Symmetries of the Dirac Equation in Flat Spacetime

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Abstract Article 08264 introduces the **spin group**, a special double cover of the group of Lorentz transformations that may be expressed as compositions of even numbers of reflections. This article introduces the **Dirac equation** in flat spacetime. This is a differential equation whose group of symmetries automatically includes the spin group. This article explores the pattern of symmetries of the Dirac equation in d-dimensional flat spacetime, including antlinear symmetries like CPT symmetry. The definition of *symmetry* used here is motivated by quantum field theory, where the Dirac equation occurs as the equation of motion for a free spinor field. This article also explores symmetries of the Weyl equation, which is defined only when d is even. This is another differential equation whose group of symmetries automatically includes the spin group.

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1 Motivation

The simplest examples of models that satisfy the general principles of quantum field theory (QFT) are models with no interactions. In these models, observables are expressed in terms of fields that satisfy linear equations of motion. Such fields are called **free fields**. The Dirac equation is one example. This article introduces the the Dirac equation for the purpose of exploring some of its symmetries.

A field whose time-dependence is governed by the Dirac equation is called a **spinor field**.[1] In QFT, the components $\psi_1(x)$, $\psi_2(x)$, ... of the spinor field at each point x in spacetime would be operators on a Hilbert space,[2] and these operators would not commute with each other. The Dirac equation can be studied without that complication, though, because it is a linear differential equation: products with more than one factor of the quantities $\psi_k(x)$ do not occur in this equation. This article treats the quantities $\psi_k(x)$ as ordinary complex-valued functions of spacetime instead of as non-commuting operators, because this is sufficient for exploring the differential equation's symmetries.

The QFT context is still important here, but only for motivation. QFT is the motive for considering the Dirac equation at all, and it also motivates the definition of *symmetry* that will be used in this article. Article 21916 reviews a definition of *symmetry* in QFT that admits both linear and antilinear transformations.[3] Both types are important in QFT. Section 4 will introduce the definition of *symmetry* used in this article, after section 2 reviews some background material to help relate that definition to the QFT context that motivates it.

[1] In this article, the word *spinor* is a synonym for *Dirac spinor*, which many authors call a *pinor* in any dimension and *spinor* only in odd dimensions. This article uses the name *chiral spinor* for what those authors would call a *spinor* in even dimensions. Article 86175 gives more context about these different dialects.

[2] In mathematically legitimate formulations of QFT, operators aren't really associated with individual points in continuous spacetime (article 44563), but this article ignores that complication.

[3] Article 21916 describes the transformations more specifically as *unitary* and *antiunitary*, respectively. Article 90771 explains what this means.

2 Automorphisms of an algebra of operators

In QFT, calling a transformation a *symmetry* typically implies two things about the transformation. First, it respects locality: it respects the association between observables and regions of spacetime, which is part of the data that defines a model.[4] Second, it respects the algebraic structure: observables are represented by linear operators an a Hilbert space, and a symmetry should be a $*$-automorphism of that algebra of operators. This section reviews the definition of $*$-*automorphism*.[5]

Let $\mathcal{A}$ be an of operators on a Hilbert space that includes the adjoint A^* of each operator $A \in \mathcal{A}$. An **automorphism** is an invertible map $\sigma : \mathcal{A} \to \mathcal{A}$ that satisfies two conditions. First, it respects products:

$$\sigma(AB) = \sigma(A)\sigma(B) \tag{1}$$

for all operators A, B. Second, it is either linear or antilinear. An automorphism σ is called **linear** if it satisfies

$$\sigma(aA + bB) = a\sigma(A) + b\sigma(B) \tag{2}$$

for all operators A, B and all complex numbers a, b. If it satisfies

$$\sigma(aA + bB) = a^*\sigma(A) + b^*\sigma(B) \tag{3}$$

instead, then it's called **antilinear**.[6] This article uses both types of automorphism, linear and antilinear. The word *automorphism* by itself usually means a linear automorphism, but in this article it can mean either type. An automorphism is called a **$*$-automorphism** if it also satisfies

$$\sigma(A^*) = \sigma(A)^*. \tag{4}$$

We can think of a $*$-automorphism as a symmetry of the operator algebra itself, ignoring the *locality* condition that is typically required in QFT.

[4] Article 21916

[5] Article 74088 gives a more complete review.

[6] Uhlmann (2016) is an introduction to antilinear operators.

3 Two forms

Let $\mathcal{A}$ be the algebra generated by the field operators. The symmetries of interest in this article are $*$-automorphisms σ of $\mathcal{A}$ whose effects on the field operators have one of these forms:[7]

$$\text{First form:} \qquad \sigma(\psi_j(x)) = \sum\nolimits_k M_{jk}\psi_k(\bar{x}) \tag{5}$$

$$\text{Second form:} \qquad \sigma(\psi_j(x)) = \sum\nolimits_k M_{jk}\psi_k^*(\bar{x}), \tag{6}$$

where $\psi_1(x), \psi_2(x), ...$ are the components of the spinor field $\psi(x)$, the coefficients M_{jk} are complex numbers, $x \to \bar{x}$ is an isometry, and σ is either linear or antilinear. These equations may be written more cleanly using matrix notation: $\psi(x)$ is the column matrix with components $\psi_k(x)$, and $\sigma(\psi(x))$ is the column matrix with components $\sigma(\psi_k(x))$. With this notation, equations (5)-(6) are[8]

$$\text{First form:} \qquad \sigma(\psi(x)) = M\psi(\bar{x}), \tag{7}$$

$$\text{Second form:} \qquad \sigma(\psi(x)) = M\psi^*(\bar{x}), \tag{8}$$

where M is the square matrix with components M_{jk}. For most choices of the matrix M, these σs are not even $*$-automorphisms, much less symmetries. For either of these to be a $*$-automorphism, the matrix M must satisfy a consistency condition. Section 4 will introduce the consistency condition.

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[7] In lattice QFT, for each point x, each component $\psi_k(x)$ of $\psi(x)$ is an operator, and σ is applied to each of those operators – separately for each x. That's why I'm using the notation $\sigma(\psi(x))$ instead of $\sigma(\psi)(x)$.

[8] Equation (8) illustrates the value of using the notation A^* (instead of $A^\dagger$) to denote the adjoint of an operator A. The field $\psi(x)$ has multiple components, each one an operator $\psi_k(x)$ with adjoint $\psi_k^*(x)$. The notation $\psi^\dagger(x)$ conventionally represents the *transpose* of the matrix whose components are $\psi_k^*(x)$, which is not what we want here. We want the column matrix to remain a column matrix, but with each component replaced by its adjoint. That's what the notation $\psi^*(x)$ represents. When the components of $\psi(x)$ are treated as complex numbers (instead of as operators), $\psi^*(x)$ is the result of taking the complex conjugate of each component, again without taking the transpose of the matrix.

4 Symmetry: perspective and definition

In QFT, equations of motion may be viewed as a way of expressing the field operators at all times in terms of those at any one time.[9] If the effect of a $*$-automorphism on the field operators at one time has been specified, then its effect on the field operators at all other times is implied by the equations of motion. That implied effect may or may not be consistent with the effect asserted by equation (7) or (8), because those equations assert an effect for all times.[10] This article is about when/how the matrix M can be chosen to make the implied effect consistent with the asserted effect, when the equation of motion is the Dirac equation.

The Dirac equation has the form $D\psi(x) = 0$, where D is a differential operator[11] that will be described later. If the field operators satisfy $D\psi(x) = 0$ and if σ is a $*$-automorphism, then clearly $\sigma(D\psi(x)) = 0$. Together with the identity $\sigma(\partial\psi(x)) = \partial\sigma(\psi(x))$,[12] this implies

$$D\sigma(\psi(x)) = 0 \qquad \text{if } \sigma \text{ is linear,} \tag{9}$$

$$D^*\sigma(\psi(x)) = 0 \qquad \text{if } \sigma \text{ is antilinear,} \tag{10}$$

where D^* is the complex conjugate of D. Use equations (7)-(8) in these to get

$$\text{First form:} \qquad DM\psi(\bar{x}) = 0 \qquad \text{if } \sigma \text{ is linear,} \tag{11}$$

$$D^*M\psi(\bar{x}) = 0 \qquad \text{if } \sigma \text{ is antilinear.} \tag{12}$$

$$\text{Second form:} \qquad DM\psi^*(\bar{x}) = 0 \qquad \text{if } \sigma \text{ is linear,} \tag{13}$$

$$D^*M\psi^*(\bar{x}) = 0 \qquad \text{if } \sigma \text{ is antilinear.} \tag{14}$$

This article will use the consistency conditions (11)-(14) to determine when/how the matrix M can be chosen so that σ has a chance of being a symmetry of the

[9] Article 22871

[10] The time coordinate is part of the argument x.

[11] Here, *differential operator* means a combination of derivatives with respect to the spacetime coordinates, not an operator on the Hilbert space. A differential operator may act on the field operators (because the field operators are parameterized by the spacetime coordinates), which in turn act on the Hilbert space.

[12] Similarly, the adjoint of $\partial\psi_k(x)$ is $\partial\psi_k^*(x)$. In contrast to the situation in article 20554, the derivatives ∂ here are not (and don't represent) operators on the Hilbert space.

quantum model. These consistency conditions might not guarantee that σ satisfies the definition of *symmetry* in article 21916, because they don't address what σ does to products of field operators, and they might not respect the spectrum condition,[13] but for the sake of being concise, the remaining sections use the word **symmetry** to mean any map of the form (7) or (8) that satisfies one of the consistency conditions (9) or (10), shown more explicitly in (11)-(14). We can think of this roughly[14] as a necessary (but not sufficient) condition for σ to be a symmetry of the full quantum model.

This way of using the word *symmetry* is also suggested by treating the components $\psi_k(x)$ of $\psi(x)$ as ordinary complex-valued functions of x instead of as operators on a Hilbert space (section 1). Then we can think of σ as a map from the space of x-dependent spinors to itself, and we can call σ a *symmetry* of the Dirac equation if applying σ to any solution of the Dirac equation always gives another solution of the Dirac equation. The resulting condition on M is the same as the condition (11). The QFT perspective described above motivates a generalization of that picture, one that also considers maps of the second form (8) and that also considers antilinear maps.

[13] Article 21916

[14] I'm using the word *roughly* to be cautious, because I'm not sure that every symmetry of the algebra of observables of the free Dirac spinor quantum field can be lifted to a $*$-automorphism of the algebra of field operators.

5 A clarification about linearity

The generic definition of *linear transformation* or *linear map* refers to a generic vector space. In this article, two different vector spaces are in play, so whenever the word *linear* is used, we need to pay attention to which vector space is being referenced. In one of these vector spaces, linear transformations cannot mix the components $\psi_k(x)$ of the spinor field with their adjoints $\psi_k^*(x)$. In the other vector space, they can.

Article 86175 introduced matrix representations of Clifford algebra, which give the Clifford algebra a vector space W on which to act. At each point x in spacetime, the spinor field $\psi(x)$ is an element of W, with components $\psi_k(x)$.[15] A linear transformation of this vector space cannot mix the components $\psi_k(x)$ with their adjoints $\psi_k^*(x)$.[16]

The other vector space is the algebra of operators generated by the field operators. Any linear combination of such operators with complex coefficients is another such operator, so this is a vector space according to the generic definition. A linear transformation of this vector space *can* mix the operators $\psi_k(x)$ with their adjoints, because their adjoints also belong to this vector space.

In this article, σ denotes a map from the algebra of operators to itself, so it can still be linear even if it has the second form (8). A linear map cannot satisfy $\sigma(A) = A^*$ for all operators A, because that would contradict (2), but it may satisfy $\sigma(A) = A^*$ for a given collection of linearly independent operators A. Similarly, a map σ can still be antilinear even if it has the first form (7). All combinations occur in equations (11)-(14).

[15]This is a vector space in the generic sense, but a spinor field is *not* a vector field in the specific sense of a vector field in spacetime (as defined in article 09894). The number of components of a spinor field is typically different from the number of dimensions of spacetime.

[16]This article does not consider Majorana spinors. For a **Majorana spinor**, the adjoints $\psi_k^*(x)$ are linear combinations of the $\psi_k(x)$s. The (non)existence of such representations of the Clifford algebra depends on the signature of the spacetime metric (article 86175).

6 Notation and conventions

In this article, spacetime is flat and topologically trivial.[17] The number of spacetime dimensions will be denoted d. Lowercase boldface symbols will be used to denote spacetime vectors – vectors that belong to the tangent space of the spacetime manifold. The spacetime metric[18] defines a scalar product $g(\mathbf{a},\mathbf{b})$ between any two vectors $\mathbf{a},\mathbf{b}$. The components of a vector $\mathbf{v}$ in a basis $\mathbf{e}_0,\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_{d-1}$ will be denoted v^a. The Einstein summation convention will be used, so $\mathbf{v}=v^a\mathbf{e}_a$ has an implied sum over the index a. The basis vectors $\mathbf{e}_a$ are orthogonal to each other ($g(\mathbf{e}_a,\mathbf{e}_b)=0$ if $a\neq b$) and satisfy $g(\mathbf{e}_a,\mathbf{e}_a)=\pm 1$. In this article, the spacetime metric has lorentzian signature: the sign of $g(\mathbf{e}_0,\mathbf{e}_0)$ will be opposite to the sign of the other $g(\mathbf{e}_k,\mathbf{e}_k)$s. The mostly-minus and mostly-plus signature conventions will both be considered.

The Clifford algebra generated by spacetime vectors will be called *the* Clifford algebra. Instead of working directly with the abstract Clifford algebra,[19] this article uses a fixed irreducible[20] matrix representation. The square matrix representing a vector $\mathbf{v}$ will be denoted $\gamma(\mathbf{v})$. These matrices satisfy

$$\gamma(\mathbf{a})\gamma(\mathbf{b})+\gamma(\mathbf{b})\gamma(\mathbf{a})=2g(\mathbf{a},\mathbf{b})I \tag{15}$$

where I is the identity matrix. The matrix representing a basis vector $\mathbf{e}_a$ will be denoted $\gamma_a\equiv\gamma(\mathbf{e}_a)$ and called a **Dirac matrix**. The matrix $\gamma(\mathbf{v})$ representing any vector $\mathbf{v}$ is a linear combination of Dirac matrices, with the vector's components as coefficients:

$$\gamma(\mathbf{v})=v^a\gamma_a.$$

[17] For the rest of this article, the condition *topologically trivial* is implied whenever flat spacetime is mentioned. Spinor fields exist only on **spin manifolds**, manifolds that satisfy a particular topological condition (Lawson and Michelsohn (1989), theorem 2.1). All parallelizable manifolds – like topologically trivial flat spacetime – satisfy that condition, and some non-parallelizable manifolds also satisfy it (page 87 in Lawson and Michelsohn (1989), and page 239 in Parker and Toms (2009)). Some flat manifolds don't satisfy it (Lutowski and Putrycz (2014) and Dekimpe *et al* (2006)), but *topologically trivial* flat manifolds do.

[18] Article 48968

[19] Articles 03910 and 08264 work directly with the abstract Clifford algebra.

[20] A representation is called **irreducible** if it doesn't contain any smaller nontrivial representation.

Equation (15) implies

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2g_{ab}I, \tag{16}$$

where g_{ab} are the components of the metric tensor, defined by

$$g_{ab} \equiv g(\mathbf{e}_a, \mathbf{e}_b) = \begin{cases} \pm 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

The components g^{ab} of the inverse metric tensor, defined by

$$g^{ab}g_{bc} = \delta^a_c \equiv \begin{cases} 1 & \text{if } a = c \\ 0 & \text{otherwise,} \end{cases}$$

may be used to define a raised-index version of each Dirac matrix:

$$\gamma^a \equiv g^{ab}\gamma_b. \tag{17}$$

These satisfy

$$\gamma_a\gamma^b + \gamma^b\gamma_a = 2\delta^b_a I \qquad\qquad \gamma^a\gamma^b + \gamma^b\gamma^a = 2g^{ab}I. \tag{18}$$

The abbreviations[21]

$$\partial_a \equiv \frac{\partial}{\partial x^a} \qquad\qquad \gamma\partial \equiv \gamma^a\partial_a$$

will also be used.

[21] The abbreviation $\not{\partial} = \gamma^a\partial_a$ is more common in the physics literature, but the notation $\gamma\partial$ has advantages.

7 The Dirac equation

Let γ be a representation of the Clifford algebra on a complex vector space of the smallest possible dimension N. Each Dirac matrix γ^a is a matrix of size $N \times N$. In flat spacetime,[22] the **Dirac equation** is either

$$(i\gamma\partial - m)\psi(x) = 0 \qquad \text{(mostly-minus signature convention)}, \tag{19}$$

or

$$(\gamma\partial + m)\psi(x) = 0 \qquad \text{(mostly-plus signature convention)}, \tag{20}$$

depending on which signature convention is used.[23,24] The **(Dirac) spinor field**[25] $\psi(x)$ is a column matrix whose N components are functions of the spacetime coordinates x. The term with no derivatives is often called a **mass term**.[26] When the mass term is zero, the Dirac equation reduces to the **massless Dirac equation**

$$\gamma\partial\psi(x) = 0. \tag{21}$$

[22] In not-necessarily-flat spacetime, the equations shown here must be generalized to account for the fact that the spacetime metric – which is implicit in the definition of the Clifford algebra – may vary from one point to the next.

[23] The sign of the mass term is also a matter of convention. The convention used here seems to be the most common one. Example: Peskin and Schroeder (1995), equation (3.31).

[24] Yet another matter of convention: some authors include a minus sign on the right-hand side of equation (15), which reverses the correlation between the signature of the metric and the presence/absence of the factor of i in the Dirac equation.

[25] The coordinate-free concept of a tensor field makes sense on any smooth manifold (article 09894). In contrast, the coordinate-free concept of a spinor field depends on extra structure, namely a **spin structure**, that is not already implicit in the general definition of a smooth manifold (footnote 17 in section 6, and definition 4.10 in Figueroa-O'Farrill (2010)). The spin structure used implicitly in this article is the only one that topologically-trivial flat spacetime admits (Genauer (2004), proposition 2), so leaving it implicit won't cause any trouble.

[26] Section 8 will explain why.

8 Why it's called a mass term

To motivate the name *mass term*, multiply equations (19) and (20) by $i\gamma\partial+m$ and $\gamma\partial-m$, respectively, and use the second identity in (18) to get

$$
\begin{aligned}
(-g^{ab}\partial_a\partial_b - m^2)\psi(x) &= 0 \qquad\qquad \text{(mostly-minus signature convention)},\\
(g^{ab}\partial_a\partial_b - m^2)\psi(x) &= 0 \qquad\qquad \text{(mostly-plus signature convention)}.
\end{aligned}
$$

This says that each component of the spinor field satisfies the same equation of motion as a free scalar quantum field whose particles have mass m.[27] The components of the spinor field are not scalar fields, because the equation of motion (the Dirac equation in this case) mixes them with each other, but m still turns out to be the mass of a single particle in a quantum model that has the Dirac equation as the field's equation of motion.

[27] Article 30983

9 Reflections in Clifford algebra

Every isometry[28] of flat spacetime may be expressed as a composition of reflections.[29] Reflections are especially easy to describe using Clifford algebra. Let $\mathbf{r}$ be a vector with $g(\mathbf{r},\mathbf{r}) = \pm 1$, and let $x \to \bar{x}$ be a reflection along the direction $\mathbf{r}$ that fixes (doesn't move the points in) a hyperplane orthogonal to $\mathbf{r}$. Using Clifford algebra, the effect of the reflection on any vector $\mathbf{v}$ may be written[30]

$$\mathbf{v} \to \bar{\mathbf{v}} \equiv \frac{-\mathbf{rvr}}{g(\mathbf{r},\mathbf{r})}.$$

Multiply this equation by $\mathbf{r}$ and use $\mathbf{r}^2 = g(\mathbf{r},\mathbf{r})$ to get

$$\mathbf{r}\bar{\mathbf{v}} = -\mathbf{vr},$$

which implies

$$\gamma(\mathbf{r})\gamma(\bar{\mathbf{v}}) = -\gamma(\mathbf{v})\gamma(\mathbf{r}) \tag{22}$$

in any matrix representation. For any differentiable function f, the quantity with components $\partial^a f \equiv g^{ab}\partial_b f$ is a vector field, so equation (22) implies

$$\gamma(\mathbf{r})(\gamma\bar{\partial} f) = -(\gamma\partial f)\gamma(\mathbf{r}). \tag{23}$$

[28] An **isometry** is a transformation $x \to \bar{x}$ for which $g_{ab}(\bar{x})d\bar{x}^a\, d\bar{x}^b = g_{ab}(x)dx^a\, dx^b$.

[29] Article 39430

[30] Article 08264

10 Some symmetries of the massless Dirac equation

The massless Dirac equation will be considered first, because it is more symmetric[31] than the Dirac equation with $m\neq 0$. This section shows that for any d, the massless Dirac equation (21) has symmetries corresponding to all reflections, which implies that it has symmetries corresponding to all isometries. Here, and in the rest of this article, *symmetry* is defined as explained in section 4.

To show that it has symmetries corresponding to all reflections, let $\mathbf{r}$ be a vector with $g(\mathbf{r},\mathbf{r})=\pm 1$, and let $x\to\bar{x}$ be a reflection along $\mathbf{r}$ that fixes a hyperplane orthogonal to $\mathbf{r}$. Consider the maps defined by

$$\sigma\big(\psi(x)\big)\equiv M\psi(\bar{x}) \qquad\qquad \sigma \text{ linear} \tag{24}$$

with

$$M=\pm\gamma(\mathbf{r}). \tag{25}$$

According to equation (23), this satisfies $\gamma\partial\, M=-M\,\gamma\bar{\partial}$, which implies

$$\gamma\partial\, M\psi(\bar{x})=-M\,\gamma\bar{\partial}\psi(\bar{x}) \tag{26}$$

with

$$\bar{\partial}_a\equiv\frac{\partial}{\partial\bar{x}^a}. \tag{27}$$

The massless Dirac equation (21) may be written as $\gamma\bar{\partial}\psi(\bar{x})=0$ just by relabelling the coordinates, so equation (26) shows that the map defined by (24)-(25) satisfies the condition (9) to be a symmetry of the massless Dirac equation (21).[32]

These symmetries are linear, not antilinear, even if the direction $\mathbf{r}$ is timelike. The time-reflection symmetries that we normally consider in quantum theory are antilinear instead. Antilinear symmetries involving time-reflection will be addressed later, starting in section 18.

[31] This article considers only symmetries corresponding to ordinary spacetime isometries, ignoring other conformal symmetries (article 38111), but the massless Dirac equation is still more symmetric (sections 17, 25, and 29).

[32] Recall footnotes 11-12 in section 4: the derivatives ∂ are not operators on the Hilbert space. The quantity $\sigma(\partial\psi(x))$ is equal to $\partial\sigma(\psi(x))$, not to $\bar{\partial}\sigma(\psi(x))$. Similarly, the coordinates x are not affected by σ. The field operators are parameterized by x, and the map σ may permute the field operators, but σ does not affect x.

11 Symmetries in the spin group

Section 10 showed that the massless Dirac equation has symmetries of the form

$$\sigma\big(\psi(x)\big) = \pm\gamma(\mathbf{r})\psi(\bar{x})$$

whenever $x \to \bar{x}$ is a reflection along the direction $\mathbf{r}$. The **spin group**[33] consists of transformations that may each be expressed as a composition of an even number of these reflections. This section shows that the group of symmetries of the Dirac equation still includes the spin group even when $m \neq 0$.[34]

Let $x \to \bar{x}$ be an isometry given by the composition of two reflections along the directions $\mathbf{r}_1$ and $\mathbf{r}_2$,[35] and consider the maps defined by

$$\sigma\big(\psi(x)\big) \equiv M\psi(\bar{x}) \qquad\qquad \sigma \text{ linear} \tag{28}$$

with

$$M = \pm\gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2). \tag{29}$$

According to equation (23), this satisfies $\gamma\partial\, M = M\,\gamma\bar{\partial}$, which implies

$$(i\gamma\partial - m)\, M\psi(\bar{x}) = M\,(i\gamma\bar{\partial} - m)\psi(\bar{x}) \tag{30}$$

with $\bar{\partial}$ defined as before.[36] The Dirac equation (19) may also be written

$$(i\gamma\bar{\partial} - m)\psi(\bar{x}) = 0 \tag{31}$$

just by relabelling the coordinates, so (30) shows that the map defined by (28)-(29) satisfies the condition (9) to be a symmetry of the Dirac equation for any value of m.

[33] Article 08264

[34] This section uses one signature convention, equation (19). The analysis for the other signature convention, equation (20), is similar.

[35] The directions $\mathbf{r}_1$ and $\mathbf{r}_2$ don't need to be orthogonal to each other.

[36] Equation (27)

12 Single-reflection symmetries when $m \neq 0$

Some transformations that were symmetries of the massless Dirac equation are no longer symmetries when $m \neq 0$.[37] If the analysis in section 10 were repeated with $m \neq 0$, equation (26) would be replaced by

$$(i\gamma\partial - m)M\psi(\bar{x}) = M(-i\gamma\bar{\partial} - m)\psi(\bar{x}).$$

Compared to the original Dirac equation (19), the derivative term on the right-hand side has the wrong sign relative to the mass term. This sign comes from the sign in equation (23). The sign didn't make any difference when $m = 0$, but it does when $m \neq 0$. As a result, the single-reflection map σ that was defined in section 10 is not a symmetry of the Dirac equation when $m \neq 0$.

Section 13 will show that when d is even, the Dirac equation with $m \neq 0$ actually does have symmetries corresponding to all individual reflections, using a different matrix M in place of (25) to cure the sign problem.

Section 14 will show that when d is odd, the Dirac equation (19) with $m \neq 0$ does not have any linear symmetries of the form (7) corresponding to individual reflections.

Single-reflection symmetries of other types, equations (12)-(14), will be considered starting in section 18.

[37]This statement refers to symmetries that correspond to ordinary isometries of spacetime, ignoring other conformal symmetries (footnote 31 in section 10).

13 Another option when d is even

Let Γ for the product of all d Dirac matrices, normalized so that $\Gamma^2 = 1$.[38] When d is even, we can satisfy the requirement $\Gamma^2 = 1$ by writing[39]

$$\Gamma \equiv \begin{cases} i\gamma_0\gamma_1\gamma_2\cdots\gamma_{d-1} & \text{when } d = 4n \quad (n \text{ denotes an integer}), \\ \gamma_0\gamma_1\gamma_2\cdots\gamma_{d-1} & \text{when } d = 4n+2. \end{cases} \tag{32}$$

When d is odd, the matrix Γ is not very useful: if γ is an irreducible representation of the Clifford algebra, then Γ is proportional to the identity matrix when d is odd.[40]

When d is even, Γ anticommutes with every Dirac matrix. We can use this to fix the sign obstacle that we encountered in section 12. Define $\bar{x}$ and $\mathbf{r}$ as in section 10, and consider the maps defined by

$$\sigma\big(\psi(x)\big) \equiv M\psi(\bar{x}) \qquad\qquad \sigma \text{ linear}$$

with

$$M = \pm\gamma(\mathbf{r})\Gamma. \tag{33}$$

According to equation (23), this satisfies $\gamma\partial\, M = M\,\gamma\bar{\partial}$, which can be used to show that this σ is a symmetry of the Dirac equation for any m when d is even.[41] Altogether, the Dirac equation has symmetries corresponding to all reflections, and therefore to all isometries, when d is even.

[38] In the context of four-dimensional spacetime, the matrix Γ is traditionally denoted γ^5 (or γ_5).

[39] To confirm that this satisfies $\Gamma^2 = 1$, use the fact that the bivectors $B_k \equiv \gamma_{2k}\gamma_{2k+1}$ all commute with each other and that they all satisfy $B_k^2 = -1$ except B_0, which satisfies $B_0^2 = 1$. If the number of B_ks with $k > 0$ is odd, then we need the factor of i to get $\Gamma^2 = 1$. Otherwise, we don't.

[40] Article 86175

[41] A symmetry whose corresponding isometry is a composition of reflections along an odd number of spacelike directions is often called **parity**, denoted P. When d is even, a symmetry that reflects along all $d-1$ of the spatial axes satisfies this definition. (Some authors might define *parity* using a reflection along all $d-1$ of the spatial axes for any d, whether even or odd, but see footnote 51 in section 19.) When d even, composing the reflection-symmetries described in this section for all $d-1$ spatial axes gives $M \propto \gamma^0$, as in equation (3.126) in Peskin and Schroeder (1995).

14 Non-existence when d is odd

The option described in section 13 only works when d is even, because Γ commutes with everything when d is odd.

In fact, when d is odd, the Dirac equation (19) does not have any linear symmetries of the form (7) corresponding to individual reflections when $m \neq 0$.[42] To prove this, consider a reflection along the spacelike[43] direction, which we can take to be $\mathbf{e}_1$ without loss of generality because rotation symmetry was already established in section 11. Let $x \to \bar{x}$ be the isometry defined by this reflection, and consider a map of the form

$$\sigma\big(\psi(x)\big) = M\psi(\bar{x}) \qquad\qquad \sigma \text{ linear}$$

for some matrix M. The condition (9) for this to be a symmetry of the Dirac equation (19) is that the quantity

$$(i\gamma\partial - m)M\psi(\bar{x})$$

must be zero. The Dirac equation implies $m\psi(\bar{x}) = i\gamma\bar{\partial}\psi(\bar{x})$, so the preceding quantity may also be written

$$i\gamma\partial\, M\psi(\bar{x}) - M\, i\gamma\bar{\partial}\,\psi(\bar{x}).$$

The field operators at a given time are all linearly independent, so this can be zero only if M satisfies the condition $\gamma\partial\, M = M\,\gamma\bar{\partial}$, which means that M must commute with every Dirac matrix except $\gamma_1 = \gamma(\mathbf{e}_1)$, and it must anticommute with γ_1. If such an M existed, then $\gamma_1 M$ would anticommute with every Dirac matrix, including γ_1. When d is odd, this implies that M would also need to anticommute with the product of all Dirac matrices, because the number of factors in that product is odd. This is impossible, though, because the product of all d Dirac matrices is proportional to the identity matrix I when d is odd,[44] and a nonzero matrix M cannot anticommute with I. This completes the proof.

[42]For $d = 3$, this is acknowledged in the text around equations (2.2)-(2.4) in Witten (2015) and in the first paragraph of Biswas and Semenoff (2022).

[43]The analysis for a timelike direction is similar.

[44]Article 86175

15 The Weyl equation

When the number d of spacetime dimensions is even, the (Dirac) spinor field ψ may be separated into two parts – two **chiral spinor** fields – that are not mixed with each other by the even subalgebra of the Clifford algebra,[45] and the massless Dirac equation separates into a pair of **Weyl equations**. The Weyl equation is the equation of motion for a single free chiral spinor field.

To work this out explicitly, suppose that d is even, and define Γ as in section 13. The matrices

$$P_\pm \equiv \frac{I \pm \Gamma}{2} \tag{34}$$

are mutually orthogonal projectors:

$$P_\pm^2 = P_\pm \qquad\qquad P_+P_- = P_-P_+ = 0 \qquad\qquad P_+ + P_- = I.$$

This implies that the massless Dirac equation (21) is equivalent to the pair of equations

$$\gamma(\mathbf{v})\gamma\partial P_+\psi = 0 \qquad\qquad \gamma(\mathbf{v})\gamma\partial P_-\psi = 0 \tag{35}$$

for any vector $\mathbf{v}$ with $\mathbf{v}^2 \neq 0$, so that the matrix $\gamma(\mathbf{v})$ is invertible.[46,47]

The overall factor of $\gamma(\mathbf{v})$ doesn't affect the equivalence of the pair (35) with equation (21), but it does accomplish something else. If W denotes the vector space on which the matrix representation γ acts, then the projectors $P_\pm$ separate W into two subspaces W_+ and W_- that each have half the number of dimensions of W. Each Dirac matrix γ^a mixes these two subspaces with each other because $\gamma^a P_\pm = P_\mp\gamma^a$, but the combination $\gamma(\mathbf{v})\gamma\partial$ does not, so including the factor of $\gamma(\mathbf{v})$ allows the first and second equations in (35) to be defined entirely within W_+ and W_-, respectively. In W_+ or W_-, the other components of $P_\pm\psi$ are nonexistent instead of merely being zero. This perspective is usually implied when the name **Weyl equation** is used.

[45] Article 86175

[46] The conventional choice is $\gamma(\mathbf{v}) = \gamma^0$ so that the matrix in the time-derivative term is the identity matrix, as in Peskin and Schroeder (1995), equation (3.40).

[47] As long as $\mathbf{v}^2 \neq 0$, we can eliminate the factor $\gamma(\mathbf{v})$ by multiplying equations (35) by $\gamma(\mathbf{v})$.

16 Some symmetries of the Weyl equation

When applied to the Weyl equations (35), the reasoning that led to the symmetry conditions (9)-(10) for the Dirac equation leads to these symmetry conditions for the Weyl equations:

$$\gamma(\mathbf{v})\gamma\partial\,\sigma\big(P_\pm\psi(x)\big) = 0 \qquad \text{if } \sigma \text{ is linear,} \tag{36}$$

$$\big(\gamma(\mathbf{v})\gamma\partial\big)^*\sigma\big(P_\pm\psi(x)\big) = 0 \qquad \text{if } \sigma \text{ is antilinear,} \tag{37}$$

with the understanding that σ should not mix the two subspaces $W_\pm$ with each other.

Section 11 showed that the symmetry group of the Dirac equation includes the spin group. This is still true for the Weyl equation, because the matrix $M = \pm\gamma(\mathbf{r}_1)\gamma(\mathbf{r}_2)$ commutes with the projections $P_\pm$, so the maps defined by

$$\sigma\big(\psi(x)\big) = M\psi(\bar{x}) \tag{38}$$

don't mix the two subspaces $W_\pm$ with each other. This shows that the symmetry group of the Weyl equation includes the spin group.

The single-reflection maps defined by (25) cannot be symmetries of the Weyl equation, because multiplication by $\gamma(\mathbf{r})$ mixes the subspaces $W_\pm$ with each other:

$$\gamma(\mathbf{r})P_\pm\psi = P_\mp\gamma(\mathbf{r})\psi.$$

The same is true for the other single-reflection maps defined by (33). If $x \to \bar{x}$ is a reflection along the direction $\mathbf{r}$, then a map of the form (38) would need to use an M satisfying $\gamma\partial M = \pm M\gamma\bar{\partial}$ in order to be a symmetry, but this same condition implies that M anticommutes with Γ. This implies that M mixes the subspaces $W_\pm$ with each other, so the Weyl equation cannot have any linear symmetries of the form (7) whose corresponding isometry consists of a single reflection. Single-reflection symmetries of other types will be considered soon, starting in section 18.

17 Summary of symmetries described so far

The preceding sections demonstrated the existence of some symmetries by explicit construction and proved the non-existence of some others. Those (non)existence results are summarized here in tables 1 and 2.

	Dirac		Weyl
	$m \neq 0$	$m = 0$	$m = 0$
d even	yes	yes	**no**
d odd	**no**	yes	

Table 1 – Summary of when linear symmetries of the form $\sigma(\psi(x)) = M\psi(\bar{x})$ exist when the isometry $x \to \bar{x}$ is a single reflection. "Yes" means that such symmetries exist for all such isometries, and "no" means that such a symmetry does not exist for any such isometry. The three columns are for the Dirac equation with $m \neq 0$, the Dirac equation with $m = 0$, and the Weyl equation, respectively.

	Dirac		Weyl
	$m \neq 0$	$m = 0$	$m = 0$
d even	yes	yes	yes
d odd	yes	yes	

Table 2 – Summary of when linear symmetries of the form $\sigma(\psi(x)) = M\psi(\bar{x})$ exist when the isometry $x \to \bar{x}$ is a composition of two reflections. "Yes" means that such symmetries exist for all such isometries. These symmetries constitute the spin group (section 11).

18 Other discrete symmetries

Section 4 mentioned four symmetry types:

- linear symmetries of the form (7),
- linear symmetries of the form (8),
- antilinear symmetries of the form (7),
- antilinear symmetries of the form (8).

All of the examples in the previous sections were of the first type: they were all linear symmetries of the form (7). The rest of this article is mostly about the other three types.

Symmetries of the other three types, and also symmetries of the first type when the number of reflections along timelike or spacelike directions is odd, are sometimes called **discrete symmetries**.[48] In spite of that name, each of these symmetries belongs to a continuum of symmetries (even if it isn't connected to the identity by any continuous path), because each one may be composed with arbitrary transformations in the spin group.

[48]Peskin and Schroeder (1995), section 3.6

19 CPT symmetry: introduction

The general principles of relativistic QFT in flat spacetime imply the existence of an antilinear symmetry called **CPT symmetry**, which involves reflections along one timelike direction and along an odd number of spacelike directions.[49,50] This general result is called the **CPT theorem**. The name CPT comes from the fact that some models have other discrete symmetries traditionally called C, P, and T whose composition gives CPT symmetry.[51] CPT symmetry is more general, though: its existence is guaranteed by the general principles of relativistic QFT in flat spacetime, whereas the existence of other discrete symmetries is not.[52]

Sections 11 and 16 showed that the Dirac equation (for any m) and the Weyl equation both have <u>linear</u> symmetries corresponding to any isometry $x \to x_{PT}$ that is composed of one reflection along a timelike direction and one reflection along a spacelike direction. Those symmetries belong to the spin group. In contrast, the symmetries whose existence is promised by the CPT theorem are <u>antilinear</u> symmetries. Section 20 will show that the Dirac and Weyl equations both have such symmetries.

[49] Witten (2015)

[50] This clearly requires $d \geq 2$, where d is the number of dimensions of spacetime.

[51] Sometimes other permutations of the letters are used, as in Streater and Wightman (1980). Witten (2015) suggests calling it **CRT symmetry** instead, using R for *(spatial) reflection* instead of P for *parity*. Whatever we call it, this symmetry is understood to involve a reflection along an odd number of spatial directions (not an even number), regardless of whether the total number of dimensions of space is even or odd. The alternate name suggested by Witten (2015) is meant to help remind people of this.

[52] Witten (2015), section 2.1.1, footnote 6

20 CPT symmetry of the Dirac equation

Let $x \to x_{PT}$ be the isometry defined by reflecting along the timelike direction $\mathbf{e}_0$ and the spacelike direction $\mathbf{e}_1$,[53] and write $\partial_a^{PT} \equiv \partial/\partial x_{PT}^a$. The matrix $M = \pm\gamma_0\gamma_1$ satisfies

$$\gamma\partial\, M = M\, \gamma\partial^{PT} \tag{39}$$

which clearly implies[54]

$$(\gamma\partial)^*\, M^* = M^*\, (\gamma\partial^{PT})^*. \tag{40}$$

Now let D denote the Dirac differential operator with either signature convention, equation (19) or (20). Use (40) to get

$$D^* M^* \psi^*(x_{PT}) = M^*(D^{PT})^*\psi^*(x_{PT}) = M^*\big(D^{PT}\psi(x_{PT})\big)^*. \tag{41}$$

with D^{PT} obtained from D by replacing ∂ with ∂^{PT}. The original Dirac equation $D\psi(x) = 0$ is equivalent to $D^{PT}\psi(x_{PT}) = 0$. Use this on the right-hand side of (41) to deduce that the transformation σ defined by

$$\sigma\big(\psi(x)\big) = M^*\psi^*(x_{PT}) \qquad\qquad \sigma \text{ antilinear} \tag{42}$$

satisfies the condition (10), so this is a symmetry of the Dirac equation for any m and any d.

This symmetry is antilinear and involves reflections along one timelike direction and an odd number (one) of spacelike directions, so this is the type of symmetry promised by the CPT theorem. The composition of this symmetry with with any linear symmetry in the spin group[55] gives another symmetry of the type promised by the CPT theorem.

[53] A similar analysis works for any pair of directions in which one is timelike and one is spacelike. They don't need to be orthogonal to each other.

[54] For any matrix, the asterisk denotes componentwise complex conjugation (or componentwise adjoint if the components are operators on a Hilbert space, as in ψ^*). Remember that $\partial^* = \partial$ (footnote 12 in section 4).

[55] Section 11

21 CPT symmetry of the Weyl equation

Define $M \equiv \pm\gamma^0\gamma^1$, as in section 20. For the Weyl equation, the condition for (42) to be a symmetry is equation (37). To apply the map (42) in this context, we can use either of two perspectives.

In one perspective, the two chiral subspaces W_+ and W_- that were defined in section 15 are viewed as two complementary subspaces of a single Dirac spinor space W. Then we can use (42) as it stands. Explicitly,

$$\sigma\big(P_\pm\psi(x)\big) = P_\pm^*\sigma\big(\psi(x)\big) = P_\pm^* M^*\psi^*(x_{PT}) = M^* P_\pm^*\psi^*(x_{PT}). \tag{43}$$

The first step uses the fact that σ is defined to be antilinear. The second step uses the effect of σ on ψ shown in (42). The third step uses the fact that M commutes with $P_\pm$ because it's a product of two Dirac matrices.

In the other perspective, the two chiral subspaces W_+ and W_- are viewed as independent entities, with no reference to W. With this perspective, the map σ should be defined only on the chiral spinors $P_+\psi$ or $P_-\psi$, without requiring it to be defined it on the Dirac spinor ψ. Explicitly, we can define σ by

$$\sigma\big(P_\pm\psi(x)\big) \equiv M^*\big(P_\pm\psi(x_{PT})\big)^* \qquad\qquad \sigma \text{ antilinear}, \tag{44}$$

and then the end result is the same as in the previous perspective.

Either way,

$$\begin{aligned}
\big(\gamma(\mathbf{v})\gamma\partial\big)^*\sigma\big(P_\pm\psi(x)\big) &= \big(\gamma(\mathbf{v})\gamma\partial\big)^*\big(MP_\pm\psi(x_{PT})\big)^* && \text{(using (43) or (44))}\\
&= \big(\gamma(\mathbf{v})\gamma\partial\, MP_\pm\psi(x_{PT})\big)^* \\
&= \big(M\,\gamma(\mathbf{v})\gamma\partial^{PT}\,P_\pm\psi(x_{PT})\big)^* && \text{(using (39))}\\
&= 0 && \text{(using (35))}.
\end{aligned}$$

This shows that σ satisfies the symmetry condition (37), and it doesn't mix the subspaces $W_\pm$ with each other, so this is a symmetry of the Weyl equation.

22 (In)equivalence of matrix representations

The remaining sections consider linear symmetries of the form (8) and antilinear symmetries of the form (7). Proofs of the existence or non-existence of a given symmetry will still be representation-independent, as in the preceding sections, but explicit constructions of symmetries that do exist will be given only for a specific matrix representation, namely the one that will be described in section 23. This section establishes some ingredients that will be used to prove existence or non-existence of these symmetries without committing to any specific representation.

Two representations γ and γ' are **equivalent** to each other if the condition

$$\gamma(\mathbf{v})M = M\gamma'(\mathbf{v})$$

is satisfied for all vectors $\mathbf{v}$ by some invertible matrix M. The matrix M is said to **intertwine** the two representations. This condition may also be written

$$M^{-1}\gamma(\mathbf{v})M = \gamma'(\mathbf{v}),$$

which says that one representation can be obtained from the other by a change of basis of the vector space W on which the representation acts.[56]

Given one irreducible representation of the Clifford algebra, replacing each Dirac matrix γ^a with $(\gamma^a)^*$ gives another one, and replacing each Dirac matrix γ^a with $-(\gamma^a)^*$ gives another one. The rest of this section answers the question: which of these representations are equivalent to each other? The answer depends on the value of d modulo 4. The answer is simplest when d is even, so that case will be treated first. When d is odd, the answer depends on whether $d = 4n + 1$ or $d = 4n + 3$,[57] and it also depends on whether the mostly-minus or mostly-plus signature convention is used.

[56] This is a vector space over $\mathbb{C}$. It should not be confused with the vector space to which the spacetime vectors $\mathbf{v}$ belong, which is a vector space over $\mathbb{R}$. These two vector spaces typically have different numbers of dimensions.

[57] n denotes an integer.

If d is even, then all irreducible representations are equivalent to each other.[58] This implies that when d is even, an invertible matrix M satisfying

$$\gamma^a M = M(\gamma^a)^* \tag{45}$$

must exist, and that another invertible matrix M satisfying

$$\gamma^a M = -M(\gamma^a)^* \tag{46}$$

must also exist. These results will be used later to prove the existence of some symmetries when d is even.

When d is odd, exactly two inequivalent irreducible representations exist.[58] When the mostly-minus signature convention is used, they are distinguished from each other by the signs in[59]

$$\gamma^0\gamma^1\gamma^2\cdots\gamma^{d-1} = \begin{cases} \pm I & \text{if } d = 4n+1, \\ \pm iI & \text{if } d = 4n+3. \end{cases} \tag{47}$$

When the mostly-plus signature convention is used, the options become

$$\gamma^0\gamma^1\gamma^2\cdots\gamma^{d-1} = \begin{cases} \pm iI & \text{if } d = 4n+1, \\ \pm I & \text{if } d = 4n+3. \end{cases} \tag{48}$$

The absense or presence of the factor of i is determined by whether the product of all Dirac matrices squares to $+1$ or -1, respectively. Changing the sign of every Dirac matrix changes the sign of the products (47)-(48) because d is odd, so two representations that differ in the signs of all of their Dirac matrices cannot be equivalent to each other. Only two inequivalent representations exist, so the representation whose Dirac matrices are γ^a must be equivalent either to the one whose Dirac matrices are $(\gamma^a)^*$ or to the one whose Dirac matrices are $-(\gamma^a)^*$. The signature convention options and the options $d \in \{4n+1,\ 4n+3\}$ define these four cases:

[58] Article 87696

[59] Shimizu (1985)

- In the mostly-minus convention with $d = 4n+1$, replacing every Dirac matrix by its complex conjugate doesn't change the sign of the product (47),[60] so the condition (45) has a solution, but the condition (46) does not.[61]

- In the mostly-minus convention with $d = 4n + 3$, replacing every Dirac matrix by its complex conjugate changes the sign of the product (47),[62] so the condition (46) has a solution, but the condition (45) does not.

- In the mostly-plus convention with $d = 4n + 1$, replacing every Dirac matrix by its complex conjugate changes the sign of the product (48), so the condition (46) has a solution, but the condition (45) does not.

- In the mostly-plus convention with $d = 4n + 3$, replacing every Dirac matrix by its complex conjugate doesn't change the sign of the product (48), so the condition (45) has a solution, but the condition (46) does not.

These results will be used later to prove the existence of some symmetries, and the non-existence of others, when d is odd.

When d is even, an irreducible representation of the Clifford algebra contains two irreducible representations of $\text{Cliff}_{\text{even}}$, the even part of the Clifford algebra. An element of the vector space on which an irreducible representation of $\text{Cliff}_{\text{even}}$ acts is a chiral spinor. The two irreducible representations of $\text{Cliff}_{\text{even}}$ are distinguished from each other by the signs in[63]

$$\gamma^0\gamma^1\gamma^2\cdots\gamma^{d-1} \sim \begin{cases} \pm iI & \text{if } d = 4n, \\ \pm I & \text{if } d = 4n+2, \end{cases} \tag{49}$$

[60] This follows from the fact that the right-hand side of (47) does not have a factor of i in this case.

[61] More explicitly: exactly two inequivalent irreducible representations exist, and flipping the sign of every Dirac matrix gives an inequivalent representation, so exactly one of the two conditions (45)-(46) has a solution, namely the one for which that replacement (either (45) or (46)) doesn't change the sign in (47) (respectively (48) if the other signature convention is used).

[62] This follows from the fact that the right-hand side of (47) does a factor of i in this case.

[63] Compare this to equation (32).

where $\sim$ denotes equality within the context of one of the two chiral subspaces $W_\pm$ that were defined in section 15. The signature convention doesn't matter here, because the even part of the Clifford algebra is the same in either convention.[64] Reasoning like what that was used above for odd d may be used again here, now for the options $d \in \{4n,\, 4n+2\}$:

- When $d = 4n$, complex conjugation changes the sign of the product (49) (because it changes the sign of the iI on the right-hand side), so it exchanges two inequivalent irreducible representations $\text{Cliff}_{\text{even}}$ with each other. This shows that an irreducible representation $\text{Cliff}_{\text{even}}$ is not equivalent to its complex conjugate when $d = 4n$.

- When $d = 4n + 2$, complex conjugation doesn't change the sign of the product (49), so it doesn't exchange two inequivalent irreducible representations $\text{Cliff}_{\text{even}}$ with each other. $\text{Cliff}_{\text{even}}$ has only two irreducible representations, so this shows that an irreducible representation $\text{Cliff}_{\text{even}}$ is equivalent to its complex conjugate when $d = 4n + 2$.

These results will be used later to prove the existence of some symmetries of the Weyl equation, and the non-existence of others.

[64] Article 03910

23 A specific matrix representation

This section describes a specific representation that will be used to construct examples of linear symmetries of the form (8) and antilinear symmetries of the form (7). The representation will be assembled using tensor products of the 2×2 matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

as explained in article 86175.

When the mostly-minus convention is used for the signature, γ^0 squares to $+1$ and the other Dirac matrices square to -1. In this case, we can use this representation for even d:

$$\begin{aligned}
\gamma^0 &= X \otimes 1 \otimes 1 \otimes \cdots \\
\gamma^1 &= Y \otimes 1 \otimes 1 \otimes \cdots \\
\gamma^2 &= Z \otimes iX \otimes 1 \otimes \cdots \\
\gamma^3 &= Z \otimes \ \ Y \otimes 1 \otimes \cdots \\
\gamma^4 &= Z \otimes Z \otimes iX \otimes \cdots \\
\gamma^5 &= Z \otimes Z \otimes \ \ Y \otimes \cdots
\end{aligned}$$

and so on. The Dirac matrices with a factor of iX have only imaginary components, and the rest have only real components. For odd d, we can use the same representation together with the additional Dirac matrix[65]

$$\begin{aligned}
\gamma^{d-1} &= \prod_{a<d-1} \gamma^a \qquad \text{if } d = 4n+1, \\
\gamma^{d-1} &= i \prod_{a<d-1} \gamma^a \qquad \text{if } d = 4n+3.
\end{aligned}$$

[65] Compare this to equation (47).

The normalization is chosen so that γ^{d-1} squares to -1. In both cases (all odd d), the resulting γ^{d-1} is imaginary.[66]

When the mostly-plus convention is used for the signature, γ^0 squares to -1 and the other Dirac matrices square to $+1$. In this case, we can obtain a representation by including an extra factor of i in each of the Dirac matrices shown above.

The number of real/imaginary Dirac matrices in each case is summarized here in tables 3 and 4.

	# real γ^as	# imag γ^as
$d = 4n$	odd	odd
$d = 4n + 1$	odd	even
$d = 4n + 2$	even	even
$d = 4n + 3$	even	odd

Table 3 – The number of real/imaginary Dirac matrices in the representation constructed above when the mostly-minus signature convention is used.

	# real γ^as	# imag γ^as
$d = 4n$	odd	odd
$d = 4n + 1$	even	odd
$d = 4n + 2$	even	even
$d = 4n + 3$	odd	even

Table 4 – The number of real/imaginary Dirac matrices in the representation constructed above when the mostly-plus signature convention is used.

[66] When applied to a Dirac matrix, the adjectives *real* and *imaginary* mean that the nonzero components of the matrix are all real numbers or all imaginary numbers (multiples of i), respectively.

24 Internal symmetries: introduction

Now consider maps of any of these forms:

$$\sigma\big(\psi(x)\big) = M\psi(x) \qquad \sigma \text{ linear,} \tag{50}$$
$$\sigma\big(\psi(x)\big) = M\psi(x) \qquad \sigma \text{ antilinear,} \tag{51}$$
$$\sigma\big(\psi(x)\big) = M\psi^*(x) \qquad \sigma \text{ linear,} \tag{52}$$
$$\sigma\big(\psi(x)\big) = M\psi^*(x) \qquad \sigma \text{ antilinear.} \tag{53}$$

The next goal is to explore when (for which d) the Dirac or Weyl equation has symmetries of these forms. When such symmetries do exist, we will determine the appropriate matrix M. These are **internal symmetries**, which means that the corresponding spacetime isometry is the identity transformation.[67]

A symmetry of the form (50) always exists, namely the trivial symmetry in which M is equal to the identity matrix. A symmetry of the form (53) also always exists, again with M equal to the identity matrix. This can be demonstrated by taking the adjoint of the whole equation of motion, as in section 20.

The more interesting forms are (51) and (52), because symmetries with these forms are not always guaranteed to exist. We only need to analyze one of these forms, because whenever a symmetry of the form (52) exists, we can compose it with a symmetry of the form (53) to get one of the form (51), and conversely. Section 26 will analyze the form (52).

Quantum electrodynamics in four-dimensional spacetime has an internal symmetry called **charge conjugation**, denoted C, whose effect on the spinor field has the form (52).[68,69] This is the heritage of the C in the name *CPT symmetry*, but remember that CPT symmetry is more general: its existence is guaranteed by the general principles of relativistic QFT in flat spacetime, but the existence of a symmetry like C is not.

[67] Notice that the argument on the right-hand side of equations (50)-(53) is x, not $\bar{x}$.

[68] Peskin and Schroeder (1995), section 3.6, pages 70-71

[69] Section 1.1 in Cordova *et al* (2018) mentions a more general definition of *charge conjugation.*

25 Internal symmetries: preview

Table 5 summarizes the existence results that will be deduced in section 26.

	Dirac		Weyl
	$m \neq 0$	$m = 0$	$m = 0$
$d = 4n$	yes	yes	**no**
$d = 4n + 1$	**no**	yes	
$d = 4n + 2$	yes	yes	yes
$d = 4n + 3$	yes	yes	

Table 5 – Summary of when internal linear symmetries of the form (52) exist. "Yes" means that such symmetries exist, and "no" means that such a symmetry does not exist.

26 Internal symmetries: analysis

For a map σ of the form (52), the condition (9) for σ to be a symmetry is shown here for both signature conventions, mostly minus and mostly plus:

$$\begin{aligned} (i\gamma\partial - m)M\psi^* &= 0 && \text{(mostly-minus convention)}, \\ (\gamma\partial + m)M\psi^* &= 0 && \text{(mostly-plus convention)}. \end{aligned}$$

Those conditions will be satisfied for all m if the matrix M satisfies the conditions shown here for every Dirac matrix γ^a, given that ψ satisfies the Dirac equation:

$$\gamma^a M = -M(\gamma^a)^* \qquad \text{(mostly-minus convention)}, \tag{54}$$

$$\gamma^a M = M(\gamma^a)^* \qquad \text{(mostly-plus convention)}. \tag{55}$$

Section 22 showed that these conditions have solutions if and only if $d \neq 4n+1$, so the Dirac equation with $m \neq 0$ has symmetries of the form (52) if and only if $d \neq 4n+1$.

For $m = 0$, the sign of the $\gamma\partial$ term in the Dirac equation no longer matters, so the condition on M becomes

$$\gamma^a M \propto M(\gamma^a)^* \tag{56}$$

with the same proportionality factor for all γ^a. Section 22 showed that this condition has a solution for every d, so the Dirac equation with $m = 0$ has symmetries of the form (52) for every d.

For the Weyl equation, which is defined whenever d is even, a symmetry of this form exists if and only if an irreducible representation of the even part of the Clifford algebra is equivalent to its complex conjugate. Section 22 showed that it is if $d = 4n+2$ but not if $d = 4n$.

Altogether, this establishes the results that were previewed in table 5.

27 Internal symmetries: construction

According to tables 3 and 4, the conditions (54)-(55) are satisfied by the matrices shown here, using the representation that was constructed in section 23:

	mostly minus	mostly plus
$d = 4n$	$M \propto \prod_{\text{imag}} \gamma^a$	$M \propto \prod_{\text{real}} \gamma^a$
$d = 4n + 1$	**no solution**	**no solution**
$d = 4n + 2$	$M \propto \prod_{\text{real}} \gamma^a$	$M \propto \prod_{\text{imag}} \gamma^a$
$d = 4n + 3$	$M \propto \prod_{\text{real}} \gamma^a$	$M \propto \prod_{\text{real}} \gamma^a$

The product is either over all of the real Dirac matrices or over all of the imaginary Dirac matrices, as indicated. To see why this works, consider the case $d = 4n$ in the mostly-minus signature. Table 3 says that the number of Dirac matrices in the product $M \propto \prod_{\text{imag}} \gamma^a$ is odd, so M commutes with all imaginary γ^as and anticommutes with all real γ^as, exactly as the condition (54) requires. The other cases can be verified in a similar way.

For the Dirac equation with $m = 0$, the condition (56) is satisfied by

$$M \propto \prod_{\text{either}} \gamma^a \tag{57}$$

where *either* means we can use either the product of all real Dirac matrices or the product of all imaginary Dirac matrices. This works for any d and for either signature convention, because the overall sign in (56) doesn't matter when $m = 0$.

For the Weyl equation, which is defined whenever d is even, this works only if the number of Dirac matrices in the product (57) is even, so that M commutes with the projections (34). According to tables 3 and 4, this implies that the solution (57) works for $d = 4n + 2$ but not for $d = 4n$. This is consistent with the existence results shown in table 5.

28 Antilinear time-reflection: introduction

Let $x \to x_T$ be an isometry consisting of a single reflection along the timelike direction $\mathbf{e}_0$, and consider maps of any of these forms:[70]

$$\sigma\big(\psi(x)\big) = M\psi(x_T) \qquad \sigma \text{ linear,} \tag{58}$$
$$\sigma\big(\psi(x)\big) = M\psi(x_T) \qquad \sigma \text{ antilinear,} \tag{59}$$
$$\sigma\big(\psi(x)\big) = M\psi^*(x_T) \qquad \sigma \text{ linear,} \tag{60}$$
$$\sigma\big(\psi(x)\big) = M\psi^*(x_T) \qquad \sigma \text{ antilinear.} \tag{61}$$

Sections 10, 12, and 16 already explored symmetries of the form (58). This section explores symmetries of the form (59). The other two forms (60) or (61) don't require a separate analysis, because they may be converted to (or obtained from) symmetries of the forms (58) or (59) by composing them with a symmetry of the form (53), which always exists.

Quantum electrodynamics in four-dimensional spacetime has a an antilinear symmetry of the form (59) called **time reflection symmetry**, denoted T. More generally, in the context of quantum physics, this name may refer to any antilinear symmetry whose corresponding isometry consists of a reflection along a timelike direction.[71,72] Such a symmetry may have the form (59) or (61).

[70]The analysis for an isometry consisting of a single reflection along a spacelike direction is similar. This section considers only reflections along $\mathbf{e}_0$, partly to keep the notation simple, and partly because antilinear time-reflection symmetries play a more prominent role in quantum physics than antilinear space-reflection symmetries do.

[71]Cordova *et al* (2018), section 1.1

[72]Section 2.6 in Weinberg (1995) and section 3.6 in Peskin and Schroeder (1995) review the motive for requiring a symmetry called *time reflection* to be antilinear in the context of quantum theory.

29 Antilinear time-reflection: preview

Table 6 summarizes the existence results that will be deduced in section 30.

	Dirac		Weyl
	$m \neq 0$	$m = 0$	$m = 0$
$d = 4n$	yes	yes	yes
$d = 4n + 1$	yes	yes	
$d = 4n + 2$	yes	yes	**no**
$d = 4n + 3$	**no**	yes	

Table 6 – Summary of when antilinear symmetries of the form (59) exist. "Yes" means that such symmetries exist, and "no" means that such a symmetry does not exist.

30 Antilinear time-reflection: analysis

For any value of m, a map of the form (59) satisfies the symmetry condition (10) if the matrix M satisfies this condition, with $\partial_a^T \equiv \partial/\partial x_T^a$:

$$(\gamma\partial)^* M = -M(\gamma\partial^T) \qquad \text{(mostly-minus convention)}, \tag{62}$$

$$(\gamma\partial)^* M = M(\gamma\partial^T) \qquad \text{(mostly-plus convention)}. \tag{63}$$

More explicitly,

$$(\gamma^a)^* M = \begin{cases} M\gamma^a & \text{for } a=0 \\ -M\gamma^a & \text{otherwise} \end{cases} \qquad \text{(mostly-minus convention)}, \tag{64}$$

$$(\gamma^a)^* M = \begin{cases} -M\gamma^a & \text{for } a=0 \\ M\gamma^a & \text{otherwise} \end{cases} \qquad \text{(mostly-plus convention)}. \tag{65}$$

If an irreducible representation is equivalent to its complex conjugate, and if M' is a matrix that intertwines[73] those two represention, then the matrix $M \equiv \gamma^0 M'$ satisfies the condition (64). On the other hand, if an irreducible representation is equivalent to the one in which each Dirac matrix is replaced by the negative of its complex conjugate, and if M' is a matrix that intertwines those two representions, then the matrix $M \equiv \gamma^0 M'$ satisfies the condition (65). According to section 22, this shows that a matrix M with the required properties exists whenever $d \neq 4n+3$. If such a matrix existed when $d = 4n+3$, then the matrix $M' \equiv \gamma^0 M$ could be used to demonstrate the equivalence of representations that must be inequivalent according to section 22, so this shows that the Dirac equation with $m \neq 0$ has symmetries of the form (59) if and only if $d \neq 4n+3$.

For the Dirac equation with $m = 0$, the overall sign in equations (62)-(63) no longer matters, so the Dirac equation with $m = 0$ has symmetries of the form (59) for all d.

[73] Section 22 defines *intertwines*.

For the Weyl equation, the matrix M needs to satisfy

$$BM = \pm MB' \tag{66}$$

for all bivectors B (with the same sign for all bivectors), where B' is obtained from B by taking the complex conjugate and reversing the sign of γ^0 (but not of any other Dirac matrices).[74] Such a matrix M exists if and only if this replacement (taking the complex conjugate and reversing the sign of γ^0) gives another representation that is equivalent to the original one. Reasoning like that used in section 22 may be used to show that these two representations of the even part of the Clifford algebra are equivalent to each other if $d = 4n$ but not if $d = 4n + 2$, so the Weyl equation has symmetries of the form (59) if $d = 4n$ but not if $d = 4n + 2$. This is consistent with the existence results shown in table 6.

[74]The condition (66) on M is expressed in terms of bivectors instead of individual Dirac matrices because, in the context of the Weyl equation, everything should be formulated using only the even part of the Clifford algebra, which does not mix the spaces W_+ and W_- that were defined in section 15 with each other.

31 Antilinear time-reflection: construction

According to tables 3 and 4, the conditions (64)-(65) are satisfied by the matrices shown here, using the representation that was constructed in section 23:[75]

	mostly minus	mostly plus
$d = 4n$	$M \propto \gamma^0 \prod_{\text{real}} \gamma^a$	$M \propto \gamma^0 \prod_{\text{imag}} \gamma^a$
$d = 4n + 1$	$M \propto \gamma^0 \prod_{\text{real}} \gamma^a$	$M \propto \gamma^0 \prod_{\text{real}} \gamma^a$
$d = 4n + 2$	$M \propto \gamma^0 \prod_{\text{imag}} \gamma^a$	$M \propto \gamma^0 \prod_{\text{real}} \gamma^a$
$d = 4n + 3$	**no solution**	**no solution**

For $m = 0$, the sign of the $\gamma\partial$ term in the Dirac equation no longer matters, so the condition on M becomes[76]

$$(\gamma\partial)^* M \propto M(\gamma\partial^T). \tag{67}$$

This condition is satisfied by

$$M \propto \gamma^0 \prod_{\text{either}} \gamma^a \tag{68}$$

where *either* means we can use either the product of all real Dirac matrices or the product of all imaginary Dirac matrices. This works for any d and for either signature convention.

For the Weyl equation, which is defined whenever d is even, this works only if the number of Dirac matrices in the product (68) (including the first factor of γ^0) is even, so that M commutes with the projections (34). According to tables 3 and 4, this implies that the solution (68) works for $d = 4n$ but not for $d = 4n + 2$. This is consistent with table 6.

[75]When d is even, these solutions can be obtained by composing other symmetries that were established in sections 13, 20, and 26.

[76]The superscript T here applies to ∂, as in section 30: $\partial_a^T \equiv \partial/\partial x_T^a$. It does not denote the transpose of a matrix.

32 Antilinear time-reflection: another formulation

Using the mostly-minus convention for the signature, we can choose a matrix representation as in section 23 so that γ^0 is hermitian and the other Dirac matrices are antihermitian:

$$(\gamma^a)^* = \begin{cases} (\gamma^a)^{\text{transpose}} & \text{if } \mu = 0 \\ -(\gamma^a)^{\text{transpose}} & \text{otherwise.} \end{cases} \tag{69}$$

The transpose of a matrix M is usually denoted M^T, but the notation M^{tranpose} is used here instead to prevent confusion with the different meaning of the superscript T in sections 30-31. Using (69), the condition (64) reduces to

$$(\gamma^a)^{\text{transpose}} M = M\gamma^a. \tag{70}$$

Using the mostly-plus convention instead gives the opposite sign in (69), and combining that with (65) gives (70) again. This shows that equation (70) is a more concise way to express the conditions that M must satisfy in order for (59) to be a symmetry of the Dirac equation with $m \neq 0$, at least in a representation where γ^0 is hermitian and the other Dirac matrices are antihermitian, like the representation described in section 23.

33 Cross-checking the (non)existence results

The preceding sections derived several results about the existence or non-existence of various types of symmetries. Some of those abstract results were already checked by constructing the symmetry transformations explicitly when they exist and observing how the constructions fail when symmetries of those types don't exist. This section describes another way of cross-checking the (non)existence results. The idea is simple: the composition of two or more symmetries is another symmetry. Suppose that a composition of symmetries of types A, B, C would give a symmetry of type D. If symmetries of types A and B exist and symmetries of type D don't, then symmetries of type C must not exist, either.

Here's one example of such a cross-check. Table 5 asserts that the Dirac equation with $m \neq 0$ does not have a symmetry of the form

$$\sigma\big(\psi(x)\big) = \text{matrix} \times \psi^*(x) \qquad\qquad \sigma \text{ linear} \tag{71}$$

when $d = 4n + 1$. If it did, then we could compose it with symmetries of the forms

$$\sigma\big(\psi(x)\big) = \text{matrix} \times \psi^*(x_{PT}) \qquad\qquad \sigma \text{ antilinear} \tag{72}$$

$$\sigma\big(\psi(x)\big) = \text{matrix} \times \psi(x_T) \qquad\qquad \sigma \text{ antilinear,} \tag{73}$$

whose existence was established in sections 20 and 30, to produce a symmetry of the form

$$\sigma\big(\psi(x)\big) = \text{matrix} \times \psi(x_P) \qquad \sigma \text{ linear,} \tag{74}$$

where the isometry $x \to x_P$ is the composition of the isometries $x \to x_{PT}$ and $x \to x_T$. Table 1 says that a symmetry of the form (74) does not exist when $d = 4n + 1$, so these (non)existence results are consistent with each other.

Here are a few more examples:

- Table 5 also asserts that the Weyl equation does not have a symmetry of the form (71) when $d = 4n$. If it did, then we could compose it with symmetries of the forms (72) and (73), whose existence was established in sections 20 and

30, to produce a symmetry of the form (74), which does not exist according to table 1, so these (non)existence results are also consistent with each other.

- Table 6 asserts that the Dirac equation with $m \neq 0$ does not have a symmetry of the form (73) when $d = 4n + 3$. If it did, then we could compose it with antilinear symmetries of the forms (71) and (72), whose existence was established in sections 26 and 20, to produce a linear symmetry of the form (74), which does not exist according to table 1, so these (non)existence results are also consistent with each other.

- Table 6 also asserts that the Weyl equation does not have a symmetry of the form (73) when $d = 4n + 2$. If it did, then we could compose it with symmetries of the forms (71) and (72), whose existence was established in sections 26 and 20, to produce a symmetry of the form (74), which does not exist according to table 1, so these (non)existence results are also consistent with each other.

34 References

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