Field Operators for Nonrelativistic Fermions and Bosons

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Abstract In quantum theory, observables are represented by linear operators on a Hilbert space. In quantum field theory (QFT), observables are expressed in terms of auxiliary operators called **field operators**. Article 28477 introduced a family of nonrelativistic models of spinless fermions and bosons. This article explains how to express those models in terms of field operators.

Contents

1 Introduction

The models introduced in articles 20554, 41522, and 28477 each had a definite number N of particles. In the real world, interactions with sufficient energy can change the number of particles. A quantum model with that behavior generally cannot have any definite number of particles, because if (say) a two-particle state can change to a three-particle state, then superpositions of those two- and three-particles states must also be allowed. Such a state has an indefinite number of particles, a concept that has no good analog in classical physics.

As a step toward learning about such models, this article describes an easier type of model, one that encompasses all different numbers of particles even though it doesn't have any interactions that change the number of particles.[1] The Hilbert space has mutually orthogonal subspaces, each with its own definite number N of particles. These subspaces are called **superselection sectors** because they are not mixed with each other by any of the model's observables. However, the model's observables are all expressed in terms of **field operators**, which do mix the superselection sectors with each other.[2] The name quantum *field* theory (QFT) alludes to operators like these.

The models constructed here are strictly nonrelativistic QFTs. In these models, each superselection sector can be treated as a model all by itself, one with a definite number of particles. The models constructed in articles 20554, 41522, and 28477 all arise this way,[3] so those models can all be regarded as parts of the quantum field theories constructed here.

Sections 4-14 introduce a model of a single fermion species. Section 15 constructs a model of a single boson species, as an easy modification of the fermion case. Sections 16-17 generalize the construction to multiple species, each of which may be fermionic or bosonic.

[1] It does have interactions that don't change the number of particles.

[2] The reason for calling them *field operators* might not be apparent here. Much of the terminology in physics comes from broad generalizations of special cases in which the words make more sense.

[3] Sections 12-13 demonstrate this for the case of a single fermion species.

2 Using a lattice to keep the math clear

In classical physics, we normally treat space as a smooth manifold. Treating space as a smooth manifold would be nice in QFT, too, except that in most cases we don't know how to do it. Even when we do know how, it involves some heavy technical details. This article uses a more straightforward appraoch: this article treats space as a discrete lattice (article 71852), one so fine that it might as well be smooth as far as all currently-feasible experiments are concerned. This is clearly artificial, and that's okay, because the models introduced here aren't supposed to be complete or exact anyway.

Here's a brief review to establish notation. The lattice is (hyper)cubic with K points along each axis, so the total number of points is K^D where D is the number of dimensions of space. Each point has $2D$ nearest neighbors. Periodic boundary conditions will be assumed. Let ϵ denote the **lattice spacing**, the distance between nearest neighbors. The lattice version of an integral over all space is

$$\int_{\mathbf{x}} f(\mathbf{x}) \equiv \epsilon^D \sum_{\mathbf{x}} f(\mathbf{x}). \tag{1}$$

The function

$$\delta(\mathbf{x}' - \mathbf{x}) \equiv \begin{cases} 1/\epsilon^D & \text{if } \mathbf{x} = \mathbf{x}' \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

has the useful property

$$\int_{\mathbf{x}} \delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}) = f(\mathbf{x}').$$

The laplacian and the kth component of the gradient are defined by

$$\nabla^2 f(\mathbf{x}) \equiv \sum_k \frac{f(\mathbf{x} + \mathbf{e}_k) + f(\mathbf{x} - \mathbf{e}_k) - 2f(\mathbf{x})}{\epsilon^2} \tag{3}$$

$$\nabla_k f(\mathbf{x}) \equiv \frac{f(\mathbf{x} + \mathbf{e}_k) - f(\mathbf{x})}{\epsilon} \tag{4}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_D$ are basis vectors for the lattice, each with magnitude ϵ.

3 Constructing a Hilbert space from an algebra

We could describe the field operators by starting with a Hilbert space and then specifying what the field operators do to it. However, sometimes an algebra-first approach is easier: we can start with an abstract algebra of operators, and then we can use the algebra to construct a Hilbert space on which the operators act.

To illustrate how this works, consider the very simple algebra generated by one operator a and its adjoint a^*, satisfying the anticommutation relations

$$\{a, a\} = 0 \qquad\qquad \{a, a^*\} = 1. \tag{5}$$

I'm using the standard notation

$$\{A, B\} \equiv AB + BA.$$

We can construct a Hilbert-space representation of these operators by starting with a vector $|0\rangle$ that satisfies

$$a|0\rangle = 0 \qquad\qquad \langle 0|0\rangle > 0$$

and then using the algebra to define other vectors. We can't get a new vector using only a, because applying a to $|0\rangle$ gives zero, but we can get a new vector using a^*, namely $|1\rangle \equiv a^*|0\rangle$. The algebra implies that this new vector is nonzero, because its inner product with itself is

$$\begin{aligned}\langle 1|1\rangle &= \langle 1|a^*|0\rangle = \langle 0|a|1\rangle^* = \langle 0|aa^*|0\rangle^* \\ &= \langle 0|(1 - a^*a)|0\rangle^* = \langle 0|0\rangle^* = \langle 0|0\rangle.\end{aligned}$$

The algebra also implies that the new vector is orthogonal to $|0\rangle$:

$$\langle 0|1\rangle = \langle 0|a^*|0\rangle = \langle 0|a|0\rangle^* = 0.$$

The two vectors $|0\rangle$ and $a^*|0\rangle$ span the whole Hilbert space, because any further application of a or a^* just mixes these two vectors with each other (or gives zero). Altogether, we have constructed a complete representation of the abstract algebra (5) on a two-dimensional Hilbert space.

4 Single fermion species: framework

To construct a model of a single fermion species with zero spin, we only need one field operator – that is, one operator $a(\mathbf{x}, t)$ for each point $\mathbf{x}$ in space and each time t, together with its adjoint $a^*(\mathbf{x}, t)$.

The **time-slice principle** (article 22871) says that the field operators $a(\mathbf{x}, t)$ for all times t should be expressible in terms of the field operators $a(\mathbf{x}, 0)$ at time $t = 0$,[4] so the algebra is generated the operators $a(\mathbf{x}) \equiv a(\mathbf{x}, 0)$. Part of the model's definition is that these operators satisfy the anticommutation relations

$$\begin{aligned} \{a(\mathbf{x}),\, a(\mathbf{y})\} &= 0 \\ \{a(\mathbf{x}),\, a^*(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}) \end{aligned} \tag{6}$$

for all points $\mathbf{x}, \mathbf{y}$ in the spatial lattice, where $\delta(\mathbf{x} - \mathbf{y})$ is the function that was defined in section 2. Now we can use the algebra-first approach to construct a Hilbert space, as illustrated in the previous section. Start with a vector $|0\rangle$ defined by the conditions

$$a(\mathbf{x})|0\rangle = 0 \tag{7}$$

for all points $\mathbf{x}$ in the lattice, together with

$$\langle 0|0\rangle > 0. \tag{8}$$

To construct other vectors in the Hilbert space, consider vectors of the form

$$a^*(\mathbf{x}_1)a^*(\mathbf{x}_2)\cdots a^*(\mathbf{x}_N)|0\rangle \tag{9}$$

where $\mathbf{x}_1, ..., \mathbf{x}_N$ is some list of N points in the lattice. Such a vector is nonzero if all of the points $\mathbf{x}_n$ are distinct. Applying the operators $a(\mathbf{x})$ and $a^*(\mathbf{x})$ to any such vector gives another such vector (or gives zero). We can use the algebra (6) together with (7)-(8) to infer the norms and inner products of all such vectors, as illustrated in the previous section. In this way, we get a complete representation of the abstract algebra (6) on a Hilbert space. The resulting Hilbert space has 2^{K^D} dimensions, where K^D is the number of points in the spatial lattice.

[4] This will be done in sections 7 and 8.

5 Single fermion species: observables

When a state is represented by a vector of the form (9), the integer N will be called the **charge** of the state. For any given N, the set of all states with charge N is a $\binom{K^D}{N}$-dimensional subspace of the full 2^{K^D}-dimensional Hilbert space.

In the present model, the rule governing which operators represent observables can be expressed concisely like this:

> A self-adjoint operator represents as an observable if and only if it doesn't mix states with different charge.

The anticommutation relations (6) imply that applying the operator $a(\mathbf{x})$ to a state with charge N gives a state with charge $N-1$, so a product of as and/or a^*s can represent an observable only if the number of as in the product is equal to the number of a^*s. Any observable is a linear combination of such products. Here are a few examples:

- $a^*(\mathbf{x})a(\mathbf{x})$ is an observable.
- $a^*(\mathbf{x})a(\mathbf{y}) + a^*(\mathbf{y})a(\mathbf{x})$ is an observable.
- The hamiltonian that will be defined in section 8 is an observable.
- $a^*(\mathbf{x}) + a(\mathbf{x})$ is *not* an observable.

Subspaces of the Hilbert space that are not mixed with each other by any of the model's observables are called **superselection sectors**. In this model, each superselection sector consists of all states with a given charge N. We can treat each superselection sector as a separate self-contained model, as in the articles cited in section 1, but treating them all together – connected to each other by the field operators – is often more convenient.

6 Single fermion species: charge operator

The rule in section 5 can also be expressed like this: a self-adjoint operator represents an observable if and only if it commutes with the **(total) charge operator**

$$C \equiv \int_{\mathbf{x}} a^*(\mathbf{x})a(\mathbf{x}). \tag{10}$$

The operator C itself obviously satisfies this condition, so it is an observable. If

$$|\psi\rangle = a^*(\mathbf{x}_1)a^*(\mathbf{x}_2)\cdots a^*(\mathbf{x}_N)|0\rangle, \tag{11}$$

then the anticommutation relations (6) and equation (7) imply

$$C|\psi\rangle = N|\psi\rangle.$$

In words: a measurement of the observable C represents a measurement of the total charge, as defined in section 5.

7 Single fermion species: local observables

Define time-dependent **field operators** by

$$a(\mathbf{x},t) = U^{-1}(t)a(\mathbf{x})U(t) \tag{12}$$

with $U(t) \equiv \exp(-iHt)$, where H is the hamiltonian that will be defined in section 8. For now, the only things we need to know about the hamiltonian are that it is self-adjoint ($H^* = H$) and that it doesn't mix states with different charge. Use equations (6) and (12) to see that the field operators satisfy these **equal-time anticommutation relations**:

$$\begin{aligned} \{a(\mathbf{x},t),\, a(\mathbf{y},t)\} &= 0 \\ \{a(\mathbf{x},t),\, a^*(\mathbf{y},t)\} &= \delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{13}$$

Section 5 (or 6) specified which operators represent observables. The next rule specifies which observables are localized in which regions of spacetime:

> An observable is localized in a spatial region R at time t if and only if it can be expressed in terms of the field operators $a(\mathbf{x},t)$ with $\mathbf{x} \in R$ and their adjoints.

Together with the rule in section 5/6, this defines everything about the model except for the hamiltonian H. The next section breathes life into the model by specifying the hamiltonian, which specifies how everything behaves in time.[5]

The nonrelativistic version of a principle called **microcausality** (article 21916) says that if two observables A and B are associated with non-overlapping regions of space at the same time, then they should commute with each other: $AB = BA$. The rules stated above are consistent with this, even though the individual field operators $a(\mathbf{x},t)$ and $a(\mathbf{y},t)$ don't commute with each other (they anticommute instead).

[5] In other words, the hamiltonian specifies the model's **dynamics**.

8 Single fermion species: hamiltonian

Let ∇ denote a discrete version of the gradient with respect to $\mathbf{x}$. In units with $\hbar = 1$, the hamiltonian for the single-species model is

$$H = H_m + H_\nabla + H_{\text{int}} \tag{14}$$

with[6,7]

$$\begin{aligned} H_m &\equiv mc^2 \int_{\mathbf{x}} a^*(\mathbf{x})a(\mathbf{x}) \\ H_\nabla &\equiv \frac{1}{2m} \int_{\mathbf{x}} \big(\nabla a(\mathbf{x})\big)^* \cdot \big(\nabla a(\mathbf{x})\big) \\ H_{\text{int}} &\equiv \frac{1}{2} \int_{\mathbf{x},\mathbf{y}} a^*(\mathbf{x})a^*(\mathbf{y})V(\mathbf{x}-\mathbf{y})a(\mathbf{y})a(\mathbf{x}) \end{aligned} \tag{15}$$

with $m > 0$ and $V(\mathbf{x}) \geq 0$. This completes the definition of the model. The physical significance of m and $V(\mathbf{x})$ can (in principle) be inferred by studying the model's behavior. The conclusion is that m is the mass of a single particle, and $V(\mathbf{x})$ specifies how the particles interact with each other.

Equation (12) says that the hamiltonian generates time evolution. According to the rule in section 5/6, the hamiltonian defined by equations (14)-(15) is an observable. By definition (article 22871), an observable that generates the model's time evolution represents the system's **total energy**. Equation (7) implies

$$H|0\rangle = 0. \tag{16}$$

The conditions $m > 0$ and $V(\mathbf{x}) \geq 0$ ensure that every term in (15) is a positive operator, and then equation (16) implies that $|0\rangle$ must be the lowest-energy state (the **vacuum state**). Equation (16) also implies $U(t)|0\rangle = |0\rangle$, and combining this with (7) gives

$$a(\mathbf{x}, t)|0\rangle = 0. \tag{17}$$

[6]The constant c relates mass to energy (section 13). We can call it the *speed of light* even though light is absent in this model.

[7]The integral is defined as in section 2, with a factor of ϵ^D for each boldface summation variable.

9 Observables that count particles

In principle, we should be able to define particles in terms of local observables that detect them. This is usually difficult, but it is easy in some special cases. Strictly nonrelativistic QFT is one of those special cases.

The usual idea of a particle is consistent with these criteria:

- Particles can be localized in space, at least approximately.
- Particles can be counted, at least when they're far enough away from each other to avoid complications.
- The lowest-energy state has none of them.

To be complete, the list of criteria should include something about how isolated particles behave over time, but here we'll focus on the criteria listed above.

Let R be a region of space, and consider the local observable

$$C(R,t) \equiv \int_{\mathbf{x}\in R} a^*(\mathbf{x},t)a(\mathbf{x},t) \tag{18}$$

where the integral is over $\mathbf{x} \in R$. According to the criteria listed above, we can interpret this observable as counting the number of particles (or, less presumptuously, the number of *charges*) in the region R at time t. To see this, consider the state

$$|\psi\rangle \equiv a^*(\mathbf{x}_1,t)a^*(\mathbf{x}_2,t)\cdots a^*(\mathbf{x}_N,t)|0\rangle. \tag{19}$$

Suppose that n of the points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N$ are within the region R and that the remaining $N - n$ point are not. Then the anticommutation relations (13) and equation (17) imply[8]

$$C(R,t)|\psi\rangle = n|\psi\rangle.$$

For any given t, states of the form (19) form a basis for the whole Hilbert space,[9] so this suggests that the local observables $C(R,t)$ count something and that the

[8] This shows that $C(R,t)$ is the same as the counting observable defined in article 28477.

[9] To see this, recall the construction in section 4 and use the fact that the time evolution operator $U(t)$ is unitary.

lowest-energy state $|0\rangle$ has none of them. That's only suggestive,[10] but the interpretation of $C(R,t)$ as counting particles can be corroborated by studying how the things it counts behave over time.[11]

In article 28477, each particle counting observable was expressed as a set of mutually orthogonal projection operators, each corresponding to a specific number of particles in the given region of space at the given time. We can do the same thing here, because the observable (18) can be written as

$$C(R,t) = \sum_{n=0}^{\infty} Q^{(n)}(R,t)\, n$$

where $Q^{(n)}(R,t)$ is the projection operator onto the subspace consisting of all states (19) (for all values of N) with exactly n of the points inside R. The observable (18) is a convenient way of packaging this set of mutually orthogonal projection operators into a single self-adjoint operator (article 03431). Each of the projection operators represents a possible outcome when this observable is measured. $Q^{(n)}(R,t)$ represents the outcome "exactly n particles are in R at time t."

When R is all of space, the observable (18) reduces to the total charge operator (10), so the total charge operator also counts the total number of particles.[12] It commutes with all observables, including the hamiltonian, so

$$\int_{\mathbf{x}} a^*(\mathbf{x},t)a(\mathbf{x},t) = \int_{\mathbf{x}} a^*(\mathbf{x})a(\mathbf{x}). \tag{20}$$

[10] It really only shows that the operator (18) has a complete set of eigenspaces, one of which is spanned by $|0\rangle$. As explained in article 03431, the eigen*values* of an observable don't really matter (at least not for its specific role as an *observable* – but some observables have other roles where the eigenvalues do matter), because nature doesn't care how we label the possible measurement outcomes. Instead of labeling the outcomes with real numbers, we could label them with words, or colors, or sounds, or whatever.

[11] More carefully: it counts the total number of *elementary* particles. If $V(\mathbf{x}) < 0$, even if $|0\rangle$ is still the vacuum state, the model may also have composite particles that are not counted by these observables (article 89695).

[12] That's a special feature of this model, not shared by most QFTs. Many QFTs have superselection sectors characterized by different values of the total charge(s), but they usually don't correspond to definite numbers of particles. That's why section 5 used the more generic word *charge* instead of *number of particles*, even though it did turn out to be the same as the number of particles in this case.

10 Fermions and the Pauli exclusion principle

Let $f(\mathbf{x})$ be a complex-valued function, and define

$$a^*(f,t) \equiv \int_{\mathbf{x}} f(\mathbf{x}) a^*(\mathbf{x},t).$$

Now consider the two-particle state

$$|\psi\rangle \equiv a^*(f,t)a^*(g,t)|0\rangle.$$

The first equation in (13) implies that $|\psi\rangle$ can also be written

$$|\psi\rangle = -a^*(g,t)a^*(f,t)|0\rangle$$

When $f = g$, this implies $|\psi\rangle = 0$, which does not represent any state. (Only nonzero vectors can be used to represent states.) This is the **Pauli exclusion principle**. If we call f the "state" of a single particle (not to be confused with the full two-particle state), then this says that two of the model's particles cannot have the same "state." Particles that respect the Pauli exclusion principle are called **fermions**.

The word *fermion* is also used more generally for a family of field operators that anticommute with each other whenever they are localized in different regions of space at the same time, even if they're not related to particles as directly as they are in the present model.

11 The Schrödinger picture

The previous sections used the Heisenberg picture, where observables are time-dependent and states are not. Here we will switch to the Schrödinger picture (article 22871), in which states are time-dependent and observables are not. Recall how it works: let

$$A(t) = U^{-1}(t)AU(t)$$

be any observable in the Heisenberg picture, where $U(t)$ are the unitary time-translation operators introduced in section 7. If we define a time-dependent version of $|\psi\rangle$ by

$$\big|\psi(t)\big\rangle \equiv U(t)|\psi\rangle, \tag{21}$$

then

$$\langle\psi'|A(t)|\psi\rangle = \big\langle\psi'(t)\big|A\big|\psi(t)\big\rangle$$

for any two vectors $|\psi\rangle$ and $|\psi'\rangle$. These are two different ways of expressing the same thing: the left-hand side is the Heisenberg picture, and the right-hand side is the Schrödinger picture.

Using $U(t) = e^{-iHt}$, the definition (21) implies

$$i\frac{d}{dt}\big|\psi(t)\big\rangle = H\big|\psi(t)\big\rangle. \tag{22}$$

This is the general **Schrödinger equation**. Section (13) derives a more explicit version of the Schrödinger equation in the N-particle sector. That will show how the hamiltonian defined in section 8 relates to the one defined in article 28477.

12 The wavefunction for N particles

Let $|\psi\rangle$ be any N-particle state, which is any state of the form

$$|\psi\rangle = \int_{\mathbf{x}_1,...,\mathbf{x}_N} \psi(\mathbf{x}_1, ..., \mathbf{x}_N) a^*(\mathbf{x}_1) \cdots a^*(\mathbf{x}_N)|0\rangle. \tag{23}$$

The quantities $\mathbf{x}_n$ in (23) are dummy integration variables, so we if we exchange two of them $\mathbf{x}_j \leftrightarrow \mathbf{x}_k$ everywhere in the integrand, then we haven't really changed anything. The operators $a(\mathbf{x}_j)$ and $a(\mathbf{x}_k)$ anticommute with each other, so if we also exchange $a(\mathbf{x}_j) \leftrightarrow a(\mathbf{x}_k)$, then the only effect on (23) is an overall change of sign. Altogether, this means that exchanging any two of the arguments of $\psi(\mathbf{x}_1, ..., \mathbf{x}_N)$ has the same effect on (23) as changing the overall sign. Therefore, we might as well take the function $\psi(\mathbf{x}_1, ..., \mathbf{x}_N)$ itself to be antisymmetric, meaning that it changes sign whenever any two of its boldface arguments $\mathbf{x}_j$ and $\mathbf{x}_k$ are exchanged.

The corresponding time-dependent state in the Schrödinger picture is defined by equation (21). In this model, time evolution doesn't change the value of N, so we can define the time-dependent N-particle **wavefunction** $\psi(\mathbf{x}_1, ..., \mathbf{x}_N, t)$ by the condition[13]

$$\int \psi(\mathbf{x}_1, ..., \mathbf{x}_N, t)\, a^*(\mathbf{x}_1) \cdots a^*(\mathbf{x}_N)|0\rangle = |\psi(t)\rangle. \tag{24}$$

Again, we can take the wavefunction to be antisymmetric under permutations of its boldface arguments. Take the inner product of both sides of equation (24) with the vector $a^*(\mathbf{x}_1) \cdots a^*(\mathbf{x}_N)|0\rangle$ and use the relationships (13) and (17) to get

$$\psi(\mathbf{x}_1, ..., \mathbf{x}_N, t) = \frac{\langle 0|a(\mathbf{x}_1) \dots a(\mathbf{x}_N)|\psi(t)\rangle}{N!\,\langle 0|0\rangle}.$$

which is manifestly antisymmetric because the operators $a(\mathbf{x}_n)$ anticommute with each other. We can use this as another (equivalent) definition of the wavefunction.

[13]The rest of this section uses $\int$ as an abbreviation for the integral over all of the boldface variables $\mathbf{x}_n$.

13 The Schrödinger equation for N particles

We can use the general Schrödinger equation (22) to derive a differential equation describing the time-dependence of the wavefunction. Start by taking the time-derivative of both sides of (24) and use (22) to get

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(\mathbf{x}_1,...,\mathbf{x}_N,t) &= i\frac{\langle 0|a^*(\mathbf{x}_1)\cdots a^*(\mathbf{x}_N)\frac{d}{dt}|\psi(t)\rangle}{N!\,\langle 0|0\rangle} \\ &= \frac{\langle 0|a^*(\mathbf{x}_1)\cdots a^*(\mathbf{x}_N)H|\psi(t)\rangle}{N!\,\langle 0|0\rangle}. \end{aligned} \tag{25}$$

Now recall the expression (14) for H. The mass term H_m is proportional to the operator (10) that counts the total number of particles, so

$$\begin{aligned} \frac{\langle 0|a^*(\mathbf{x}_1)\cdots a^*(\mathbf{x}_N)H_m|\psi(t)\rangle}{N!\,\langle 0|0\rangle} &= Nmc^2\,\frac{\langle 0|a^*(\mathbf{x}_1)\cdots a^*(\mathbf{x}_N)|\psi(t)\rangle}{N!\,\langle 0|0\rangle} \\ &= Nmc^2\,\psi(\mathbf{x}_1,...,\mathbf{x}_N,t). \end{aligned}$$

For the gradient term H_∇, the lattice version of integration-by-parts (article 71852) implies

$$H_\nabla = \int_{\mathbf{x}} a^*(\mathbf{x})\frac{-\nabla^2}{2m}a(\mathbf{x}), \tag{26}$$

which is a special case of the general form

$$J \equiv \int_{\mathbf{x},\mathbf{y}} a^*(\mathbf{x})j(\mathbf{x}-\mathbf{y})a(\mathbf{y}).$$

For any such operator, the the anticommutation relations (13) and (17) imply

$$\begin{aligned} \frac{\langle 0|a(\mathbf{x}_1)\cdots a(\mathbf{x}_N)J|\psi\rangle}{N!\,\langle 0|0\rangle} &= \int_{\mathbf{y}} j(\mathbf{x}_1-\mathbf{y})\psi(\mathbf{y},\mathbf{x}_2,\mathbf{x}_3,...,\mathbf{x}_N,t) \\ &+ \int_{\mathbf{y}} j(\mathbf{x}_2-\mathbf{y})\psi(\mathbf{x}_1,\mathbf{y},\mathbf{x}_3,...,\mathbf{x}_N,t) \\ &+ \text{and so on,} \end{aligned}$$

where the pattern should be clear. For the special case (26), this gives

$$\frac{\langle 0|a(\mathbf{x}_1)\cdots a(\mathbf{x}_N)J|\psi\rangle}{N!\,\langle 0|0\rangle} = \sum_{n=1}^{N} \frac{-\nabla_n^2}{2m}\,\psi(\mathbf{x}_1,...,\mathbf{x}_N,t)$$

where ∇_n is the gradient with respect to $\mathbf{x}_n$. For the interaction term H_{int}, the relations (13) and (17) imply

$$\frac{\langle 0|a(\mathbf{x}_1)\cdots a(\mathbf{x}_N)H_{\text{int}}|\psi\rangle}{N!\,\langle 0|0\rangle} = \sum_{j<k} V(\mathbf{x}_j-\mathbf{x}_k)\,\psi(\mathbf{x}_1,...,\mathbf{x}_N,t).$$

Altogether, this gives

$$i\frac{\partial}{\partial t}\psi(\mathbf{x}_1,...,\mathbf{x}_N,t) = \left(Nmc^2 + \sum_{n=1}^{N}\frac{-\nabla_n^2}{2m} + \sum_{j<k} V(\mathbf{x}_j-\mathbf{x}_k)\right)\psi(\mathbf{x}_1,...,\mathbf{x}_N,t). \tag{27}$$

This is the Schrödinger equation for the N-particle wavefunction. This matches the Schrödinger equation for the single-species model constructed in article 28477, except for the extra term Nmc^2. This term represents the contribution of the total mass (or **rest energy**) of the N particles to the total energy. A term like this appears whenever a nonrelativistic model is derived as an approximation to a relativistic model.[14] This term can be important even in the purely nonrelativistic context, because when the superselection sectors for different values of N are all regarded as parts of a single quantum field theory, the mass term Nmc^2 is necessary for ensuring that the state with no particles is the lowest-energy state, especially when the model may have states representing composite particles (section 18).[15]

[14] Article 77597 illustrates this in the case of a single classical particle.

[15] If we didn't account for the rest energy (mass), then a bound state of two particles could appear to have lower energy than the state with no particles, because the binding energy is negative.

14 Approximating continuous space

To avoid distracting mathematical technicalities, the preceding sections defined the model using a discrete lattice for space. That's clearly artificial: we have no reason to think that space is anything like such a lattice. It's also harmless, though, because we can take the lattice spacing to be so tiny that the model's predictions for real experiments are completely insensitive to it.

More precisely, if we consider only states of sufficiently low energy,[16] and if we consider only observables that respect that constraint, then the model's predictions are not sensitive to the lattice spacing. The low-energy constraint works because constraining the total energy implies constraining the contribution of the gradient term H_∇, and constraining the contribution of the gradient term H_∇ is the same as requiring the wavefunction (section 12) to vary only very gradually in space, compared to the lattice spacing ϵ. Predictions that involve only such functions are not sensitive to the lattice spacing ϵ.

We can take the continuum limit $\epsilon \to 0$ of the N-particle Schrödinger equation (27),[17] but we can't take $\epsilon \to 0$ in the algebra generated by the field operators (equations (6)), because the function (2) that appears in that algebra is undefined in that limit. The restriction to low-energy states described above achieves essentially the same goal – namely eliminating lattice artifacts – without requiring $\epsilon \to 0$.

[16] This requires that the coefficients in the hamiltonian (14) all have sufficiently small magnitudes – barring delicate cancellations, which do occur in many interesting QFTs, but not in this one.

[17] This is why article 28477 didn't treat space as a lattice.

15 Single boson species

To construct a model of a single boson species with zero spin, we only need a slight modification of the construction that was used for the fermion model: just replace the anticommutation relations (6) with

$$\begin{aligned} [a(\mathbf{x}),\, a(\mathbf{y})] &= 0 \\ [a(\mathbf{x}),\, a^*(\mathbf{y})] &= \delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{28}$$

I'm using the standard notation

$$[A, B] \equiv AB - BA.$$

The resulting Hilbert space is infinite-dimensional, even on a finite lattice, because the Pauli exclusion principle does not apply: the charge N of a state can be arbitrarily large, even on a lattice with only one point. The time-dependent field operators satisfy the correspondingly modified version of (13), with anticommutators $\{\cdot,\cdot\}$ replaced by commutators $[\cdot,\cdot]$. The rules used in sections 5-7 to specify which operators can represent observables in which regions of spacetime are still appropriate here.[18] The hamiltonian is the same as in section 8, and the analysis in section 9 is unchanged. The sign-change that led to the Pauli exclusion principle in section 10 is absent in the boson case, so the state

$$a^*(f,t)a^*(f,t)\cdots a^*(f,t)|0\rangle$$

is nonzero no matter how many factors of $a^*(f,t)$ are applied, all with the same f. The equations shown in sections 11-13 are unchanged, except that now the wavefunction is symmetric instead of antisymmetric: it is invariant under arbitrary permutations of its boldface arguments. The comments about approximating continuous space in section 14 are unchanged.

[18] In the bosonic case, the restriction imposed in section 5/6 could be relaxed without violating microcausality (section 7), but the restriction in section 5/6 arises naturally when the model is derived as an approximation to a more complete model in which the particles interact with each other via the electromagnetic field. In that context, the charge N really is the total *electrostatic* charge, and **gauge invariance** requires that states with different total electrostatic charge belong to different superselection sectors in the more complete model.

16 Multiple species: field operators and hamiltonian

The generalization to multiple species is easy. Instead of just one field operator $a(\mathbf{x}, t)$ (and its adjoint) at each point $\mathbf{x}$ at each time t, we have a separate field operator $a_s(\mathbf{x}, t)$ for each species s. For any given species, the field operator algebra is just like before, with anticommutators for fermions and commutators for bosons. The field operators associated with different species $s \neq s'$ satisfy

$$[a_s(\mathbf{x}, t),\, a_{s'}(\mathbf{y}, t)] = 0 \qquad\qquad [a_s(\mathbf{x}, t),\, a^*_{s'}(\mathbf{y}, t)] = 0$$

if at least one of the species is a boson, and they satisfy

$$\{a_s(\mathbf{x}, t),\, a_{s'}(\mathbf{y}, t)\} = 0 \qquad\qquad \{a_s(\mathbf{x}, t),\, a^*_{s'}(\mathbf{y}, t)\} = 0$$

if both species are fermions. In the hamiltonian, each species can have a different mass $m_s > 0$, and each pair of species (same or different) can have a different interaction function $V_{s,s'}(\mathbf{x})$. Explicitly: $H = H_m + H_\nabla + H_{\text{int}}$ with

$$\begin{aligned}
H_m &\equiv \sum_s m_s c^2 \int_{\mathbf{x}} a^*_s(\mathbf{x}) a_s(\mathbf{x}) \\
H_\nabla &\equiv \sum_s \frac{1}{2m_s} \int_{\mathbf{x}} \big(\nabla a_s(\mathbf{x})\big)^* \cdot \big(\nabla a_s(\mathbf{x})\big) \\
H_{\text{int}} &\equiv \frac{1}{2} \sum_{s,s'} \int_{\mathbf{x},\mathbf{y}} a^*_s(\mathbf{x}) a^*_{s'}(\mathbf{y}) V_{s,s'}(\mathbf{x} - \mathbf{y}) a_{s'}(\mathbf{y}) a_s(\mathbf{x}).
\end{aligned}$$

Section 18 addresses the signs of the interaction functions $V_{s,s'}$.

17 Multiple species: local observables

An observable is localized in a given region R at a given time t if and only if it can be expressed entirely in terms of the field operators $a_s(\mathbf{x}, t)$ with $\mathbf{x} \in R$, together with their adjoints. The right way to generalize the rule in section 5/6 depends on the model's heritage as an approximation to a more complete model. The possibilities include:

- The model's observables all commute with all of the single-species charge operators

$$C_s \equiv \int_{\mathbf{x}} a_s^*(\mathbf{x}) a_s(\mathbf{x}), \tag{29}$$

  for every s.

- The only restriction on the model's observables is that they all commute with the operator

$$C \equiv \sum_s \int_{\mathbf{x}} a_s^*(\mathbf{x}) a_s(\mathbf{x}). \tag{30}$$

  This can be appropriate if all of the species are fermions (or all bosons) with the same mass and same interactions, so that they are all related to each other by symmetry,[19] because then they can all be regarded as a single species with multiple components.

Section 18 considers an important case that is intermediate between these two possibilities. That case uses two species to represent electrons with nonzero spin (footnote 19) and other species to represent various types of atomic nucleus.[20] In that case, observables should commute with the charge operator that is summed

[19] In this model, the symmetry is an **internal symmetry**, meaning that it doesn't mix observables associated with different regions of spacetime. In a more complete model, which this one approximates, the symmetry could come from a spacetime symmetry: the different species in this approximate model could correspond to different spin-components of a single species in the more complete model.

[20] Accounting for the nonzero spin of a nucleus is not important in most applications of this model.

over the two electron species (like (30)) and also with the individual charge operators for each nucleus species (like (29)). States that differ from each other in the value of one or more of these charge operators belong to different superselection sectors.

18 The spectrum condition

The family of models defined in sections 16-17 includes an important special case, defined by these conditions:

- Space is three-dimensional: $D = 3$.
- The interactions are Coulomb interactions: $V_{s,s'}(\mathbf{x}) = e_s e_{s'} / \big(4\pi\epsilon_0 |\mathbf{x}|\big)$.
- Two species,[21] both called *electrons*, are fermions with equal values of $e_s < 0$ and the same single-particle mass m.
- The rest of the species, called *nuclei*, are fermions or bosons with $e_s > 0$ and masses that are $\gg m$ but otherwise arbitrary.

This model captures some of the qualities of electrons and nuclei that are most important for matter under ordinary conditions.[22] Article 28477 mentioned that if the rest energy term H_m is ignored, then the lowest possible energy E_0 of a system with N_- electrons of charge e_- and N_+ nuclei is bounded by[23]

$$E_0 \geq -\kappa(N_- + N_+) \qquad \text{with } \kappa \sim m e_-^4 / (\epsilon_0 \hbar)^2. \tag{31}$$

The lower bound is negative because the model ignores the particles' rest energies (footnote 15). If the rest energy term H_m is included, then the bound becomes

$$E_0 \geq mc^2 N_- + m_{\text{nucleus}} c^2 N_+ - \kappa(N_- + N_+).$$

Now increasing the number of particles increases the total energy, because $mc^2 \gg \kappa$, with an even more extreme inequality for the N_+ terms. This ensures that the model satisfies the spectrum condition (article 21916) even when all superselection selectors are included, and it ensures that the state with no particles has the lowest energy overall, as it should (section 9).

[21] The two species account for the electron's spin (footnote 19).

[22] It doesn't include electromagnetic interactions (except for the static Coulomb interactions), but it can still exhibit complex phenomena that resemble real molecules, fluids, and solids.

[23] When every nucleus is a single proton and $N_+ = N_-$, equation (55) in Lieb (1976) says $E_0 \geq -\kappa N_-$ with $\kappa = 23$ times the Rydberg energy. This excludes the protons' kinetic energy, which can only increase the lower bound.

19 References

Lieb, 1976. "The stability of matter" *Reviews of Modern Physics* **48**: 553-569

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