The Free Nonrelativistic Quantum Particle

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Abstract Many introductions to quantum mechanics study a model of a single nonrelativistic quantum particle with zero spin. This article re-introduces that model in a way that emphasizes the general principles introduced in articles 03431 and 21916.

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1 Introduction

This article (re)introduces a simple model of an isolated nonrelativistic[1] quantum particle with no spin. For the rest of this article, I'll call it **the single-particle model**.[2] This model is already treated in detail in many existing introductions to quantum mechanics,[3] but the goal here is different. Here, the goal is to present the model in a way that helps clarify how it relates to the general principles of quantum theory introduced in articles 03431 and 21916, especially how it relates to quantum field theory (QFT). In general, QFT doesn't assign observables to particles. It assigns (sets of) observables to regions of spacetime instead. Such observables are called **local observables**. This article introduces the single-particle model with an emphasis on its *local* observables.

To specify a model, we specify its observables – things that can be measured. The single-particle model has local observables that detect the presence/absence of the particle in a given region of space at a given time.[4] I'll call them **detection observables**. They will be constructed explicitly in section 6. First, section 2 previews a few of the model's important properties with a perspective that is a little different than most of the other existing introductions to this model.

Throughout this article, D denotes the number of spatial dimensions,[5] and boldface denotes a quantity with D components, as in $\mathbf{x} = (x_1, ..., x_D)$. The standard abbreviations

$$\mathbf{x} \cdot \mathbf{y} \equiv x_1 y_1 + \cdots + x_D y_D \qquad\qquad \mathbf{x}^2 \equiv x_1^2 + \cdots + x_D^2$$

will be used.

[1] **Nonrelativistic** means moving very slowly compared to the speed of light.

[2] This isn't the only single-particle model. Other single-particle models differ from this one in various ways, like including a background field, including nonzero spin, and having Lorentz symmetry.

[3] One such introduction is *Introduction to Quantum Mechanics* by Griffiths, 1995.

[4] This article mostly uses the Heisenberg picture (article 22871).

[5] Space is three-dimensional in the real world, but sometimes we may want to consider easier examples in lower-dimensional space, such as one-dimensional space.

2 Preview

This section previews a few important properties of the model that will be constructed explicitly in section 6.

For any region R of space and any time t, the model has a projection operator $Q(R,t)$. The complementary projection operator $1 - Q(R,t)$ will be abbreviated $\overline{Q}(R,t)$. A detection observable is represented by a pair $\{Q(R,t), \overline{Q}(R,t)\}$. These projection operators represent the two possible outcomes when the observable is measured: the outcome $Q(R,t)$ means that the particle is (at least momentarily) localized entirely inside R at time t, and the outcome $\overline{Q}(R,t)$ means it's localized entirely outside of R at time t.[6,7]

One important property of these observables is that for any time t, they satisfy

$$Q(R_1,t)Q(R_2,t) = Q(R_1 \cap R_2, t), \tag{1}$$

where $R_1 \cap R_2$ is the intersection of the two regions. This implies that the detection observables at time t all commute with each other, because $R_1 \cap R_2 = R_2 \cap R_1$.

Another important property of the model is its symmetry under translations in space and time: it has a continuous family of unitary operators $U(\mathbf{x},t)$ for which[8]

$$Q(R + \delta\mathbf{x}, t + \delta t) = U^{-1}(\delta\mathbf{x}, \delta t)Q(R,t)U(\delta\mathbf{x}, \delta t) \tag{2}$$

and

$$U(\mathbf{x}_1,t_1)U(\mathbf{x}_2,t_2) = U(\mathbf{x}_1 + \mathbf{x}_2,\ t_1 + t_2). \tag{3}$$

Equation (3) implies that all of these unitary operators commute with each other, because $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$ and $t_1 + t_2 = t_2 + t_1$. Thanks to **Stone's theorem**,[9]

[6] The particle was not necessarily localized in either of these two regions before time t. In quantum theory, the properties that a system has can depend on what is measured (article 70833).

[7] These observables correspond to detecting the presence/absence of the particle in a region R with a perfectly sharp boundary. This is only an idealization, because real detectors don't have perfectly sharp boundaries. That idealization shouldn't bother us too much, though, because any model of a *single particle* is obviously much too simplistic to describe the microscopic complexities of real detectors anyway.

[8] $R + \delta\mathbf{x}$ denotes the region obtained by translating R by an amount $\mathbf{x}$. In other words, $R + \delta\mathbf{x}$ consists of the points obtained by adding $\delta\mathbf{x}$ to the coordinates of the points in R.

[9] Article 22871

they can all be written as[10,11]

$$\begin{aligned} U(\mathbf{x},t) &= \exp\left(\frac{-i}{\hbar}(\mathbf{x}\cdot\mathbf{P}+tH)\right) \\ &\equiv \exp\left(\frac{-i}{\hbar}(x_1P_1+\cdots+x_DP_D+tH)\right) \end{aligned} \tag{4}$$

for some fixed set of self-adjoint operators $P_1, ..., P_D$ and H that all commute with each other. The operator H that generates time translations is called the **hamiltonian** or **energy operator** (article 22871), and the operators P_k that generate space translations are called the **momentum operators**.

Another important property of the model is this relationship between the energy and momentum operators:

$$H = \frac{\mathbf{P}^2}{2m} \equiv \frac{P_1^2+\cdots+P_D^2}{2m}, \tag{5}$$

where m is the particle's mass. This looks like the familiar relationship between the energy and momentum of a nonrelativistic particle in classical physics, but here the quantities H and P_k are operators. They can be regarded as observables. Equation (2) says that they don't commute with the local observables $Q(R,t)$. The operators $Q(R,t)$ don't commute with each other, either, except at equal times (equation (1)). The fact that the model's observables don't all commute with each other is what makes it *quantum*.

Section 6 constructs the observables $Q(R,t)$ as operators on a Hilbert space. These are not the model's only local observables, but we can use them to define all of the others. This is the rule:[12] for an observable A to qualify as being localized in region R at time t, it must satisfy $AQ(\overline{R},t)|\psi\rangle \propto Q(\overline{R},t)|\psi\rangle$ for all $|\psi\rangle$, where $\overline{R}$ is the complement of R (the largest region that does not intersect R).

[10] I won't review the precise definition of the exponential of an operator here, but the key properties are $\frac{d}{dx}\exp(xA) = A\exp(xA)$ and $\exp(A+B) = \exp(A)\exp(B)$ for any real parameter x and any operators A, B that commute with each other.

[11] The minus sign and the factor of $\hbar$ are conventional.

[12] This rule is consistent with the general principle that observables localized in non-intersecting regions of space at the same time should commute with each other (article 21916).

3 The Hilbert space, part 1

In quantum theory, observables are represented by operators on a Hilbert space $\mathcal{H}$.[13] The single-particle model uses a specific Hilbert space, one that is infinite-dimensional and separable. This article describes the Hilbert space in a way that is convenient for the single-particle model, but remember that the same Hilbert space can be described in many very different-looking ways. A model is defined by its observables, not by the Hilbert space.

An element of $\mathcal{H}$ will be represented by a complex-valued function $\psi(\mathbf{x})$ of D real variables $\mathbf{x} = (x_1, ..., x_D)$.[14] A function ψ is called **normalizable** if the quantity

$$\int d^D x \, \Big|\psi(\mathbf{x})\Big|^2 \tag{6}$$

is well-defined (not infinite). Only normalizable functions are used to represent elements of $\mathcal{H}$.

The quantity (6) can be zero even if the function ψ is not zero. In particular, if $\psi(\mathbf{x})$ is zero everywhere except at a finite number of points $\mathbf{x}$, then the quantity (6) is still zero. Any function for which (6) is zero is said to have **zero norm**. Any function with zero norm represents the *unique* zero element of $\mathcal{H}$. If the difference two functions has zero norm, then they both represent the same element of $\mathcal{H}$.

[13] The symbol $\mathcal{H}$ for the Hilbert space should not be confused with the symbol H for the hamiltonian.

[14] The Greek letter ψ is spelled "psi" and pronounced "sigh," like the "sci" in *sci*ence. In contrast, the Greek letter ϕ (spelled "phi") is pronounced "fee." Some communities might use different dialects.

4 The Hilbert space, part 2

When thinking of the Hilbert space in abstract terms, we can use the symbol $|\psi\rangle$ to denote the element of $\mathcal{H}$ represented by the function $\psi(\mathbf{x})$. A Hilbert space is a vector space, so $|\psi\rangle$ can also be called a **vector**.[15] Elements of $\mathcal{H}$ can be added. The vector $|\psi\rangle + |\phi\rangle$ is represented by the function

$$\psi(\mathbf{x}) + \phi(\mathbf{x}).$$

If z is a complex number, then the vector $z\,|\psi\rangle$ is represented by the function

$$z\,\psi(\mathbf{x}).$$

The zero vector is represented by the zero function.

A Hilbert space is more than just a vector space: it also has an inner product. In this case, the **inner product** is defined by[16]

$$\langle\, \phi \,|\, \psi \,\rangle \equiv \int d^D x\ \phi^*(\mathbf{x})\psi(\mathbf{x}), \tag{7}$$

where

$$\int d^D x\ f(\mathbf{x}) \equiv \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_D\ f(x_1,\, ...,\, x_D).$$

This has the required property

$$\big(\langle\phi|\psi\rangle\big)^* = \langle\psi|\phi\rangle. \tag{8}$$

This Hilbert space is infinite-dimensional. It's also **separable** (article 90771), as required in quantum theory.

[15] The word "vector" here refers to a vector in the abstract vector space $\mathcal{H}$, which is infinite-dimensional.

[16] The notation $A \equiv B$ means that A is defined by B.

5 Notation

For any operator A on the Hilbert space, this article uses the notation $A\psi(\mathbf{x})$ for a function that represents $A|\psi\rangle$, given that the function $\psi(\mathbf{x})$ represents $|\psi\rangle$. Think of "$A\psi$" as the name of a new function whose value at $\mathbf{x}$ is $A\psi(\mathbf{x})$.

With that notation, using $A^\dagger$ for the adjoint of A is recommended, instead of using A^* to denote the adjoint as in articles 74088 and 03431. To see why, consider the (unbounded) operator P_k defined by[17]

$$P_k\psi(\mathbf{x}) = -i\hbar\nabla_k\psi(\mathbf{x}), \tag{9}$$

where ∇_k is the derivative with respect to the kth component of $\mathbf{x}$. The operator P_k is self-adjoint, but the complex conjugate of equation (9) is

$$\big(P_k\psi(\mathbf{x})\big)^* = i\hbar\nabla_k\psi^*(\mathbf{x}),$$

so

$$\big(P_k\psi(\mathbf{x})\big)^* \neq P_k^\dagger\psi^*(\mathbf{x}).$$

If the adjoint of P_k were denoted P_k^*, then the two sides of this inequality might mistakenly be equated with each other. Writing $P_k^\dagger$ for the adjoint of P_k can help prevent that mistake.

[17] This is equation (11) in section 6.

6 Construction of the model

This section completes the construction of the model that was previewed in section 2. The first step is to define the detection observables and the unitary translation operators at the arbitrary reference time $t = 0$. The projection operator $Q(R, 0)$ is defined by

$$Q(R,0)\psi(\mathbf{x}) = \begin{cases} \psi(\mathbf{x}) & \text{if } \mathbf{x} \in R \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

for all functions ψ.[18] The unitary translation operator $U(\mathbf{x}, 0)$ is defined by

$$U(\mathbf{y},0)\psi(\mathbf{x}) = \psi(\mathbf{x}-\mathbf{y}).$$

Using equation (4), this implies that the momentum operators P_k are given by[19]

$$P_k\psi(\mathbf{x}) = -i\hbar\nabla_k\psi(\mathbf{x}), \tag{11}$$

where ∇_k is the derivative with respect to the kth component of $\mathbf{x}$. The operators P_k are not defined on the whole Hilbert space,[20] but they are defined on a dense subset: any element of the Hilbert space can be approximated arbitrarily well by one on which the momentum operators are defined.[21]

To finish the construction, use equation (5) and (11) to define the hamiltonian H, use Stone's theorem to define the unitary time translation operators, and then use (2) and (10) to define $Q(R, t)$ for all other times t.

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[18] Recall that any function with zero norm represents the zero vector in $\mathcal{H}$, so if the region R is such that every function of the form (10) has zero norm, then $Q(R, 0) = 0$. In particular, if R consists of a single point, then $Q(R, 0) = 0$.

[19] To deduce this, consider the effect of the translation operator on a real analytic function, which is defined by its Taylor expansion: $\psi(\mathbf{x} - \mathbf{y}) = \exp(-\mathbf{y} \cdot \nabla)\psi(\mathbf{x})$ with $\exp(\omega) \equiv \sum_{n\geq 0} \omega^n/n!$.

[20] They are not defined on functions that have sharp edges or discontinuities.

[21] The fact that the set of smooth (infinitely differentiable) functions is dense in the Hilbert space can be deduced by starting with any function in the Hilbert space and convolving it with an arbitrarily narrow **mollifier** (https://en.wikipedia.org/wiki/Mollifier) to get a smooth function that approximates the original one arbitrarily closely. The function (33) used for a different purpose in section 15 can also be used a mollifier.

7 Ideal measurement of a detection observable

Let $|\psi\rangle$ be a state-vector representing whatever we know about how the physical system was prepared before the measurement, and use the abbreviation

$$\rho(\cdots) \equiv \frac{\langle\psi|\cdots|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{12}$$

Starting with this state, consider a perfect measurement of the detection observable $\{Q(R,t), \overline{Q}(R,t)\}$. The general principles of quantum theory (article 03431) say that the quantity

$$p \equiv \rho\big(Q(R,t)\big) \tag{13}$$

is the probability that we should assign to the possible outcome $Q(R,t)$ of the measurement. In other words, this is the probability that the outcome of the measurement will be "the particle is localized entirely within R." The probability of the opposite outcome, "the particle is localized entirely outside of R," is

$$\rho\big(\overline{Q}(R,t)\big) = 1 - p.$$

If the state-vector $|\psi\rangle$ is represented by a function $\psi(\mathbf{x})$, then equations (2) and (10) give this expression for the probability (13):

$$p = \frac{\int_R d^D x \, \Big|\psi(\mathbf{x},t)\Big|^2}{\int d^D x \, \Big|\psi(\mathbf{x},t)\Big|^2} \tag{14}$$

where the subscript R in the numerator specifies the domain of integration, and $\psi(\mathbf{x},t)$ is a function representing the state-vector $U(t)|\psi\rangle$ with

$$U(t) \equiv e^{-iHt/\hbar}. \tag{15}$$

Thanks to equation (14), we can analyze the particles' behavior by analyzing the time-dependence of the function $\psi(\mathbf{x},t)$. This way of representing the model's time-dependence is called the **Schrödinger picture**.

8 The Schrödinger equation

Suppose that the state-vector $|\psi\rangle$, which can be represented by a function $\psi(\mathbf{x})$, represents[22] everything we know about how the physical system was prepared. When we do a measurement that asks "is the particle localized in R at time t," the probability that the outcome is "yes" is given by equation (14). The function $\psi(\mathbf{x}, t)$ on the right-hand side of that equation is defined to be a function that represents the state-vector $U(t)|\psi\rangle$, which depends on time through the unitary operator $U(t)$ defined in equation (15). According to equations (5) and (11), this function satisfies

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = -\frac{(\hbar\nabla)^2}{2m}\psi(\mathbf{x}, t). \tag{16}$$

Equation (16) is one example of a **Schrödinger equation**. After solving equation (16) subject to the given initial condition $\psi(\mathbf{x}, 0) = \psi(\mathbf{x})$, we can use equation (14) to read off the probability that a measurement of the observable $\{Q(R, t), \overline{Q}(R, t)\}$ will give the outcome $Q(R, t)$.

[22]The word "represent" is used twice in this sentence, with two different meanings. The first refers to using one mathematical object to represent another mathematical object. The second refers to using a mathematical object to represent a non-mathematical concept.

9 Example of a solution

Here's one example of a function $\psi(\mathbf{x}, t)$ that satisfies equation (16):[23]

$$\psi(\mathbf{x}, t) = N(t) \exp\left[-\frac{(\mathbf{x} - \mathbf{x}_0)^2}{2\sigma^2 \eta(t)}\right] \tag{17}$$

where $\mathbf{x}_0$ is a given point in space (independent of t) and where

$$N(t) = \Big(\eta(t)\Big)^{-D/2} \qquad\qquad \eta(t) \equiv 1 + \frac{i\hbar t}{\sigma^2 m}.$$

For this solution, the integrand in equation (14) is

$$|\psi(\mathbf{x}, t)|^2 \propto \exp\left[-\frac{(\mathbf{x} - \mathbf{x}_0)^2}{\sigma^2 |\eta(t)|^2}\right] \tag{18}$$

with

$$\sigma^2 |\eta(t)|^2 = \sigma^2 + \frac{1}{\sigma^2}\left(\frac{\hbar t}{m}\right)^2.$$

At any given time t, the function (18) is mostly concentrated in a region of width $\sim \sigma|\eta(t)|$ centered on the point $\mathbf{x}_0$. The width is minimized at $t = 0$ and grows without bound as $t \to \infty$.

[23] Section 14 shows a slightly more general example.

10 Dispersion

Suppose that the single-particle system is prepared so that the state is given by equation (17) at time $t = 0$, when the function's width is $\sim \sigma$. After that time, the width grows without bound as time passes, so equation (14) says that the probability of detecting the particle in *any* region of size $\sim \sigma$ declines toward zero as time passes. To detect the particle with certainty (probability ≈ 1), we need to consider a region of size $\sim \sigma|\eta(t)|$, which grows without bound as t increases. This is called **dispersion**. Here are a few examples of the time needed for the function's width to grow from 10^{-10} meters (roughly the size of an atom) to 10^{-3} meters (roughly the width of a line drawn by a typical pencil), for various values of m and σ:[24]

Mass m	Initial width σ	Final width $\sigma\|\eta(t)\|$	Elapsed time t
10^{-3} kg (pill)	10^{-10} m	10^{-3} m	10^{18} s
10^{-9} kg (speck)	10^{-10} m	10^{-3} m	10^{12} s
10^{-30} kg (electron)	10^{-10} m	10^{-3} m	10^{-9} s

These examples are based on the special solution described in section 9, but the conclusion is more general: every solution of equation (16) eventually disperses. The larger the particle's mass, the slower the dispersion.

The examples shown above are still relevant even if the mass m is much larger than anything we would normally call a *particle*, because this same model turns out to be valid also for a (nonrelativistic) composite object if we ignore the object's internal structure. In that context, measuring the observable $Q(R, t)$ corresponds to asking whether the object's *center of mass* is within R at time t, even if the whole object is much larger than R.

[24] To put these examples in perspective: 10^{12} seconds is more than 10,000 years.

11 Position as an observable

Section 2 described the model in terms of the detection observables $Q(R,t)$, which are local observables. Traditional presentations of the single-particle model package those local observables into nonlocal observables called **position operators**, one for each dimension of space. This is often useful, even though it tends to obscure the model's relationship to QFT (section 1).

For motivation, suppose that we partition space into cells of some tiny size $\sim \epsilon$, and let R_n denote the nth cell. For any given time t, the corresponding detection observables satisfy $Q(R_n,t)Q(R_{n'},t) = 0$ for $n \neq n'$ because the cells don't overlap. We can think of this set of projection operators as a single observable, with each projection operator $Q(R_n,t)$ representing a possible outcome ("the particle is in R_n at time t") when the observable is measured. Such a measurement amounts to measuring the particle's coordinates with resolution $\sim \epsilon$.

Now consider the operators $X_1(t), ..., X_D(t)$ defined by

$$X_k(t) \equiv U^{-1}(t) X_k U(t) \qquad\qquad X_k\psi(\mathbf{x}) = x_k\psi(\mathbf{x}), \tag{19}$$

where x_k is the kth component of $\mathbf{x}$ and $U(t)$ is given by (15). If the cells in the preceding paragraph are small enough, then

$$X_k(t) \approx \sum_n (R_n)_k Q(R_n,t) \tag{20}$$

where $(R_n)_k$ denotes the kth coordinate of the center of the nth cell. For this reason, the operators $X_k(t)$ can be regarded as observables whose measurement – which necessarily has limited resolution – returns the kth coordinate of the particle's location. These the **position operators**.[25] They can be regarded as observables, but equation (20) shows that they are not local observables: they are not associated with any one bounded region of space.

[25] I would prefer to call them **location operators** because, in common speech, the word *position* is also used to mean *orientation*.

12 Momentum and velocity

This section derives the relationship[26]

$$P_k = m \frac{d}{dt} X_k(t), \tag{21}$$

where P_k are the momentum operators defined in equation (11). This is an operator-valued version of the relationship between momentum and velocity that is already familiar from nonrelativistic classical physics. The interpretation of $dX_k(t)/dt$ as a velocity observable is justified by equations (14) and (20).

Equation (15) implies

$$i\hbar \frac{d}{dt} U(t) = H\, U(t). \tag{22}$$

Take the time-derivative of the first of equations (19) and use (22) to get

$$i\hbar \frac{d}{dt} X_k(t) = \big[X_k(t),\, H\big], \tag{23}$$

where $[A, B] \equiv AB - BA$ denotes the **commutator** of two operators A and B. The operators X_k defined in the second of equations (19) satisfy[27]

$$[X_j, P_k] = i\hbar\, \delta_{jk} \equiv \begin{cases} i\hbar & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and the fact that the momentum operators commute with the hamiltonian then implies

$$\big[X_j(t), P_k\big] = i\hbar\, \delta_{jk}.$$

Use this in (23) to get the result (21).

[26] This model doesn't know about the existence of a limiting speed. The nonrelativistic approximation is built into the model itself, thanks to the relationship (5).

[27] This follows from the identity $x_j \nabla_k \psi(\mathbf{x}) - \nabla_k \big(x_j \psi(\mathbf{x})\big) = \delta_{jk} \psi(\mathbf{x})$.

13 Boost operators

Let $\mathbf{p} = (p_1, ..., p_D)$ be a list of real numbers. For each $\mathbf{p}$, define an operator $U(\mathbf{p})$ by

$$U(\mathbf{p})\psi(\mathbf{x}) \equiv e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}\psi(\mathbf{x}). \tag{24}$$

These operators are unitary. Use the definition (11) of the momentum operators P_k to deduce

$$P_k U(\mathbf{p})|\psi\rangle = U(\mathbf{p})(P_k + p_k)|\psi\rangle \tag{25}$$

for all $|\psi\rangle$ on which P_k is defined. In words: the unitary operator $U(\mathbf{p})$ implements a **boost** – a transformation that shifts all momenta by the amount $\mathbf{p}$.

The boost operators may also be written

$$U(\mathbf{p}) = \exp\left(\frac{i}{\hbar}\,\mathbf{p}\cdot\mathbf{X}\right)$$

where $\mathbf{X} = (X_1, ..., X_D)$ is the list of operators X_k defined by the second of equations (19).

14 Boost operators: example

To illustrate the effect of the boost operators, consider this solution of (16):[28]

$$\tilde{\psi}(\mathbf{x},t) = N(t)\exp\left[-\frac{\left(\mathbf{x}-\mathbf{x}_0-i\frac{\mathbf{p}\sigma^2}{\hbar}\right)^2}{2\sigma^2\eta(t)}\right] \tag{26}$$

where $\mathbf{x}_0$ and $\mathbf{p}$ are constants (independent of t) and where $N(t)$ and $\eta(t)$ are defined as in section 9. When $\mathbf{p}=0$, this solution reduces to the solution $\psi(\mathbf{x},t)$ that was shown in equation (17). At time $t=0$, the functions (26) and (17) are related to each other by the boost operator (24):

$$\tilde{\psi}(\mathbf{x},0) \propto U(\mathbf{p})\psi(\mathbf{x},0). \tag{27}$$

According to equations (21) and (25), the centroid of this function should move with constant velocity $\mathbf{p}/m$ as time passes. To confirm that it does, consider the magnitude of (26):

$$|\tilde{\psi}(\mathbf{x},t)|^2 \propto \exp\left[-\frac{(\mathbf{x}-\mathbf{x}_0-\mathbf{v}t)^2}{\sigma^2|\eta(t)|^2}\right] \qquad \mathbf{v}\equiv\mathbf{p}/m. \tag{28}$$

The centroid of this function moves with constant velocity $\mathbf{p}/m$, as expected from (27).

This example showed that the centroid of the function $U(t)U(\mathbf{p})\psi(\mathbf{x},0)$ moves with constant velocity $\mathbf{p}/m$. In contrast, the centroid of the function $U(\mathbf{p})\psi(\mathbf{x},t) = U(\mathbf{p})U(t)\psi(\mathbf{x},0)$ does not move at all: the magnitude of this function is the same as the magnitude of $\psi(\mathbf{x},t)$. This illustrates the fact that the boost operators don't commute with the time evolution operators, which is clear from equations (5) and (25).

[28] Problem 2.22 in Griffiths (1995), and page 64 in Cohen-Tannoudji *et al* (1977)

15 The state-update rule for indirect measurements

Suppose that the observable $\{Q(R,t), \overline{Q}(R,t)\}$ is perfectly measured, and suppose that the outcome is $Q(R,t)$ – the particle is in R at time t. If the state before the measurement was represented by a state-vector $|\psi\rangle$, then the **state-update rule** (article 03431) says that we should use the projected state-vector $Q(R,t)|\psi\rangle$ for making predictions about subsequent measurements.

But recall footnote 7: the detection observables $Q(R,t)$ correspond to detecting the presence/absence of the particle in a region R with a perfectly sharp boundary. This is only an idealization, because real detectors don't have perfectly sharp boundaries. In reality, observables like $Q(R,t)$ are never measured directly. Real measurements involve interactions of the particle with a larger complex system. The single-particle model doesn't know anything about such interactions, but we can at least emulate the smooth-edges quality of such indirect measurements by generalizing the state-update rule. Instead of using mutually orthogonal projection operators to represent the possible outcomes of the measurement, the generalized rule (article 03431) uses a list of operators $M_1, M_2, ...$ satisfying

$$\sum_n M_n^\dagger M_n = 1, \tag{29}$$

where $M_n^\dagger$ is the adjoint of M_n. If the outcome M_n is obtained, then we account for this by replacing the original state $|\psi\rangle$ with $M_n|\psi\rangle$.

Here's a contrived example in one-dimensional space ($D = 1$) to illustrate how this generalization can emulate the smooth-edges quality of an indirect measurement with resolution $\sim \epsilon$. For each integer n, define an operator M_n by

$$M_n\psi(x) = f(x - n\epsilon)\psi(x) \tag{30}$$

where $f(x)$ is a real-valued function satisfying

$$\sum_n f^2(x - n\epsilon) = 1. \tag{31}$$

This clearly satisfies (29). Many functions $f(x)$ satisfy this condition. The most obvious example is the window function

$$f(x) = \begin{cases} 1 & \text{if } -\epsilon/2 < x < \epsilon/2 \\ 0 & \text{otherwise,} \end{cases}$$

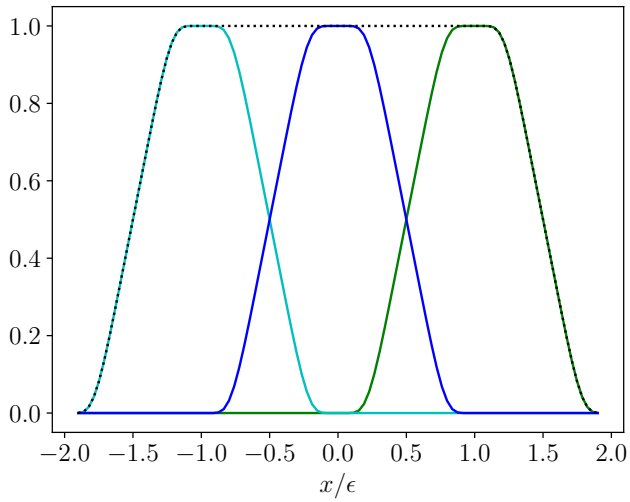
but in this case M_n is one of the detection observables defined earlier, which isn't what we want. We want an example with smooth edges, like this one:

$$f(x) = \left(\frac{g(x)}{g(x+\epsilon) + g(x) + g(x-\epsilon)} \right)^{1/2} \tag{32}$$

with

$$g(x) = \begin{cases} \exp\left(\frac{-2}{1-(x/\epsilon)^2}\right) & \text{if } -\epsilon < x < \epsilon \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

The following picture shows graphs of $f^2(x)$, $f^2(x-\epsilon)$, and $f^2(x+\epsilon)$ as solid lines, and their sum as a dotted line:



This picture demonstrates that the operators defined by (30) and (32) satisfy the condition (29). This shows that the generalized state-update rule can emulate the smooth-edges quality of an indirect measurement.

16 References

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