Conserved Currents and Gauge Invariance

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Abstract In electrodynamics, the same local conservation law for the charge/current density can be derived using either of two different methods. One method relies on the equation of motion for the electromagnetic field, and the other method uses only the equation of motion for the charged matter. This article explains why both methods lead to the same conservation law.

Contents

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1 Introduction

Consider classical electrodynamics in N-dimensional flat spacetime. The electromagnetic field has components

$$F_{ab}(x) \equiv \partial_a A_b(x) - \partial_b A_a(x), \tag{1}$$

where $A_a(x)$ are the components of the **gauge field**. The equation of motion (EoM) for the gauge field has the form

$$\partial_b F^{ba}(x) + J^a(x) = 0, \tag{2}$$

where J^a are the components of a quantity called the **current density** or just **current**.[1] The current is expressed in terms of matter fields or particles,[2] each of which is subject to its own EoM.

For any behavior of the gauge field, whether or not it satisfies the EoM (2), the Faraday tensor F^{ab} automatically satisfies the identity

$$\partial_a F^{ab} = 0. \tag{3}$$

When combined with the EoM (2), this implies the local conservation law

$$\partial_a J^a = 0 \tag{4}$$

for the current. When viewed this way, the conservation law is a condition that the matter's EoMs must satisfy in order to be consistent with the gauge field's equation of motion.

This same current conservation law may also be derived in a different way, without using the identity (3). It can be derived instead using only the matter's EoMs, as long as the action from which they are derived is **gauge invariant**.

This article explains why these two seemingly different approaches – one relying on the equations of motion for the matter, and one not – both lead to the same conservation law (4) for the current.

[1] In the usual Minkowski coordinate system (article 48968), the component J^0 is the charge density.

[2] In this article, any field that is not constructed from the gauge field A is called "matter." In the context of general relativity, everything except the metric field is called "matter," including the electromagnetic field.

2 Current conservation from the gauge-field EoM

This section shows how to derive the conservation law (4) using the EoM for the gauge field, *without* using any EoMs for the matter. This approach is based on these assumptions:

- The EoM for the gauge field satsifies an action principle. In other words, it may be derived from an **action**, using the principle of stationary action.
- The action is the sum of two parts, each of which makes a non-zero contribution to the EoM: One part S_A involving only the gauge field A, and one part S_m that involves both the matter and the gauge field.
- The gauge-only part S_A is invariant under gauge transformations, whose effect on the gauge field is $A_a \to A_a + \partial_a \theta$ for any smooth function θ.

Aside from these basic properties, the explicit form of the action doesn't matter. Most importantly, this approach does not assume anything about the EoMs for the matter.

To explore the consequences of those assumptions, consider an action of the form

$$S = S_A + S_m \tag{5}$$

with S_A and S_m as described above. According to the action principle, the EoM for the gauge field is

$$\frac{\delta S}{\delta A_a(x)} = 0. \tag{6}$$

This may also be written

$$\frac{\delta S_A}{\delta A_a(x)} + J^a(x) = 0 \tag{7}$$

with

$$J^a(x) \equiv \frac{\delta S_m}{\delta A_a(x)}. \tag{8}$$

Classical electrodynamics uses

$$S_A = -\frac{1}{4}\int d^N x\; F_{ab}F^{ab} \qquad \text{(we won't need this here),}$$

which gives

$$\frac{\delta S_A}{\delta A_a(x)} = \partial_b F^{ba}(x) \qquad \text{(we won't need this here, either).}$$

This specific version of S_A gives equation (2), but we won't need such details here. Using only the gauge invariance of S_A, we will deduce

$$\partial_a \frac{\delta S_A}{\delta A_a(x)} = 0, \tag{9}$$

which in turn implies the conservation law (4) as explained in section 1.

The remaining task is to derive equation (9). For this, we only need the assumption that S_A is invariant under the gauge transformation

$$A_a \to A_a + \partial_a\theta \tag{10}$$

for any smooth function $\theta(x)$. This may also be written

$$\delta A_a = \partial_a\theta. \tag{11}$$

For an arbitrary variation of the gauge field (not necessarily a gauge transformation), we have the identity

$$\delta S_A = \int d^N x\; \frac{\delta S_A}{\delta A_a(x)}\delta A_a(x). \tag{12}$$

If we take the transformation $\delta A_a(x)$ to be a gauge transformation (11), then S_A is invariant, so we have

$$0 = \int d^N x\; \frac{\delta S_A}{\delta A_a(x)}\partial_a\theta(x). \tag{13}$$

If $\theta(x)$ has compact support in spacetime, then we can use integration-by-parts to get

$$0 = \int d^N x\ \theta(x)\partial_a \frac{\delta S_A}{\delta A_a(x)}. \tag{14}$$

The fact that equation (13) (and therefore equation (14)) holds for all compactly-supported smooth function $\theta(x)$ implies equation (9).[3] This completes the derivation of the conservation law (4) from the assumptions that were listed at the beginning of this section. Most importantly, this derivation did *not* use any equations of motion for the matter. Instead, we deduced something (equation (4)) about how the matter must behave.

[3] In a little more detail: suppose that $f(x)$ is an unknown smooth function satisfying the condition that $\int_a^b dx\ \theta(x)f(x) = 0$ for all intervals $[a,b]$, for all smooth functions $\theta(x)$ that are zero in a neighborhood of a and b. Now choose any interval $[a,b]$ in which $f(x)$ is either positive or negative everywhere in that interval. Then no cancellations can occur if $\theta \geq 0$ everywhere in that interval, so the assumed condition cannot hold unless $f(x) = 0$ everywhere. The same idea works for higher-dimensional integrals, too.

3 Current conservation from the matter-field EoM

This section shows that if the action for the matter is invariant under gauge transformations in the sense defined below, then the conservation law (4) holds. In this approach, the conservation law is a consequence of the EoMs for the matter alone. The EoM for the gauge field (equation (6) or (7)) is not used. This approach is based on these assumptions:

- The EoMs for the matter collectively satsify an action principle. In other words, they may all be derived from a single action S_m, using the principle of stationary action. (The EoM for the gauge field is not used, so it doesn't need to satisfy the action principle.)

- The action S_m is invariant under all gauge transformations, as long as the gauge transformation is applied to the gauge field as well as to the matter.

Aside from these basic properties, the explicit form of the action doesn't matter. Most importantly, this approach does not assume anything about the EoM for the gauge field. The gauge field may simply be prescribed, and it may even be "pure gauge" ($F_{ab} = 0$).

Before continuing, I'll clarify what *gauge invariance* means when the gauge field is prescribed. In a model where the gauge field is one of the dynamic fields whose behavior is governed by the model's equations of motion (instead of being prescribed), *gauge invariance* implies that any gauge transformation applied to any solution gives another solution. The word **gauge** alludes to the fact that two solutions which can be obtained from each other by a gauge transformation are regarded as being physically equivalent to each other. In contrast, in a model where the gauge field is merely prescribed, the result of applying an arbitrary gauge transformation to a solution is typically *not* another solution of the original equations of motion – unless we also apply that same gauge transformation to the gauge field. In this context, *gauge invariance* assumes that we also apply the same gauge transformation to the gauge field that we apply to the matter, even though the gauge field is one of the model's prescribed inputs.

To simplify the notation, suppose that the matter consists of fields instead of particles. Let $\phi_1, \phi_2, ...$ denote the list of matter fields, and suppose that the matter action S_m is invariant under gauge transformations in the sense described above. Consider any gauge transformation that leaves the matter action S_m invariant and that affects the gauge field according to $\delta A_a(x) = \partial_a \theta(x)$. Exactly how the gauge transformation affects the matter isn't important in this proof, so we can just write the effect as $\delta\phi_n$. For an arbitrary variation of the fields (not necessarily a gauge transformation), including the gauge field, we have the identity

$$\delta S_m = \int d^N x \, \left(\sum_n \frac{\delta S_m}{\delta\phi_n(x)} \delta\phi_n(x) + \frac{\delta S_m}{\delta A_a(x)} \delta A_a(x) \right).$$

If we take the transformation to be a gauge transformation, then the assumption that S_m is invariant gives

$$0 = \int d^N x \, \left(\sum_n \frac{\delta S_m}{\delta\phi_n(x)} \delta\phi_n(x) + \frac{\delta S_m}{\delta A_a(x)} \partial_a \theta(x) \right). \tag{15}$$

This is analogous to the result (13), which is a consequence of the gauge invariance of S_A, but now we have an extra term because the action S_m involves matter fields fields as well as the gauge field. If the matter fields satisfy their own EoMs, namely

$$\frac{\delta S_m}{\delta\phi_n(x)} = 0,$$

then equation (15) reduces to

$$0 = \int d^N x \, \frac{\delta S_m}{\delta A_a(x)} \partial_a \theta(x). \tag{16}$$

This is just like equation (13), but with S_m in place of S_A, so we get the same result (9) but with J^a in place of $\partial_b F^{ba}$. This gives the conservation law (4), but here we derived it without using any EoM for the gauge field. We used only gauge invariance of the matter part of the action combined with the EoM for the matter fields.

4 Why both approaches give the same result

The preceding sections derived the same conservation law (4) using two different methods. The difference between the two methods is that one method uses the EoM for the gauge field, while the other method uses only the EoMs for the matter fields.

Now we can understand why both approaches give the same result. In electrodynamics, the action has the form $S_A + S_m$, and S_A and S_m are each separately invariant under gauge transformations. The term S_A involves only the gauge field, so the gauge invariance of S_A gives the identity (9), and then consistency with the gauge field's equation of motion (7) requires that the matter fields satisfy the conservation law (4). The term S_m involves both matter fields and the gauge field, so the gauge invariance of S_m gives the identity (15), which reduces to the conservation law (4) when the matter fields satisfy their equations of motion. This explains why the same conservation law can be derived either way, at least if explaining "why" means finding a short list of basic conditions that make the coincidence inevitable.

5 Non-abelian gauge fields

The previous sections considered only an abelian gauge field – one for which the gauge group is commutative. This section briefly considers the case of the non-commutative (nonabelian) gauge group $SU(N)$. (This new "N" is not related to the "N" in previous sections, which denoted the number of spacetime dimensions.) The derivations are similar, so the presentation here will be brief.

When the gauge group is $SU(N)$, each spacetime component $A_a(x)$ of the gauge field is an $N \times N$ matrix, as is the gauge transformation function $\theta(x)$. Each field ϕ_n is an $N \times 1$ matrix.[4] The effect of a gauge transformation is

$$\delta A_a(x) \propto [D_a, \theta(x)] \qquad\qquad \delta\phi_n(x) \propto \theta(x)\phi_n(x)$$

with

$$D_a \equiv i\partial_a - eA_a(x).$$

The analog of equation (12) includes a sum over matrix indices, and equation (14) becomes

$$0 = \int d^N x \; \text{Trace}\left(\theta(x)\left[D_a, \frac{\delta S_A}{\delta A_a(x)}\right]\right) \tag{17}$$

This is valid for all $\theta(x)$ that comply with the definition of the group $SU(N)$, which is enough to imply the conservation law

$$[D_a, J^a(x)] = 0$$

for the current

$$J^a(x) \equiv \frac{\delta S_A}{\delta A_a(x)},$$

which is now an $N \times N$ matrix for each value of the spacetime index a. The derivation described in section 3 may also be adapted for $SU(N)$, with similar modifications.

[4] This assumes the fields belong to the **fundamental representation** of $SU(N)$.

6 References

Kosmann-Schwarzback, 2011. *The Noether Theorems.* Springer

7 References in this series

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