsilon}[\phi_1] \approx\ & \epsilon^d \sum_x \left(\frac{\big(\partial\phi_1'(x)\big)^2}{2} + g_2\lambda^2\Lambda_0^2\,\frac{\big(\phi_1'(x)\big)^2}{2} + g_4\lambda^{4-d}\Lambda_0^{4-d}\,\frac{\big(\phi_1'(x)\big)^4}{4!}\right) \\ & + \epsilon^d \sum_x \left(-\frac{g_4}{1+g_2}\,\lambda^2\Lambda_0^2\,\frac{\big(\phi_1'(x)\big)^2}{2} + \left(\frac{g_4}{1+g_2}\right)^2 \lambda^{4-d}\Lambda_0^{4-d}\,\frac{\big(\phi_1'(x)\big)^4}{8}\right) \frac{\delta\Lambda}{\Lambda_0}\,\omega_d \\ & + \text{constant}. \end{aligned}$$

[53]One factor of $1/2$ from each term is included in the definition of ω_d, equation (50).

[54]Factors of Λ_0 are *not* replaced with Λ_0/λ. Even though we chose Λ_0 to be closely related to $1/\epsilon$ in this example, the quantities ϵ and Λ_0 are independent of each other (section 7).

The definition of λ implies $\delta\Lambda/\Lambda_0 = 1 - 1/\lambda$, so comparing this to the original action (21) gives these expressions for the λ-dependent coefficients $g_n(\lambda)$ in terms of the original coefficients g_n:

$$
\begin{aligned}
g_2(\lambda) &\approx \lambda^2 \times \left(g_2 - \omega_d \frac{g_4}{1+g_2}\left(1 - \frac{1}{\lambda}\right)\right) \\
g_4(\lambda) &\approx \lambda^{4-d} \times \left(g_4 + 3\omega_d \left(\frac{g_4}{1+g_2}\right)^2 \left(1 - \frac{1}{\lambda}\right)\right).
\end{aligned}
$$

This implies

$$\lambda \frac{d}{d\lambda} g_n(\lambda) = \beta_n\big(\vec{g}(\lambda)\big) \tag{58}$$

with[55]

$$
\begin{aligned}
\beta_2(\vec{g}\,) &\approx 2g_2 + \omega_d \frac{g_4}{1+g_2} \\
\beta_4(\vec{g}\,) &\approx (4-d)g_4 - 3\omega_d \left(\frac{g_4}{1+g_2}\right)^2
\end{aligned}
\tag{59}
$$

after neglecting terms of order $(1-1/\lambda)^2$, which is valid in the thin-shell approximation. These ordinary first-order differential equations for the functions $g_n(\lambda)$ are called the **renormalization group (RG) equations** or **flow equations**,[56] and the functions $\beta_n(\vec{g})$ defined by (58) are called the **beta functions**.[57] Sections 24-26 will explain the significance of this result.

[55]This assumes that the initial momentum domain Γ_0 is spherical (section 7). Using a domain with a different shape may modify the relative coefficients in (59), as explained in section 6.

[56]Section 27 explains that in the complete system of flow equations, the derivatives $\lambda\, dg_{2k}/d\lambda$ are nonzero for all $k \geq 1$, thanks to the terms of order $(\delta V)^k$ in equation (31).

[57]The name comes from the fact that it is conventionally abbreviated β. Sometimes the opposite sign convention is used (Fradkin (2021), text below equation 16.20). Regardless of sign conventions, beware the name *beta function* (and the notation β) is also used for related-but-different quantities that are typically only approximately equal to the ones defined here – equal only up to some low order in a small-coupling expansion. One example is highlighted in Montvay and Münster (1997), equation (1.268) and the text below it. In the math literature, the name *beta function* is used for something completely different.

23 Dimensionless coefficients

Factors of Λ_0 that otherwise would have been present in the flow equations (58)-(59) were eliminated by writing the coefficients c_n in terms of the quantities g_n that were defined in equations (57). When considering more general perturbations with N_ϕ factors of ϕ and N_∂ derivatives, factors of Λ_0 can be eliminated from the flow equations by writing the coefficients c_n as[58]

$$c_n = \frac{\Lambda_0^d}{\Lambda_0^{(d-2)N_\phi/2}\Lambda_0^{N_\partial}}\, g_n. \tag{60}$$

A system of units in which g_n is dimensionless is *natural*[59] in the sense that it simplifies the form of the flow equations by eliminating factors of Λ_0.

This system of units can also be defined by[60] requiring the unperturbed action

$$\epsilon^d \sum_x \frac{(\partial\phi)^2}{2}$$

to be dimensionless, in which case ϕ has the same units as $\Lambda_0^{(d-2)/2}$. Requiring all other terms in the action to be dimensionless then implies that c_n may be written as in (60) with dimensionless coefficients g_n.

[58]This generalizes equations (57).

[59]Article 37431

[60]In this article, I chose to avoid any glib references to *dimensional analysis* in the derivation of equations (58)-(59). Units are arbitrary (even if some are more natural than others), so any legitimate use of *dimensional analysis* in a derivation clearly must involve more than just a natural choice of units. Many applications of *dimensional analysis* in physics are simple enough that unpacking those implicit ingredients is relatively straightforward, but the subject of this article is more intricate, so I chose to make those implicit ingredients explicit from the beginning.

24 Significance: first-order approximation

To first order in the coefficients g_n, the flow equations (58)-(59) are

$$\begin{aligned} \lambda\frac{dg_2}{d\lambda} &\approx 2g_2 + \omega_d g_4 + O(\vec{g}^2) \\ \lambda\frac{dg_4}{d\lambda} &\approx (4-d)g_4 + O(\vec{g}^2). \end{aligned} \tag{61}$$

The solution of this system of equations is

$$\begin{bmatrix} g_2(\lambda) \\ g_4(\lambda) \end{bmatrix} \approx \lambda^M \begin{bmatrix} g_2(1) \\ g_4(1) \end{bmatrix}$$

with

$$M = \begin{bmatrix} 2 & \omega_d \\ 0 & 4-d \end{bmatrix}.$$

Use the identities

$$\lambda^M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad \lambda^M \begin{bmatrix} \omega_d/(2-d) \\ 1 \end{bmatrix} = \lambda^{4-d} \begin{bmatrix} \omega_d/(2-d) \\ 1 \end{bmatrix}$$

to deduce that a ϕ^2 term is a relevant perturbation for all d and that

$$\phi^4 - \frac{\omega_d}{d-2}\,\phi^2 \tag{62}$$

is relevant if $d = 3$ and irrelevant if $d \geq 5$. Including the g_4^2 term in equations (59) modifies the rate at which the effects of these terms increases or decreases as a function of λ, but it doesn't change these conclusions about (ir)relevance.[61] This is all consistent with what we would expect based on scaling dimensions,[62] as reviewed in section 1.

[61] Section 25 will show that when $d = 4$, the perturbation corresponding to (62) is just barely *irrelevant*, thanks to terms of order g_4^2 that were neglected here.

[62] Thanks to this agreement, *scaling dimensions* are sometimes defined to be the eigenvalues of the linearized RG flow (linearized about a fixed point), as in section 1.5 in McGreevy (2021). Section 2.1 in Codello *et al* (2018) extends the definition beyond this approximation but acknowledges that those generalized scaling dimensions "take precise values only in the vicinity of a fixed point of the RG flow."

25 Significance: the almost-marginal case

To determine the effect of the ϕ^4 term when $d = 4$, we need to include the g_4^2 term in equations (58)-(59). To second order in the coefficients g_n, the flow equations (58)-(59) are

$$\lambda\frac{dg_2}{d\lambda} \approx 2g_2 + (1-g_2)\omega_d g_4 + O(\vec{g}^3)$$
$$\lambda\frac{dg_4}{d\lambda} \approx (4-d)g_4 - 3\omega_d g_4^2 + O(\vec{g}^3). \tag{63}$$

When $d = 4$, the solution of the second equation is

$$g_4(\lambda) = \frac{1}{(\log\lambda + \kappa)3\omega_d}$$

with an arbitrary constant κ. This is a decreasing function of λ, which suggests that some combination of ϕ^4 and ϕ^2 should be an irrelevant perturbation. The fact that equations (63) are nonlinear implies that the relative coefficient in this combination must depend on the overall coefficient. Explicitly, if we choose

$$g_2(1) = \frac{-\omega_d g_4(1)/2}{1+\omega_d g_4(1)},$$

then equations (63) with $d = 4$ imply

$$\lambda\frac{d}{d\lambda}\begin{bmatrix} g_2 \\ g_4 \end{bmatrix} = -3\omega_d g_4 \begin{bmatrix} g_2 \\ g_4 \end{bmatrix} \qquad \text{at } \lambda = 1.$$

This shows that the effect of a perturbation

$$g_4 \times \left(\phi^4 - \frac{\omega_d/2}{1+\omega_d g_4}\phi^2\right) \tag{64}$$

starts to decrease when λ starts to increase from its initial value $\lambda = 1$, indicating that this perturbation is irrelevant when $d = 4$. When terms of order g_4^2 are neglected, the combination (64) becomes proportional to (62) (when $d = 4$).

26 Significance: a hint of a nontrivial fixed point

When $d = 3$, equations (58)-(59) say that if g_4 is initially zero, then it starts to *increase* in proportion to λ. This eventually invalidates the small-g_4 approximation, so conclusions that we get by ignoring the nonlinear terms terms in equations (58)-(59) cannot be quantitatively correct, but suppose for a moment that they were at least qualitatively correct. When $d = 3$, a positive value of $g_4(\lambda)$ exists for which the derivative of $g_4(\lambda)$ is zero according to equation (59),[63] suggesting the existence of a scale-invariant model with $g_4 > 0$ when $d = 3$. This qualitative conclusion turns out to be correct, and improved computational techniques can be used to obtain quantitatively accurate results about this scale-invariant model,[64] which is called the **Wilson-Fisher fixed point**.[65] It is one of the most well-studied examples of a scale-invariant model in quantum field theory in $d \geq 3$ dimensions, not including models whose fields obey linear equations of motion.

In the context of quantum field theory, scale-invariant models are often called **fixed points**. They are *points* in the space of possible models, and they are *fixed* in the sense that they are invariant under the renormalization group flow.[66]

[63] Even if the ϕ^4 term is the only nonzero term in (22) when $\lambda = 1$, other terms are generated in the effective action when $\lambda > 1$, and that generates new terms in $\lambda\, dg_4/d\lambda$. Those generated terms are of order $O(g_4^3)$ (article 79649), so they are included in the "$+O(\vec{g}^{\,3})$" bucket in equation (63).

[64] Kleinert and Schulte-Frohlinde (2001), starting in section 10.12, and Pelissetto and Vicari (2002), section 3

[65] The first two sections Liendo (2017) give a concise introduction.

[66] Article 10142

27 Generation of higher-order terms

In section 11, the expansion in powers of δV was truncated at second order. If we had included terms of order $(\delta V)^k$ with $k \geq 3$, then we would have ended up with nonzero beta functions $\beta_{2k}(\vec{g})$ for all $k \geq 3$ in addition to the beta functions shown in equations (59). The expressions for these beta functions would involve only the coefficients g_2 and g_4, because these are the only terms included in the original action (22), but the fact that the beta functions β_{2k} are all nonzero says that terms of order ϕ^{2k} will be generated[67] in the effective action for all k. Article 79649 will explore the consequences of these higher-order terms.

Sections 28-29 will derive the beta functions that will be used in that article, starting with an action that already includes terms of all orders ϕ^{2k}. Setting the initial values of the coefficients c_{2k} to zero for all $k \geq 3$ gives the special case (22) that was used in the preceding sections.

[67]Remember that the flow equations were derived here only for an infinitesimal shell $\delta\Lambda/\Lambda_0 \ll 1$, so they only describe the *initial* values of the derivatives $\lambda\, dg_{2k}/d\lambda$.

28 A more general family of actions

The preceding sections, starting with section 9, studied an action of the form[68]

$$S[\phi] = \epsilon^d \sum_x \left(\frac{\big(\partial\phi_0(x)\big)^2}{2} + v\big(\phi_0(x)\big) \right) \qquad\qquad v(\phi) \equiv c_2\frac{\phi^2}{2} + c_4\frac{\phi^4}{4!}$$

using both the no-derivatives and one-loop approximations. This section describes an efficient way to handle the more general case

$$v(\phi) \equiv c_2\frac{\phi^2}{2} + c_4\frac{\phi^4}{4!} + c_6\frac{\phi^6}{6!} + \cdots$$

using the same approximations.

Start with the definition of the effective action, equation (20), and expand the quantity $S[\phi_1 + \chi]$ on the right-hand side in powers of χ. Section 18 showed that in the no-derivatives approximation, the linear-in-χ term can be discarded. When the no-derivatives and one-loop approximations are both used, the terms involving χ^k with $k \geq 3$ can also be discarded. To deduce this, we need to show two things. First, we need to show that terms generated by the quadratic part cannot involve more than one loop. Second, we need to show that that terms generated by the higher-order-in-χ parts necessarily involve at least two loops.

Suppose that only the quadratic-in-χ part is retained in the exponent on the right-hand side of (20) and that the exponential is then expanded in powers of χ. Then all of the expectation values of products of χs involve only products of $\chi^2(x)$s at different points. The disconnected terms cancel in the log, and the only way to get a completely-connected term is to connect the points in a single loop. Example:

$$\langle \chi^2(x)\chi^2(y)\chi^2(z)\rangle \propto \langle \chi(x)\chi(y)\rangle\langle \chi(y)\chi(z)\rangle\langle \chi(z)\chi(x)\rangle + \text{disconnected.}$$

This demonstrates that terms generated by the quadratic part cannot involve more than one loop.

[68] In the second equation (the definition of $v(\phi)$), ϕ is a single real variable.

Now suppose that the χ^k part were retained in the exponent on the right-hand side of (20), for some $k \geq 3$, and that the exponential is then expanded in powers of χ. Then all of the expectation values to which that term contributes would include a factor of $\chi^k(x)$. Section 13 showed that the expectation value may be written as a sum of products of two-point correlation functions. In the no-derivatives approximation, products with a factor of $\langle\chi(x)\chi(y)\rangle$ cannot contribute unless both points x and y also occur in at least one other factor. In that case, the only way to get a graph with only one loop is for each vertex to join exactly two edges, but terms involving $\chi^k(x)$ join k edges. This demonstrates that terms generated by the higher-order-in-χ parts necessarily involve at least two loops when the no-derivatives approximation is used.

Altogether, this shows that when the no-derivatives and one-loop approximations are both used, equation (20) for the effective action reduces to

$$\exp\big(-S_{1,\epsilon}[\phi_1]\big) = \exp\big(-S[\phi_1]\big)\exp\big(-R[\phi_1]\big) \tag{65}$$

with

$$\exp\big(-R[\phi_1]\big) \approx \int [d\tilde{\phi}]_{\text{shell}}\ \exp\left(-\epsilon^d\sum_x\left(\frac{\big(\partial\chi(x)\big)^2}{2} + v''\big(\phi_1(x)\big)\frac{\chi^2(x)}{2}\right)\right),$$

where $v''(\phi) \equiv d^2v/d\phi^2$. In the thin-shell approximation with a spherical shell of radius Λ_0, we can use $(\partial\chi)^2 \approx \Lambda_0^2\chi^2$ to get

$$\exp\big(-R[\phi_1]\big) \approx \int [d\tilde{\phi}]_{\text{shell}}\ \exp\left(-\epsilon^d\sum_x\big(\Lambda_0^2 + v''\big(\phi_1(x)\big)\big)\,\frac{\chi^2(x)}{2}\right).$$

To evaluate this, use equation (16) to write $\chi(x)$ in terms of the independent integration variables $\tilde{\phi}(p)$, which gives

$$\exp\big(-R[\phi_1]\big) \approx \int [d\tilde{\phi}]_{\text{shell}}\ \exp\left(-\frac{1}{L^d}\sum_{p,q\,\in\,\text{shell}}\tilde{\phi}^*(p)M(p,q)\tilde{\phi}(q)\right)$$

with

$$M(p,q) \equiv \frac{1}{L^d}\epsilon^d \sum_x \frac{\Lambda_0^2 + v''\big(\phi_1(x)\big)}{2} e^{i(p-q)\cdot x}.$$

If we don't take the infinite-volume limit, then the number momenta in the shell is finite, so $M(p,q)$ is an ordinary matrix with indices p and q. Thanks to the factor of L^d in the denominator, the matrix M^2 has components

$$M^2(p,q) = \sum_\ell M(p,\ell)M(\ell,q) = \frac{1}{L^d}\epsilon^d \sum_x \left(\frac{\Lambda_0^2 + v''\big(\phi_1(x)\big)}{2}\right)^2 e^{i(p-q)\cdot x}. \tag{66}$$

More generally,

$$M^n(p,q) = \frac{1}{L^d}\epsilon^d \sum_x \left(\frac{\Lambda_0^2 + v''\big(\phi_1(x)\big)}{2}\right)^n e^{i(p-q)\cdot x}. \tag{67}$$

Similarly,

$$(\log M)(p,q) = \frac{1}{L^d}\epsilon^d \sum_x \log\left(\frac{\Lambda_0^2 + v''\big(\phi_1(x)\big)}{2}\right) e^{i(p-q)\cdot x}. \tag{68}$$

The identity $M^*(p,q) = M(q,p)$ implies that the matrix M is diagonalizable, with real-valued eigenvalues, so we can use the identity

$$\int d\varphi_R\, d\varphi_I\; e^{-(\varphi_R+i\varphi_I)^*(\varphi_R+i\varphi_I)m} = \frac{\pi}{m} \propto e^{-\log m} \tag{69}$$

to get[69,70]

$$\exp\big(-R[\phi_1]\big) \approx \exp\left(-\frac{1}{2}\text{trace}\,(\log M) + \text{constant}\right) \tag{70}$$

[69] To get this from (69), let m be an eigenvalue of M and let φ be the amplitude of the corresponding eigenvector, and use the fact that the trace of a matrix is the sum of its eigenvalues.

[70] The factor 1/2 comes from the fact that the trace is over all momenta in the shell, but p and $-p$ were already counted separately in the identity (69) because $\tilde{\phi}(p) = \varphi_R(p) + i\varphi_I(p) = \tilde{\phi}^*(-p)$.

with

$$\text{trace}\,(\log M) = \sum_{p\,\in\,\text{shell}} (\log M)(p,p).$$

Using equation (68) and the thin-shell approximation with a spherical shell of radius Λ_0 gives[71]

$$\text{trace}\,(\log M) \approx \epsilon^d \sum_x \frac{\delta\Lambda}{\Lambda_0}\Lambda_0^d\,\omega_d\,\log\left(\frac{\Lambda_0^2 + v''\big(\phi_1(x)\big)}{2}\right).$$

Altogether, the effective action is

$$S_{1,\epsilon}[\phi_1] \approx \epsilon^d \sum_x \left(\frac{\big(\partial\phi_1(x)\big)^2}{2} + v^{\text{eff}}\big(\phi_1(x)\big)\right)$$

with

$$v^{\text{eff}}(\phi) \equiv v(\phi) + \frac{\delta\Lambda}{\Lambda_0}\Lambda_0^d\,\omega_d\,\log\left(\frac{\Lambda_0^2 + v''(\phi)}{2}\right) + \text{constant}. \tag{71}$$

After rescaling ϵ and ϕ as described in section 7, the effective action becomes

$$S_{1,\lambda\epsilon}[\phi_1] \approx \epsilon^d \sum_x \left(\frac{\big(\partial\phi_1(x)\big)^2}{2} + \lambda^d v^{\text{eff}}\big(\phi_1'(x)/\lambda^{(d-2)/2}\big)\right).$$

Equation (13) says that $c_n(\lambda)$ is the coefficient of $(\phi_1')^n/n!$ in the second term in large parentheses, so

$$\begin{aligned} c_n(\lambda) &= \left(\frac{d}{ds}\right)^n \lambda^d v^{\text{eff}}\big(s/\lambda^{(d-2)/2}\big)\bigg|_{s=0} \\ &= \lambda^{d-(d-2)n/2}\left(\frac{d}{ds}\right)^n v^{\text{eff}}(s)\bigg|_{s=0}. \end{aligned} \tag{72}$$

Section 29 uses this to derive RG flow equations for the dimensionless coefficients g_n that were defined in equation (60).

[71] The factor 1/2 from equation (70) is included in the definition of ω_d, equation (50).

29 Flow equations for the dimensionless coefficients

Use (72) to get

$$\lambda\frac{d}{d\lambda}c_n(\lambda) \approx \big(d-(d-2)n/2\big)c_n(\lambda) + \lambda^{d-(d-2)n/2}\left(\frac{d}{ds}\right)^n \Lambda_0^d\,\omega_d\,\log\big(\Lambda_0^2+v''(s)\big)\Big|_{s=0}.$$

Now use $\lambda \approx 1$ and equation (60) to get

$$\lambda\frac{d}{d\lambda}g_n(\lambda) \approx \big(d-(d-2)n/2\big)g_n(\lambda) + \left(\frac{d}{ds}\right)^n \Lambda_0^{(d-2)n/2}\,\omega_d\,\log\big(\Lambda_0^2+v''(s)\big)\Big|_{s=0}.$$

Equation (60) also implies

$$\Lambda_0^2+v''(\phi)=\Lambda_0^2\left(1+\sum_{k\geq 0} g_{2(k+1)}\frac{(\phi/\Lambda_0^{(d-2)/2})^{2k}}{(2k)!}\right),$$

so[72]

$$\lambda\frac{d}{d\lambda}g_n(\lambda) \approx \big(d-(d-2)n/2\big)g_n(\lambda) + \left(\frac{d}{ds}\right)^n \omega_d\log\big(1+u''(s)\big)\Big|_{s=0}$$

for all $n \in \{2,4,6,8,...\}$, with

$$u''(s)=\left(\frac{d}{ds}\right)^2 u(s) \qquad\qquad u(s)\equiv\sum_{k\geq 1} g_{2k}\frac{s^{2k}}{(2k)!}.$$

Article 79649 analyzes these flow equations in detail.

[72] This agrees with equation (5.56) in Skinner (2016).

30 References

Bagnuls and Bervillier, 2001. "Exact Renormalization Group Equations. An Introductory Review" *Phys. Rept.* **348**: 91-157, https://arxiv.org/abs/hep-th/0002034

Codello *et al*, 2018. "Functional perturbative RG and CFT data in the ϵ-expansion" *Eur. Phys. J.* **C78**: 30, https://arxiv.org/abs/1705.05558

Fradkin, 2021. *Quantum Field Theory: an Integrated Approach.* Princeton University Press

Fradkin, 2022. "Chapter 15: The Renormalization Group" https://eduardo.physics.illinois.edu/phys583/physics583.html

Hellwig *et al*, 2015. "Scaling and superscaling solutions from the functional renormalization group" *Phys. Rev. D* **92**: 085027, https://arxiv.org/abs/1508.02547

Kleinert and Schulte-Frohlinde, 2001. *Critical Properties of ϕ^4-Theories.* World Scientific, http://users.physik.fu-berlin.de/~kleinert/kleiner_reb8/psfiles/phi4.pdf

Lepage, 1989. "What is renormalization?." Pages 483-508 in *Proceedings of TASI '89: From Actions to Answers*, edited by DeGrand and Toussaint (World Scientific) https://arxiv.org/abs/hep-ph/0506330

Liendo, 2017. "Revisiting the dilatation operator of the Wilson-Fisher fixed point" https://arxiv.org/abs/1701.04830

McGreevy, 2021. "Physics 217: The Renormalization Group" https://mcgreevy.physics.ucsd.edu/f18/

Montvay and Münster, 1997. *Quantum Fields on a Lattice.* Cambridge University Press

Pelissetto and Vicari, 2002. "Critical Phenomena and Renormalization-Group Theory" *Phys.Rept.* **368**: 549-727, https://arxiv.org/abs/cond-mat/0012164

Rota *et al*, 1973. "On the Foundations of Combinatorial Theory. VIII. Finite Operator Calculus" *Journal of Mathematical Analysis and Applications* **42**: 684-760, https://www.sciencedirect.com/science/article/pii/0022247X739017

Skinner, 2016. "The Renormalization Group" https://www.damtp.cam.ac.uk/user/dbs26/AQFT/Wilsonchap.pdf

31 References in this series

Article **10142** (https://cphysics.org/article/10142):
"Universality and Continuum Limits with Scalar Quantum Fields" (version 2023-02-10)

Article **30983** (https://cphysics.org/article/30983):
"The Free Scalar Quantum Field: Particles" (version 2022-08-23)

Article **37431** (https://cphysics.org/article/37431):
"How to Think About Units" (version 2022-02-05)

Article **63548** (https://cphysics.org/article/63548):
"Scalar Quantum Fields on a Spacetime Lattice" (version 2023-02-10)

Article **71852** (https://cphysics.org/article/71852):
"Treating Space as a Lattice" (version 2022-08-21)

Article **79649** (https://cphysics.org/article/79649):
"Renormalization Group Flow Near the Trivial Fixed Point" (version 2023-02-10)